An unphysical result for the Landau-Lifshitz equation of motion for a charged particle

G. Ares de Parga[a], R. Mares[a], and S. Domínguez[b]
[a]*Escuela Superior de Física y Matemáticas,*
[b]*Escuela Superior de Cómputo,*
Instituto Politécnico Nacional,
U.P. Adolfo López Mateos, Zacatenco,
México D.F., C.P. 07738, México.

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An unphysical result for the Landau-Lifshitz equation of motion for a charged particle is presented. The similarity with the Lorentz-Dirac equation is discussed. Indeed the reaction force obtained for the uniform electric field vanishes when the motion is parallel to it in both cases. A discussion of this unphysical result is given and the need for of an expression for the radiation rate of energy for the Landau-Lifshitz theory is emphasized.

Keywords: Special relativity; classical field theories; radiation reaction force.

Se presenta un resultado no físico en la ecuación de movimiento de Landau-Lifshitz para una partícula cargada. Se discute la similitud con la ecuación de Lorentz-Dirac. En efecto, la fuerza de reacción a la radiación obtenida para el caso de un campo eléctrico constante y paralelo al movimiento se anula en ambos casos. Se realiza un análisis del resultado no físico y se hace énfasis en encontrar una expresión de la taza de energía radiada en la teoría de Landau-Lifshitz.

Descriptores: Relatividad espacial; teorías de campo clásicas; fuerza de reacción a la radiación.

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1. Introduction

The search for an equation of motion for a point charged particle which considers a term due to radiation effects began with the work carried out by Abraham, Lorentz, and Planck [2] practically at the same time at the end of the nineteen century and beginning of the twentieth. The appearance of quantum mechanics left the problem to one side. Nevertheless, since Dirac [1] obtained, the Lorentz-Dirac equation [LD] of motion for a charged point particle in 1938, many discussions about its validity have appeared. Indeed, it is one of the most controversial equations in the history of physics [2]. The third order time derivative term leads to runaway and preaccelerated solutions. Asymptotic conditions or appropriate boundary conditions are imposed on the equation in order to neglect the non-classical results [3], leaving the corrections then to the quantum domain. Moreover, the development of quantum electrodynamics by the middle of the last century left this problem aside. Nevertheless, during the seventh decade of the last century, Shen [4,5] showed that there is a region over an Energy vs Field diagram where quantum effects can be neglected and a classical equation is required. Indeed, this region corresponds to the order of magnitude analyzed in Plasma Physics and Astrophysics. Moreover, in this same order of ideas, this region permits us to design an experiment to ascertain which is the equation for a charged particle [6].

New proposals have appeared in the last four decades but none with any appreciable impact, except for Sphon's proposal [16]. As an example, the Mo-Papas equation [7] has been criticized by Shen [8], and the Cook series representation [9] was rejected by Peter [10] and Ares de Parga [11]. Bonnor proposed a radiating mass [12] and he criticized the idea himself in the same paper, and it was discarded by Ares de Parga [13] later on. The list of such examples is uncountable, but the result is that each time a promising idea appears, there is always a counterpart and the problem remains open. The failure of an alternative equation and the formal works realized by Synge [14] and Teitelboim [15] supporting the LD equation, indicate that the solution consists in an adequate interpretation of it. Recently, Sphon [16] presented a mathematical work in which he proved that the old LD equation must be restricted to its critical surface, yielding the Landau-Lifshitz [17] equation [LL]. Indeed, Dirac's asymptotic condition forces the solution to be on the critical manifold. So even if Landau and Lifshitz deduced their equation as the first order iteration of the LD equation, it has to be taken into account that the solutions to this last equation are the exact solutions to the old problems of the LD equation, within the Shen region [5]. It must be pointed out that Herrera [18] obtained a particular equation which coincides with the LL equation for fields with

$$\frac{\partial F\mu\nu}{\partial x^{\sigma}} = 0.$$

The Herrera equation has been solved for different cases [18,19], giving apparently physical results. In the same order of ideas Rorlhich [20] asserts, about the LL equation, "The result is an equation free of unphysical solutions. The deeper mathematical meaning of this approximation can be learned from Kunze and Spohn [21]". Finally, we can conclude that nowadays the LL equation, within the Shen [5] region, supported by the mathematical work done by Kunze and Sphon [16,20], represents the solution for the description of the motion of a classical point charge. Unlike the LD equation, an important result is that the LL equation eliminates the runaway solutions and the preaccelerations. Preaccelerations survive even if we consider asymptotic conditions for the LD equation. So the solutions for the LL equation or the LD equation with asymptotic conditions, are different. In this order of ideas, although we know that the physical solutions will correspond to the LL equation, it will be interesting to consider the differences between both equations for critical situations. One of the critical situation, where unphysical results may appear, is for the simple case of a constant electric field. Indeed, the LD equation [22,23] reaction force vanishes when a constant electric field is applied in the same direction as the initial motion. It should be noted, as Parrott has mentioned [23], that the LD equation and other equations present the same problem. It is expected that, for the LL equation, the result will be repeated. We shall discuss why this is an unphysical result, and propose that the problem may not consist in considering the solution on a critical manifold of the LD equation, but in analyzing the classical deduction of the LD equation.

2. Landau-Lifshitz equation

The Lorentz-Dirac [1] equation of motion for a charged particle is:

$$m\,\frac{du^\mu}{ds} = e\,F^{\mu\nu}u_\nu + \frac{2}{3}e^2\left[\frac{d^2u^\mu}{ds^2} - u^\mu u^\nu\frac{d^2u_\nu}{ds^2}\right]. \quad (1)$$

Here u is the four-velocity of a charged particle of mass m and charge e, s denotes its proper time, F is the field, tensor for an external electromagnetic field and the velocity of light is taken as unity. Solutions of this equation for some physical situation appear physically unreasonable. Many authors have proposed modifications which might result in physically reasonable solutions, including an equation proposed in the classical text of Landau and Lifshitz [16,17,19,21]. As we mentioned above, although Landau and Lifshitz deduced the equation by means of an iteration, for Sphon the solutions of the equation, must be considered to be the exact physical results. The Landau-Lifshitz equation for a charged particle is:

$$m\frac{du^\mu}{ds} = eF^{\mu\nu}u_\nu + g^\mu_{LL}\ , \quad (2)$$

where g^μ_{LL} represents the Landau-Lifshitz reaction force and is expressed by:

$$g^\mu_{LL} = \frac{2}{3}\frac{e^3}{m}\frac{\partial F^{\mu\nu}}{\partial x^\gamma}u_\nu u^\gamma - \frac{2}{3}\frac{e^4}{m^2}F^{\mu\gamma}F_{\nu\gamma}u^\nu + \frac{2}{3}\frac{e^4}{m^2}\left(F_{\nu\gamma}u^\gamma\right)\left(F^{\nu\alpha}u_\alpha\right)u^\mu. \quad (3)$$

For a constant electric field $(\partial F^{\mu\nu}/\partial x^\gamma = 0)$, the last expression reduces to Herrera [18] reaction force,

$$g^\mu_{LL} = g^\mu_H = \frac{2}{3}\frac{e^4}{m^2} \times \left(-F^{\mu\gamma}F_{\nu\gamma}u^\nu + \left(F_{\nu\gamma}u^\gamma\right)\left(F^{\nu\alpha}u_\alpha\right)u^\mu\right). \quad (4)$$

Since $F^{\mu\nu}$ is skew-symmetric, for any vector u^ν,

$$F^{\mu\nu}u_\nu u_\mu = F_{\nu\mu}u^\nu u^\mu = 0. \quad (5)$$

Thus the first tensor of Eq. (2), $eF^{\mu\nu}u_\nu$, is orthogonal to u. The left side of Eq. (2), $m\,(du^\mu/ds)$, is also orthogonal to u. Hence, for the sake of consistency, g^μ_{LL} must also be orthogonal to u. If we consider a charged particle moving in the direction of a constant electric field, $(\partial F^{\mu\nu}/\partial x^\gamma = 0)$, for purposes of calculating the motion, Minkowskian space is effectively two dimensional. From Eq. (5), it follows that the first term of the Landau-Lifshitz reaction force, Eq. (4), namely

$$\frac{2}{3}\frac{e^4}{mc^3}F^{\mu\gamma}F_{\nu\gamma}\,u^\nu \quad (6)$$

must be in the direction of u, that is, proportional to u. This is because

$$\omega := F^{\alpha\beta}u_\beta$$

is orthogonal to u by Eq. (5), so that

$$F^{\mu\gamma}F_{\nu\gamma}u^\nu$$

is orthogonal to ω. In a two-dimensional space with a non-degenerate inner product, as is the case, if ω is orthogonal to u and v is orthogonal to ω, then v must be proportional to u. Since the second term of the Landau-Lifshitz reaction force is in the direction of u, we can conclude that g^μ_{LL} is proportional to u, which is also orthogonal to u. Hence it must vanish. In others words, the reaction force vanishes in this special case. As we mentioned above, the same result is obtained for the LD equation and other equations. In the next section we will explain why we consider this result to be unphysical behavior.

3. Unphysical result

If a classical charged particle is accelerated, a momentum is transferred to the field; thus, from momentum balance, a reaction force must act on the charged particle. Indeed, a reaction force will be needed to describe a Bremsstrahlung effect ("braking radiation"), which is physically observed when charged particles are decelerated by a force in the direction of their motion (*e.g.* when a beam of charged particles hits a target). Even if the electric field is not constant, it can be considered to be constant for a short time or simply that the beam of particles is exposed in a uniform electric field. We should note that the result is similar for the LD equation [22,23], as we predicted in the introduction. Moreover, in his famous paper about classical radiation of accelerated electrons, Schwinger [24] expressed the ratio of power lost in radiation to power gained from external sources as:

$$\frac{P}{\frac{dE}{dt}} = \frac{2e^2}{3mc^2}\frac{dE}{mc^2 dx}, \tag{7}$$

for high energy electrons. He commented: "It is evident that radiation losses in a linear accelerator are negligible, unless the accelerating field supplies energy of the order mc^2 in a distance equal to the classical radius of the electron!". In the same order of ideas, for typical linear accelerators, Jackson [25] numerically calculated the energies gained, showing that radiation losses are completely negligible. The same will happen if the motion of the particle obeys the LL equation, since the difference between the LL trajectory and the LD trajectory is very small for the cases of linear accelerators. Namely, the parameter which describes the difference is the characteristic time of the electron,

$$\tau_o = \frac{2}{3}\frac{e^2}{mc^3} = 6.26 \times 10^{-24}\text{s},$$

and we can assert that the radiation losses, for linear cases, are also negligible when the LL equation is considered. Nevertheless, the unphysical result persists since radiation losses are connected to the Larmor formula, which represents a large distance of energy radiated by the particle; and for the LL or LD equation, the reaction force vanishes and will not explain the balance of energy, even if for certain cases the energy losses are small.

4. Radiation rate of energy

The radiation rate of energy for a point charged particle must be analyzed in order to understand the possible paradox that we have just cited above. Indeed if we review the Lorentz-Dirac theory, we can note that there are two kinds of radiation and they are present in the Lorentz-Dirac equation of motion [2]. That is: the total radiation rate of energy leaving the particle in its neighborhood will consist in the so-called bounded or attached radiation energy rate to the particle, which will proceed from fields that decay for large distances and the detached radiation energy rate which comes from fields that are not attached to the particle, that is, the radiation fields which contribute to large distance radiation. The first one is related to the Schott term, which is an exact differential, and is equal to:

$$P_b^o = \tau_o mc\frac{da^o}{d\tau}; \tag{8}$$

and the second which is related to the relativistic generalization of the Larmor formula and is expressed as:

$$P_d^o = \tau_o m\frac{1}{c}a^\mu a_\mu v^o. \tag{9}$$

When the Lorentz-Dirac equation is used in order to describe the motion of a charged particle, it must be taken into account that the total radiation rate of energy is the sum of P_b^o and P_d^o. So the Larmor formula will not give us the total radiation rate of energy, but just the part of the energy radiated to a large distance. The above paradox could be explained in the case of Lorentz-Dirac theory since there exists another radiation rate of energy. But for the Landau-Lifshitz theory, it is not clear what the total energy radiation rate is.

5. Conclusion

More than a century after Abraham, Lorentz, Planck, and later Dirac, claimed a third order derivative equation of motion, it is time to think that drastic changes must be made to deal with the problem. Indeed, hyperacceleration is responsible for this whole issue. Although the mathematical work done by Sphon is undeniable, this does not mean that the result is physically acceptable, since the point of departure may be wrong. Indeed, the reasoning for obtaining the LD equation is based on the use of the Maxwell stress tensor. This latter is defined from electric and magnetic fields which are made meaningful by the use of an equation of motion. This equation of motion is a Lorentz equation and not the LD equation. So we depart from the Lorentz equation of motion for a charged particle and, after a mathematical process, we obtain another equation of motion for the charged particle. Something is misunderstood. In this order of idea, it is convenient to mention Galeriu's comment [26]: "The physical origin of this 4-force, which gives the acceleration energy, is not clear, and the mechanism by which a charged particle acquires rest mass from the field needs more investigation". In the case of the Landau-Lifshitz theory, an expression for the radiation rate of energy must be improved.

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