

# A limit-cycle solver for nonautonomous dynamical systems

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A numerical technique for finding the limit cycles of nonautonomous dynamical systems is presented. This technique uses a matrix representation of the time derivative obtained through the trigonometric interpolation of periodic functions. This differentiation matrix yields exact values for the derivative of a trigonometric polynomial at uniformly spaced points selected as nodes and can therefore be used as the main ingredient of a numerical method for solving nonlinear dynamical systems. We use this technique to obtain some limit cycles and bifurcation points of a sinusoidally driven pendulum and the steady-state response of an electric circuit.

*Keywords:* Nonautonomous dynamical systems; nonlinear circuits; limit cycles; differentiation matrices; trigonometric polynomials.

Se presenta una técnica numérica para encontrar los ciclos límite de algunos sistemas dinámicos no autónomos. Esta técnica usa una representación matricial de la derivada temporal obtenida mediante interpolación de funciones periódicas. Produce valores exactos para la derivada de un polinomio trigonométrico en puntos equiespaciados y puede ser usada como elemento principal de un método numérico para resolver sistemas dinámicos no lineales. Usamos esta técnica para obtener algunos ciclos límite y puntos de bifurcación de un péndulo forzado sinusoidalmente y la respuesta estacionaria de un circuito eléctrico.

*Descriptores:* Sistemas dinámicos no-autónomos; circuitos no-lineales; ciclos límites; matrices de diferenciación; polinomios trigonométricos.

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## 1. Introduction

Dynamical systems are usually given by a set of ordinary differential equations and some initial conditions. Such problems are solved by conventional procedures such as Runge-Kutta methods, multistep methods or general linear methods (see for example Ref. 1). Shooting methods [2] and extrapolation methods [3] have been applied to accelerate the convergence of the solution to the periodic steady-state of electric systems. Some of these implementations require a considerable computational effort and therefore an alternative and more simple technique may be welcomed.

In a series of papers (see Ref. 4 and references therein), a Galerkin-collocation-type method for solving differential boundary value problems has been developed. In some simple cases this technique yields exact results, *i.e.* the numerical output can be interpolated to obtain the functions that exactly solve the problem, and therefore it has been used to implement a numerical scheme to solve boundary-value problems. Such a method consists basically in the substitution of the derivatives that appear in the differential equations with finite-dimensional matrix representations of the derivative (differentiation matrices) whose entries depend on the nodes in a simple form. In some cases, the nodes must be chosen according to some criterion in order to accelerate the convergence to the solution. The differentiation matrices used in this technique arise naturally in the context of the interpolation of functions (see for instance [5]).

The aim of this paper is to present a novel method for obtaining numerical approximants to the steady-state solutions of nonautonomous systems, based on the use of the differentiation matrix for periodic functions given in Refs. 5 and 6. The main idea of the method is established in Sec. 3 and two examples are given in Sec. 4.

## 2. Discrete formalism

In this section, we present only the main results of our discretization scheme for periodic functions; proofs and further applications can be found in [5] and [6]. Let  $D$  be the  $N \times N$  matrix whose components are given by

$$D_{jk} = \begin{cases} \sum_{l=1}^N \frac{1}{2} \cot \frac{t_j - t_l}{2}, & j = k, \\ \frac{\tau_j}{2\tau_k} \csc \frac{t_j - t_k}{2}, & j \neq k, \end{cases}$$

where  $t_k, k = 1, 2, \dots, N$  are arbitrary (but different) points of  $[-\pi, \pi)$ , *i.e.*  $-\pi < t_1 < t_2 < \dots < t_N \leq \pi$  and

$$\tau_j = \frac{d}{dt} \left[ \prod_{l=1}^N \sin \frac{t - t_l}{2} \right]_{t_j}.$$

Let  $x(t)$  be a trigonometric polynomial of degree at most  $n$  and  $x$  and  $x'$  denote the  $N \times 1$  vectors whose elements are  $x(t_j)$  and  $x'(t_j)$ , *i.e.* the values of the polynomial and its derivative at the nodes respectively. Then, the matrix  $D$

applied to  $x$  becomes equal to  $x'$  whenever  $N = 2n + 1$ . Since the derivative of a trigonometric polynomial is again a trigonometric polynomial of the same degree, we have that

$$x^{(k)} = D^k x, \quad k = 0, 1, 2, \dots, \quad (1)$$

where  $x^{(k)}$  is the vector whose elements are given by the  $k$ th derivative of  $x(t)$  evaluated at the nodes.

The differentiation matrix  $D$  takes the simple form

$$D_{jk} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{j+k}}{2 \sin \frac{\pi}{N}(j-k)}, & j \neq k, \end{cases} \quad (2)$$

if the nodes are chosen to be the  $N$  equidistant points<sup>i</sup>

$$t_j = -\pi + \frac{2\pi j}{N}, \quad j = 1, 2, \dots, N. \quad (3)$$

If the periodic function  $x(t)$  is not a polynomial, a residual vector depending on  $x(t)$ ,  $k$ , and  $N$  must be added to the right-hand side of (1). However, it is expected that the norm of such a vector will approach zero as the number of nodes is increased, since  $x(t)$  can be expanded in a Fourier series. Therefore, if  $N$  is great enough, the residual vector can be ignored and the function  $x^{(k)}(t)$  can be approximated by an interpolation of the elements of  $D^k x$ .

### 3. Method

Let us consider now a nonautonomous system of  $m$  components described by

$$\dot{x} = f(x, \omega t), \quad (4)$$

where  $x$  is the vector of components  $x_1(t), x_2(t), \dots, x_m(t)$  and  $f$  is a nonlinear vector function of  $m + 1$  variables, periodic in  $t$ , with period  $T = 2\pi/\omega$ . A very important problem is to find the response of the system in the steady-state regime. Taking into account the periodicity of this response, we can use the differentiation matrix (2) to obtain approximants to the steady-state solution of (4) according to the following scheme.

First of all, we make  $\omega t \rightarrow t$  in (4) in order to change the period of  $f$  to  $2\pi$ . Thus, (4) becomes

$$\omega \dot{x} = f(x, t). \quad (5)$$

Let us take an odd number  $N$  of points  $t_j$  as given by (3) and evaluate (5) at each node to form an equality between  $Nm \times 1$  vectors in such a way that the first  $N$  entries of the left-hand side are the components of the vector  $\dot{x}_1$ , *i.e.*  $\dot{x}_1(t_1), \dot{x}_1(t_2), \dots, \dot{x}_1(t_N)$ ; the following  $N$  entries are the components of  $\dot{x}_2$ , *i.e.*  $\dot{x}_2(t_1), \dot{x}_2(t_2), \dots, \dot{x}_2(t_N)$ , and so on. Now we can approximate the vector blocks  $\dot{x}_k$  by  $Dx_k$  to obtain the discrete form of (5), which can be written as

$$\omega \sum_{l=1}^N D_{jl} x_k(t_l) = f_k(x_1(t_j), x_2(t_j), \dots, x_m(t_j), t_j), \quad (6)$$

where  $j = 1, 2, \dots, N$ , and  $k = 1, 2, \dots, m$ , or in the more compact form

$$\omega \mathbf{D}\mathbf{X} = \mathbf{F}, \quad (7)$$

where  $\mathbf{F}$  denotes the  $Nm \times 1$  vector whose elements are given by the right-hand side of (6) first running  $j$  and then  $k$ ,  $\mathbf{X}$  is the vector whose elements are given by  $x_k(t_j)$  (the unknown solution) ordered in a similar way, and  $\mathbf{D} = \mathbf{1}_m \otimes D$ , where  $\mathbf{1}_m$  is the identity matrix of dimension  $m$ .

The points on which our method is based are the following:

1. If (4) has a limit cycle, then the solution of (7) is an approximation of the steady-state solution of (4).
2. Since (7) [or equivalently (6)] is a system of  $Nm$  nonlinear equations with  $Nm$  unknowns  $x_k(t_j)$ , its solution can be obtained by using a standard procedure (Newton's method for instance).

This shows that a nonautonomous system in the steady-state regime can be described approximately by a system of nonlinear algebraic equations. The procedure sketched in these statements is not concerned at all with the initial conditions of the system and yields simultaneously all the values of  $x_k(t_j)$  (at all times). Thus, this method is quite different in essence from those designed as initial-value-problem solvers.

To obtain the solution of (6), we can use the Newton method or some variation of it with global convergence. As is well-known, it is not always easy to give a good initial approximation  $\mathbf{X}_0$  to attain convergence in Newton's method. For dynamical systems depending on a parameter  $p$ , this problem can be circumvented by building  $\mathbf{X}_0$  with the values of a known solution of the system for a certain value of the parameter  $p_0$  (this is the case of many dynamical systems). By using such an initial approximation, it is possible to obtain the steady-state solution for a value of  $p$  close to  $p_0$ . By iteration of this procedure, the solution for any value of  $p$  can be obtained. For the case in which a solution is not known, a few Runge-Kutta integrations of the system yield a good initial approximation, but this procedure can be time-consuming.

Due to the nature of our method for approximating the steady-state solution of (5), an algorithm for this technique will consist necessarily in the algorithm selected to solve the set of nonlinear equations.

In the following section, we solve two important nonautonomous dynamical systems and we find, according to the case, the limit cycles and bifurcation points.

### 4. Test cases

We have chosen a chaotic mechanical system and a very common electrical circuit as test cases. To find their steady-state solutions, we rewrite the equations describing the dynamics of these problems in the discretized form (7) and use a FORTRAN90 program with standard libraries running on a personal computer to solve them.

**4.1. Test case 1: the driven pendulum**

In spite of the fact that the classical pendulum is a very old problem, interest in it is still growing. The pendulum becomes a chaotic system when it is driven at the pivot point. Let us consider a rigid and planar pendulum consisting of a mass attached to a light rod of length  $l$  which is vertically driven by a sinusoidal force of the form  $-A \cos \omega t$  and damped by a linear viscous force with damping  $\mu$  as in [7]. If  $\theta$  denotes the angular displacement of the pendulum measured from the vertically downward position, the equation of motion is

$$\frac{d^2\theta}{dt^2} + a \frac{d\theta}{dt} + (1 + b \cos \omega t) \sin \theta = 0, \quad (8)$$

where

$$a = \frac{2\mu}{\sqrt{lg}}, \quad b = \frac{A\omega^2}{l}.$$

To compare the steady-state solutions yielded by our procedure with those obtained by other authors, we take  $0 \leq b \leq 200$  (the driving amplitude) as the control parameter,  $a = 0.1$  and  $\omega = 17.5$ . For values of  $b$  near zero, a good initial guess is a vector  $\mathbf{X}_0$  whose entries are a sinusoidal deviation from  $\pi$ . Once we have found a solution  $\mathbf{X}(p)$ , we proceed to obtain  $\mathbf{X}(p + \delta p)$  by using  $\mathbf{X}_0 = \mathbf{X}(p)$ . We have taken  $N = 101$  nodes of the form (3) in this case.

As is known, this system presents period-doubling and bifurcation. Figure 1 displays these phenomena for solutions of (8) close to the inverted state  $\theta = \pi$ . To make the drawing more simple, we only plot the maximum and minimum values of the angular displacement  $\theta(t)$  as functions of the control parameter  $b$ . Other solutions, such as the hanging state, are not considered here.

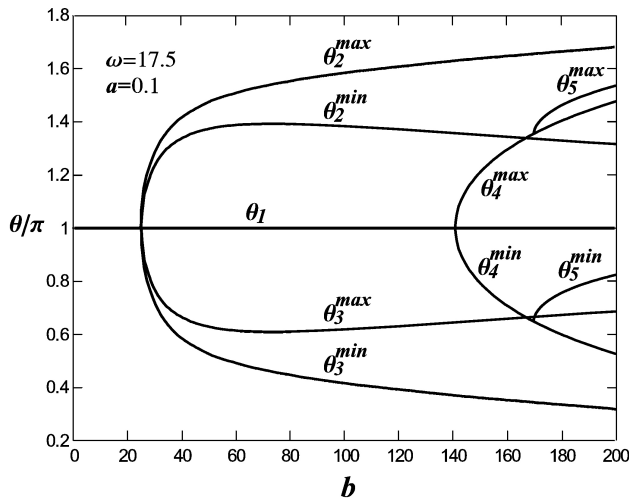


FIGURE 1. Maximum and minimum values of the angular displacement  $\theta$  of the vertically driven pendulum (8) vs. the driven amplitude  $b$  for  $a = 0.1$  and  $\omega = 17.5$ . For any value of  $b$ , the solution  $\theta_1 = \pi$  (the inverted pendulum) was always found. The curves labeled by  $\theta_k$ ,  $k = 2, \dots, 5$  correspond to different solutions.

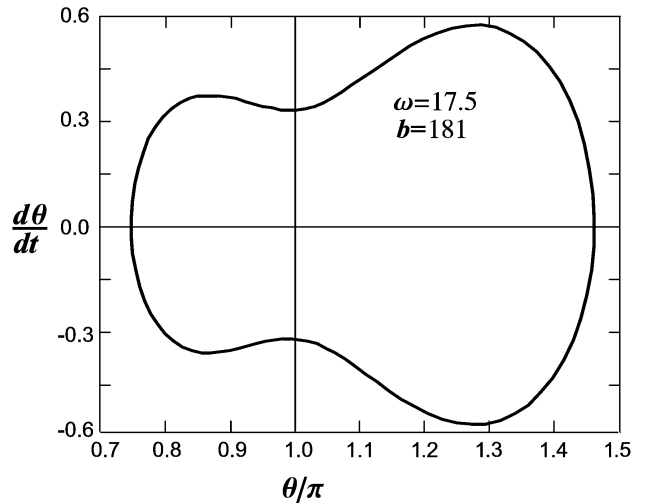


FIGURE 2. Phase-space diagram of the period-2 “mariachi” solution  $\theta_4$  for  $a = 0.1$ ,  $\omega = 17.5$  and  $b = 181$ .

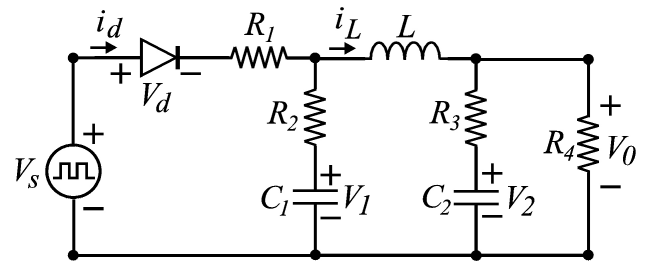


FIGURE 3. A typical commutation circuit.

The information displayed is in line with the findings of other authors (see for example [7]), but in our case the problem of stability of the solution is not revised. In Fig. 1,  $\theta_1$  is the solution corresponding to the inverted state,  $\theta_2$  and  $\theta_3$  correspond in essence to the same period-1 solution (one can get the second from the first by a reflection through the axis  $\theta = \pi$ ), and  $\theta_4$  and  $\theta_5$  are period-2 solutions both of which oscillate about the vertical. The asymmetrical way in which  $\theta_5$  oscillates is illustrated in Fig. 2.

**4.2. Test case 2: a commutation circuit**

Figure 3 shows a rectifier-filter circuit and a resistive load, excited by a squared wave pulse voltage source. This circuit is a simplified model for a typical output stage of a switching power supply.

The equations for the state variables  $V_1 = x_1$ ,  $V_2 = x_2$  and  $i_L = x_3$  can be simplified to the form

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C_1 R_2} \left[ V_s - x_1 - V_d - i_s R_1 \left( \exp\left(\frac{V_d q}{\eta k T}\right) - 1 \right) \right], \\ \dot{x}_2 &= \frac{1}{C_2 (R_3 + R_4)} \left( -x_2 + R_4 x_3 \right), \end{aligned}$$

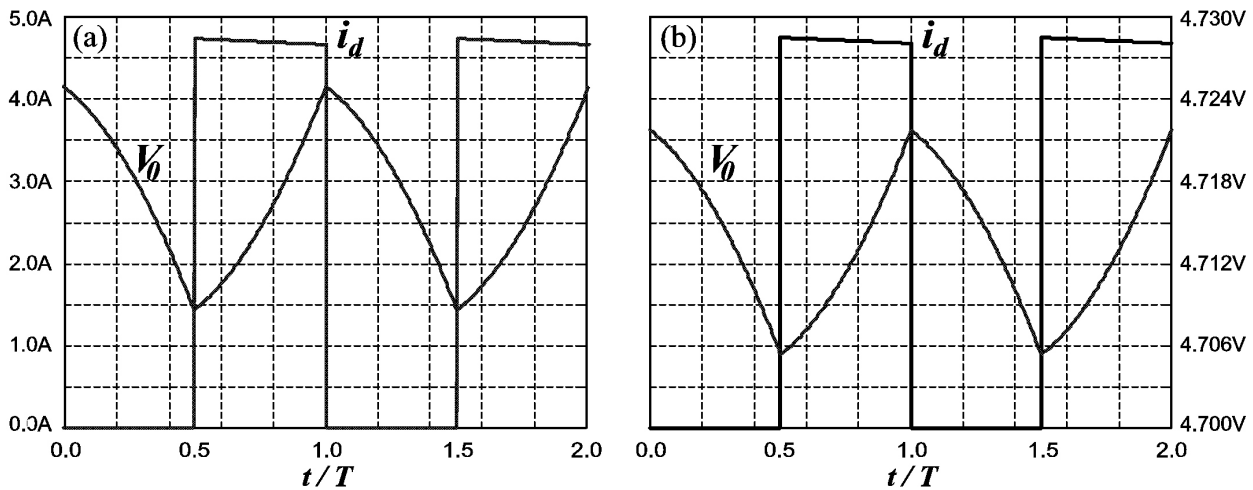


FIGURE 4. Steady-state solution for the diode current  $i_d = x_3 + C_1x_1$  and the voltage  $V_0 = C_2R_3x_2 + x_2$  along two periods obtained by the present method with  $N = 251$  nodes (4a) and by a Spice simulation (4b). The left-hand and right-hand vertical axes are used for different scales.

$$x_3 = \frac{1}{L} \left[ V_s - \frac{R_4}{(R_3 + R_4)}x_2 - \frac{R_3R_4}{(R_3 + R_4)}x_3 - V_d - i_sR_1 \left( \exp\left(\frac{V_dq}{\eta kT}\right) - 1 \right) \right], \quad (9)$$

where

$$V_d = V_s - C_1(R_1 + R_2)x_1 - x_1 - R_1x_3,$$

$k$  is the Boltzmann constant,  $q$  is the electron charge,  $T$  the absolute temperature which is taken at the standard value  $T=300^\circ\text{K}$ , and  $\eta$  is the emission coefficient. The driving voltage is chosen as  $V_s(t) = A_m \text{sgn}(2\pi t/T)$ ,  $t \in [-T/2, T/2]$ , and the remaining parameters of this problem are taken at typical values:  $A_m = 5.6\text{V}$ ,  $T = 10^{-5}\text{s}$ ,  $i_s = 10^{-8}\text{A}$ ,  $R_1 = 0.0149\Omega$ ,  $R_2 = 0.15\Omega$ ,  $R_3 = 0.2\Omega$ ,  $R_4 = 2.0\Omega$ ,  $C_1 = 470.0 \times 10^{-6}\text{F}$ ,  $C_2 = 20.0 \times 10^{-6}\text{F}$ ,  $L = 20.0 \times 10^{-6}\text{H}$ ,  $\eta = 0.8953$ .

In order to compare the performance of the technique presented in this paper, we have simulated the circuit using an LT version of the Spice program which is currently available for free from Linear Technology. For this circuit, the response of the system was computed by the Spice simulation over 150 cycles to achieve the steady-state. A maximum step size of  $0.04\mu\text{s}$  was specified. In our case, we use 251 equispaced time values  $t_j$  of the form (3) (yielding a constant step size approximately equal to  $0.04\mu\text{s}$ ), and all of the steady-state

response values were computed at once. Since Spice uses an adaptive strategy for the step size, it is not possible to compare both of the results on a point-by-point basis. However, it turns out that they agree generally up to  $10^{-2}$ . It is worth to notice that the present method is able to handle a discontinuous driving voltage pretty well.

The outputs yielded by the present method and by the Spice simulation for  $i_d = x_3 + C_1x_1$  and  $V_0 = C_2R_3x_2 + x_2$  are displayed in Figs. 4a and 4b, respectively.

### 5. Final remarks

As the simple tests of the previous section show, the method presented in this paper gives a novel approach to obtaining the limit cycles of nonautonomous systems and represents an alternative to the conventional methods of integration. Based on a well-behaved differentiation matrix for periodic functions, it departs from standard procedures in the way in which the limit cycle is found: the usual integration of the system for long periods of time is replaced by the problem of finding the solution to a set of nonlinear algebraic equations. This feature makes this method a tool for studying dynamical systems from a new point of view.

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*i.* The same result is obtained for the set of points  $t_j = \pi(2j - N - 1)/N$ .

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