Algebraic approach for the reconstruction of Rössler system from the x_3- variable

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In this paper we propose a simple method to identify the unknown parameters and to estimate the underlying variables from a given chaotic time series $\{x_3(t_k)\}_0^{k=n}$ of the three-dimensional Rössler system (**RS**). The reconstruction of the **RS** from its x_3- variable is known to be considerably more difficult than reconstruction from its two other variables. We show that the system is observable and algebraically identifiable with respect to the auxiliary output $\ln(x_3)$, hence, a differential parameterization of the output and its time derivatives can be obtained. Based on these facts, we proceed to form an extended re-parameterized system (linear-in-the -parameters), which turns out to be invertible, allowing us to estimate the variables and missing parameters.

Keywords: Chaotic systems; inverse problem; estimation of parameters and variables.

Este articulo se presenta un método sencillo para recuperar el los parámetros del modelo y para recuperar las variables no disponibles del sistema caótico de Rossler, a partir de el conocimiento de una serie de tiempo $\{x3(t_k)\}_0^{k=n}$. Es muy bien sabido, que reconstruir este sistema a partir de la variable x_3 es mas difícil que tratar de reconstruirlo a partir de las otras variables. Usando el hecho que este sistema es identificable y algebraicamente observable con respecto a la transformación $\ln(x3)$, se procede a obtener una parametrización diferencial de la salida. Esta parametrización puede ser invertible bajo ciertas condiciones. Permitiéndonos estimar parámetros y variables desconocidas del modelo.

Descriptores: Sistemas caóticos; problema inverso; estimación de parámetros y variables.

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1. Introduction

In the last two decades, considerable attention has been paid to the reconstruction of chaotic attractors from one or more available variables (see the pioneering works by Taken [1], Packard *et al.* [2] and Sauer *et al.* [3]). This is an interesting and challenging topic that allows us to test the accuracy of some empirically derived models [4, 5]. This inverse problem consists in recovering the underlying variables and unknown parameters from a partial knowledge of a particular chaotic system. There are two ways to approach this problem. The first approach is based on embedding a time series of the observed variables in a phase space. Roughly speaking, the vector state is constructed with the time delayed values of the measured scalar quantity [6–11]. The other approach exploits control theoretical ideas, such as inverse system design and system identification, generally using Kalman's filters, Luenberger's observers and high gain observers [12–19].

According to the second approach, we recover (approximately) the set of non-available parameters and the remaining states of the **RS**, based on the knowledge of a recorded time series, which is the sampling variable x_3 from the **RS** [i]. We emphasize that the observability index of the **RS** with respect to variable x_3 is the smallest of the three states. A small index indicates little information content, which implies great difficulties [20]. That is, identification of the **RS** from x_3-variable poses more problems than utilizing any of the other two variables x_1 and x_2. So, we approach the identification problem using the algebraic properties of observability and identifiability of the **RS**. These properties allow us to find a differential parametrization of the recorded data and a finite number of its time derivatives. Then, based on this parameterization, we show that it is possible to recover the missing states and the unknown parameters. This approach requires the time derivatives (from first to third) of the data set, which are solved with a digital differentiator [21].

The rest of this paper is organized as follows. Section 2 is devoted to studying some important algebraic properties of the **RS.** In Sec. 3, we establish the framework of the identification problem and introduce a digital filter for estimation of the time derivatives of the recorded data set. Section 4 presents the results of the simulations. Section 5 is devoted to giving some conclusions. Finally, in the Appendix we provide a proof of the proposition.

2. Problem Definition

Consider the **RS** which is defined by a set of three differential equations,

$$\dot{x}_1 = -(x_2 + x_3),$$
$$\dot{x}_2 = x_1 + ax_2,$$
$$\dot{x}_3 = b + x_3(x_1 - c), \qquad (1)$$

where the coefficients a,b, an c are adjustable constants. Originally, this system, introduced by Otto Rössler, arose out of work in chemical kinetics [22]. This system presents a chaotic behavior in a large neighborhood of $\{a = b = 0.2, c = 5\}$, and it is considered to exhibit one of the simplest possible strange attractors [23].

It is well known that state x_3 shows a highly complex behavior, which consists of a set of spikes with irregular amplitude[ii], so that a convenient nonlinear coordinate transformation for numerical purpose is presented as:

$$z_1 = x_1, z_2 = x_2, z_3 = \ln x_3,$$

with $x_3 \neq 0$. Hence, in this new coordinate system, (1) becomes

$$\begin{aligned}
\dot{z}_1 &= -z_2 - \exp(z_3),\\
\dot{z}_2 &= z_1 + az_2,\\
\dot{z}_3 &= -c + z_1 + b\exp(-z_3). \qquad (2)
\end{aligned}$$

This system, referred to as transformed **RS**, has only one nonlinear term, $\exp(z_3)$. Evidently it is easier to study and analyze than the original system (1), which involves two variables in the nonlinear term x_3x_1.

Remark 1: Because the estimation of the states $\{x_1\,,\,x_2\}$ from variable x_3 has a very small observability index, then, intuitively, it is not possible to recover (with high accuracy) the underlying states around a valley or a crest of the recorded signal x_3; that is, an information portion is lost. This inconvenience is partially solved (numerically) using the nonlinear transformation $z_3 = \ln x_3$, which has the advantage of smoothing the variable x_3 peaks. Therefore, it is easier to estimate and synchronize, numerically, the states x_1 and x_2. [25].

The problem addressed in this paper consists in determining the unknown parameters a, b and c, from a given recorded set $\{x_3(t_k)\}_0^{k=n^{iii}}; t_k \in \Im\}$ where $\Im$ is a discrete set of observation times

$$\begin{aligned}
\Im &= (t_1, t_2, \ldots, t_n); t_{j+1} - t_j = T\\
j &= \{1, 2, \ldots, n-1\}. \qquad (3)
\end{aligned}$$

Lainscsek *et al*. [5] recover a global model from the x_3- variable, by means of an Ansatz library. They employ embedding methodology as a tool to derive a model in space spanned by the state variable of the time-series itself, while generic functions of the other two state variables are formed. One disadvantage of their method is the use of the Genetic Algorithm to obtain the inverse of some nonlinear transformations. In contrast, we solve the problem in a straightforward way by using some algebraic properties, which we discuss in the next section.

2.1. Some Algebraic properties

We introduce two useful properties that the transformed **RS** satisfies [18].

Definition 1: *Consider an undetermined system of ordinary differential equations*

$$G\left(t, X, \dot{X}, P\right) = 0, \qquad (4)$$

where $X^{\ T} = (x_i)_1^{i=n} \in R^n$ is a state vector and $P^{\ T} \in R^l$ is a constant parameter vector. Suppose that there exists a smooth, local and one to one correspondence between solution $X(t)$ of system (4) *and an arbitrary function $y(t) = h(t, X(t)) \in R$; then, state x_i is said to be algebraically observable with respect to $y(t)$ if it satisfies*

$$x_i = \frac{f_i(y, \ldots, y^{(m)}, P)}{g_i(y, \ldots, y^{(s)}, P)},$$

where f_i, g_i and h are smooth maps, $y^{(k)}$ is the k^{th} derivative of y, l, m and s are integers, with $m \leq s$. Variable y is the output. If x_i is observable for every $i = 1, \ldots, n$, then we say that the system is completely observable.

Definition 2: *Under the same conditions as Definition 1. If we can find a smooth map $W : R^j \rightarrow R^l$ such that*

$$0 = W(y, \dot{y}, \ldots, y^{(j)}, P),$$

then the parameter vector P is said to be algebraically identifiable with respect to the output y.

That is, a system is algebraically observable if there exists a suitable variable y (output) such that all the variables can be differentially parameterized solely in terms of y and its respective time derivatives. Moreover, if vector P is a root of a differential parametric function of y, we say that the system is algebraically identifiable.

Indeed, we show that system (2) satisfies the previous definitions when the output $y = z_3$. Clearly, variables z_1 and z_2 can be rewritten as

$$\begin{aligned}
z_1 &= c - b\exp(-y) + \dot{y};\\
z_2 &= -\exp(y) - b\exp(-y)\dot{y} - \ddot{y},
\end{aligned}$$

hence, system (2) is algebraically observable with respect to the selected output. Moreover, from the third equation of (2), we obtain

$$\begin{aligned}
y^{(3)} = &-c - (1 + \exp(y))\dot{y} + a(\exp(y) + \ddot{y})\\
&+ b\left(1 + a\dot{y} + \dot{y}^2 - \ddot{y}\right)\exp(-y). \qquad (5)
\end{aligned}$$

Finally, we conclude that system (2) is identifiable with respect to the output y, because the above differential parameterization of the output y can be written as

$$0 = W(y, \dot{y}, \ddot{y}, y^{(3)}, p)$$

with $p \stackrel{\triangle}{=} [a, b, c]$.

3. Model parameter estimation

The differential parameterization (5) can be rewritten as:

$$S(t) = -c + aF_a(t) + bF_b(t) + abF_{ab}(t) \tag{6}$$

where

$$S(t)=y^{(3)}(t)+(1+e^{y(t)})\dot{y}(t); \quad F_a(t)=e^{y(t)}+\ddot{y}(t),$$
$$F_b(t)=(1+\dot{y}^2(t)-\ddot{y}(t))e^{-y(t)}; \quad F_{ab}(t)=\dot{y}(t)e^{-y(t)}. \tag{7}$$

This makes it possible to build an extended re-parameterized linear system of the output and its time derivatives, which is formed evaluating (6) at different times $\{t_k, t_{k-1,} t_{k-2}, t_{k-3}\} \subset \Im$. This yields

$$\Phi[t_k : t_{k-3}]Q = \Sigma[t_k : t_{k-3}] \tag{8}$$

where

$$\Phi[t_k : t_{k-3}]=\begin{bmatrix} -1 & F_a(t_{k-3}) & F_b(t_{k-3}) & F_{ab}(t_{k-3}) \\ -1 & F_a(t_{k-2}) & F_b(t_{k-2}) & F_{ab}(t_{k-2}) \\ -1 & F_a(t_{k-1}) & F_b(t_{k-1}) & F_{ab}(t_{k-1}) \\ -1 & F_a(t_k) & F_b(t_k) & F_{ab}(t_k) \end{bmatrix}, \tag{9}$$

and

$$Q = \begin{bmatrix} c \\ a \\ b \\ ab \end{bmatrix} \quad ; \quad \Sigma[t_k : t_{k-3}] = \begin{bmatrix} S(t_{k-3}) \\ S(t_{k-2}) \\ S(t_{k-1}) \\ S(t_k) \end{bmatrix}. \tag{10}$$

Now, the following proposition allows us to estimate vector Q, by computing a simple inverse matrix, under the following basic assumptions:

A.1) The set of equations (1) has a chaotic behaviour, where the trajectories of the **RS** are asymptotic to a compact attractor **A**.

A.2) The time derivatives (from first to third) of the output are always available.

Proposition 1: Consider the system (1) under assumptions *A.1* and *A.2*. Then, the inverse of matrix (9) exists almost for any time.

Proof: (the proof is given in the Appendix).

Remark 2: In order to simplify the following identification method, we prefer to consider the relaxed case when the system exhibits a chaotic behavior, instead of the case when its behavior is periodical or quasiperiodical. If we considered the second case, then it would be necessary to use the Poincaré maps, which leads to a highly sophisticated and elaborated analysis. Also, a characteristic of the attractor of the **RS** is that the signal x_3 is positive and is formed by a set of spikes with irregular amplitude. Consequently, the time series $y(t_k) = \ln(x_3(t_k))$ is well defined in the attractor **A**. *A2* will be relaxed by numerical calculation of the derivatives of the recorded signal $\{y(t_k)\}_0^{k=n}$ (from first to third); this can be done since $y(t_k) = \ln(x_3(t_k))$; hence, the time derivatives of y can be computed using finite derivatives, as we show in the next section.

Remark 1: Another possibility for solving the problem is the least-squares method. For instance, a convenient quadratic function may be:

$$M(p)=\sum_0^{k=n} \left[-c+aF_a(t_k)+bF_b(t_k)+abF_{ab}(t_k)-S(t_k)\right]^2 .$$

with $t_k \in \Im$. In other words, finding the vector p is equivalent to minimizing $M(p)$ for $p \in R^3$. However, it is evidently more efficient to recover the unknown parameters a, b, and c by means of **Proposition 1**.

3.1. Numerical Differentiators

A suitable method for estimating the time derivatives on a discrete set of recorded data was developed in Ref. 21. The method consists in approximating a window of data $\{y(t_{k-W}), \ldots, y(t_{k)}\}$ by means of an interpolating polynomial

$$\widetilde{y}(t) = \sum_0^{k=N} a_k(t-(k-W)T)^k, \tag{11}$$

where the coefficients $\{a_0, \ldots, a_N\}$ are computed from the least squares solution of

$$\begin{bmatrix} 1 & 0 & .... & 0 \\ 1 & T & .... & T^N \\ : & : & : & : \\ 1 & WT & & (WT)^N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ : \\ a_N \end{bmatrix} = \begin{bmatrix} y(t_{k-W}) \\ : \\ : \\ y(t_k) \end{bmatrix}. \tag{12}$$

N is the order of the interpolating polynomial, $W+1$ is the window points number, and T is the sampling time. Thus, the time derivatives of $\widetilde{y}$ at time s are

$$\widetilde{y}^{(j)}(s) = \sum_0^{k=N} a_k \frac{d^j}{dt^j} \left\{(t-(k-W)T)^k\right\}\Big|_{t=s}, \tag{13}$$

with $t_{k-W} \leq s \leq t_{k.}$ Notice that it is convenient to implement centered differentiators, because the set $\{t_j, y(t_j)\}_{j=k-W}^N$ is available. Hence, it is possible to estimate the time derivatives in the center of the moving window given by $s = kT - WT/2$.

We select the spline-based interpolating polynomial to approximate the selected set of data windows, since a lower-order polynomial can be more accurate than higher-order polynomials[iv]. In practice, decreasing the window size allows a higher-frequency noise to pass. For a very small T, the window size should be increased to capture more information about the signal y in order to smooth out the calculated derivatives. For higher noise levels, we need to increase the window size in order to filter out most of the noise. This works up to a certain limit, after which the error becomes independent of the window size [15]. Also, an advantage of this method over other differentiators is its convenient transient behaviour.

4. Numerical Simulations

The proposed identification scheme, (8) to (10), in conjunction with the selected spline interpolate method, (11) to (13), is illustrated with some numerical simulations. For generation of the chaotic time series, we used a fourth-order Runge-Kutta algorithm, with a precision of 6 decimal numbers, from $t = 0$ to $t = 5$ seconds. The step size in the numerical method was set to 5×10^{-4}seconds. The parameter values were set as $a = 0.25$, $b = 0.3$ and $c = 8$, and the initial conditions were set as $x_1(0) = 4.56$, $x_2(0) = -1.69$ and $x_3(0) = 0.07$. The parameter values of the spline were selected as $N = 5$ and $W = 6$. The evaluation of the time derivatives was implemented at the moving time $s = (k-3)T$. The estimation process was started after $t \geq 0.5$ seconds.

Figures 1 to 3 show the error evolution of each output's time derivatives, defined by $e_j = y^{(j)} - \widetilde{y}^{(j)}$; for $j = 1, 2, 3$, for the time sampling $T = 0.06$ [s] and $T = 0.025$ [s], respectively. The behaviour of the method's solution is consistent with the motion, *i.e.* a better performance is obtained with smaller sampling time.

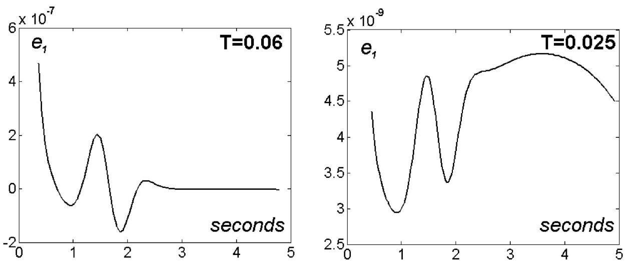

FIGURE 1. Error evolution of the first output's time derivative, for two time samplings.

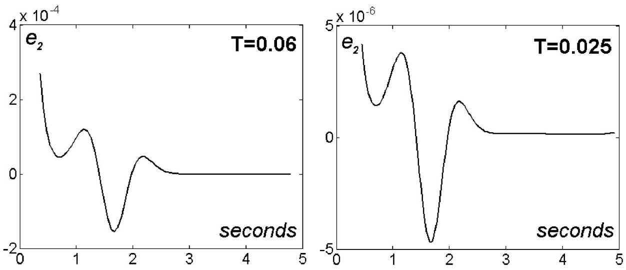

FIGURE 2. Error evolution of the second output's time derivative, for two time samplings.

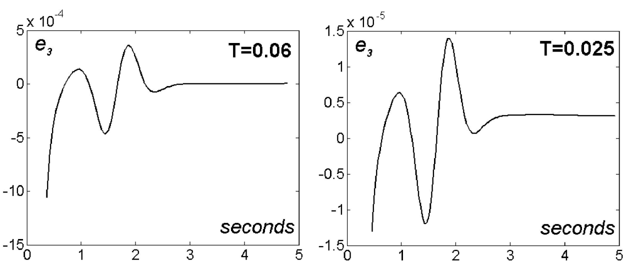

FIGURE 3. Error evolution of the third output's time derivative, for two time samplings.

Figures 4 to 6 show the numerical values of the parameters a, b, and c, for the same time sampling $T = 0.06$ [s] and $T = 0.025$ [s].

The obtained parameters are quite reasonable, particularly for time sampling $T = 0.025$ [s]. However, to have a better estimation of the parameters, the window size must be increased in order to avoid the ill-condition of the least-squares method.

The second experiment was the same as the first one, except for the following abrupt variations in the values of the parameters: if $t \leq 2.5[s]$, then $\{a = 0.25, b = 0.3, c = 8\}$, or else $\{a = 0.25, b = 0.25, c = 6\}$ for a sampling time $T = 0.02$[s].

Figure 7 shows the values of the parameters obtained by the numerical simulation in the second experiment. Notice that in the time interval $2.5 \leq t \leq 2.8$ the estimation fails, because the abrupt variations in the parameters were introduced when $t = 2.5$[s].

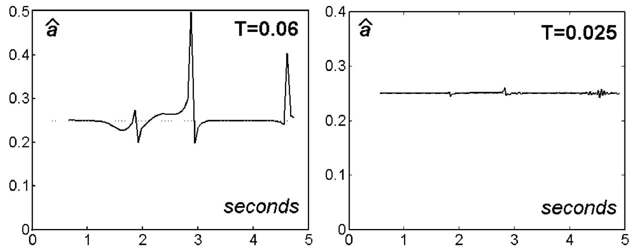

FIGURE 4. Estimation of parameter a.

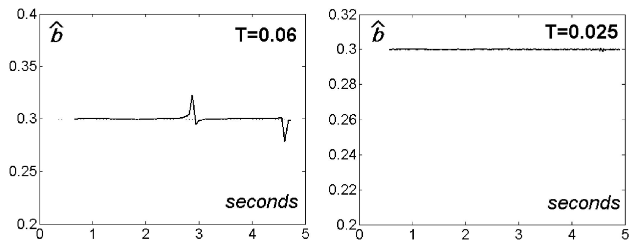

FIGURE 5. Estimation of parameter b.

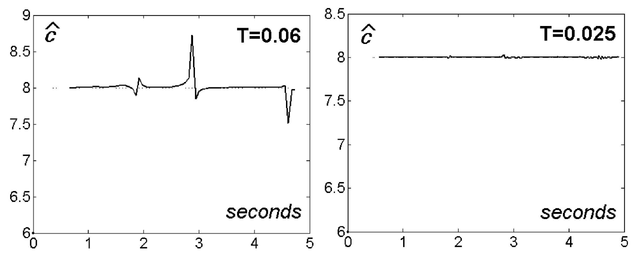

FIGURE 6. Estimation of parameter c.

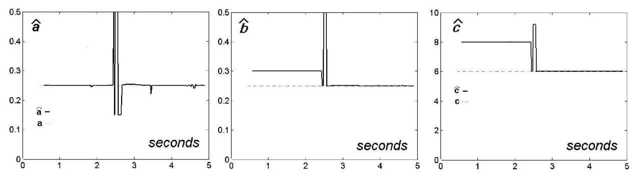

FIGURE 7. Estimation of the parameter set when an abrupt variation is introduced in the model.

5. Conclusions

The differential algebraic approach allows us to recover the parametric model of the **RS** from the knowledge of a given time series $\{x_3(t_k)\}_{k=0}^{n}$. We exploit the algebraic properties of observability and identifiability that Rössler's model fulfills with respect to the auxiliary output $y = ln(x_3)$. This facts permits us to obtain a differential parameterization to the output and its time derivatives (from first to third). The differential parameterization to the output contains the information necessary for determining the remaining states and the unknown parameters. So, we evaluate in different times this parameterization to form an extended over-parameterized linear system, which turns out to be invertible with respect to the new parameters. The identification approaches in combination with the spline method (to evaluate the time derivatives) are illustrated by numerical simulations.

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Appendix

Proof of proposition 1:

The proof follows from the well-known Poincaré-Bendixon theorem and the following Lemma [19].

Lemma: If the real set of functions $\{\Phi_i(t)\}_{i=1}^{m}$ are linearly independent in a time interval $t_i < t < t_f$, then the following matrix

$$\begin{bmatrix} \Phi_1(t_1) & ..... & \Phi_m(t_1) \\ : & & : \\ : & & : \\ \Phi_1(t_m) & & \Phi_m(t_m) \end{bmatrix}$$

is nonsingular for $t_i \leq t_1 < t_2 < \ldots < t_m \leq t_f$.

Let us begin to prove the proposition. Suppose that the set of real functions $\{1, F_a(t), F_b(t), F_{ab}(t)\}$ is linearly dependent on a time interval $I = [t_i, t_f]$ (t_i is the time when the trajectories of the **RS** lie in the attractor **A**. In practice, t_i is very small), where the functions $F_a(t)$, $F_b(t)$ and $F_{ab}(t)$ are given in (7). There are nonzero constants c_1, c_2, c_3, and c_4, such that

$$c_2\left(e^{y(t)} + \ddot{y}(t)\right) + c_3\left(1 + \dot{y}^2(t) - \ddot{y}(t))e^{-y(t)}\right) + c_1 + c_4\dot{y}(t)e^{-y(t)} = 0, \quad (14)$$

since $y = \ln(x_3)$, so that $\dot{y} = \dot{x}_3/x_3$ and $\ddot{y} = (\ddot{x}_3 - \dot{x}_3^2)/x_3^2$, which are well defined by *A1*. Substituting the latter three relations into (14), we have, after some manipulation, the following differential equation:

$$\ddot{x}_3 = \frac{2c_3\dot{x}_3^2 + c_4x_3\dot{x}_3 - c_2x_3\dot{x}_3^2 + c_3x_3^2 + c_1x_3^3 + c_2x_3^4}{c_3 - c_2x_3}.$$

It should be noticed that $c_3 - c_2x_3$ must be different from zero, because the entiny $\ddot{x}_3$ is well defined. Hence, x_3 is a solution of a second order differential equation in the time interval I. But this is a contradiction because by the Poincaré-Bendixon theorem [6], it is well known that a second order differential equation cannot exhibit a chaotic behaviour (recalling *A1*). Therefore, the real functions $\{1, F_a(t), F_b(t), F_{ab}(t)\}$ are linearly independent in a time interval I.

Of course, we need to select the time series $\{x_3(t_k)\}_{k=0}^{n}$ with $t_k \in \Im = (t_1, t_2, ..., t_n)$, such that $[t_1, t_n] \subset I$. It is necessary to take $n = 4$ (see 6).

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i. Other authors describe the **RS** by using the states x,y and z. Here, we use the variables x_1,x_2 and x_3 , because we use the symbol y to refer to the observed variable (available variable).

ii. Much of this behavior is described by one-dimensional logistic map, that is,the chaotic behavior of x_3 can be approximated to the map $x_{3,k+1} = \lambda x_{3,k}\,(1 - x_{3,k})$; with $x_0 > 0$. Besides, the initial condition $x_3(0) > 0$ leads to $x_3(t) > 0$ for all $t > 0$, hence, $z_3(t)$ is well-defined [24, 25].

iii. $\{x_3(t_k)\}_0^{k=n}$ is the single noise-free time series observed from system (1).

iv. Above all if the data set $\{t_j, y(t_j)\}_{j=k-W}^{N}$ includes local abrupt changes in the values of $y(t)$ for a steady change in the value of t, then high-order interpolating polynomial produces more oscillations around the abrupt changes.

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