Optimization of an irreversible Carnot engine in finite time and finite size

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In this work, we consider the class of irreversible Carnot engines that results from combining the characteristics of two models found in the literature: the model in finite time and the model in finite size. The performance of the resulting model, including three irreversibilities, was doubly-optimized in finite time and finite size. The first optimization of power and efficiency, maintaining the thermal conductances fixed, was performed in finite time. Since the optimum time ratio from the first optimization, is the same for both maximum power and maximum efficiency, this means that the model can be newly optimized but now in finite size. Then, the second optimization, maintaining the overall heat transfer coefficient constant, was performed. For both optimizations, analytical expressions for the efficiency that maximizes the power and maximum efficiency were obtained. Changing the order in which partial optimizations were carried out, a remarkable optimal property was obtained: the resources of total contact time and the total area of heat transfer are proportional.

Keywords: Internal and external irreversibilities; heat engines; finite time and size thermodynamics.

En este trabajo consideramos la clase de ciclos de Carnot irreversible que resultan de combinar las características de dos modelos: Modelo en tiempo finito y Modelo en dimensión finita. El desempeño del modelo resultante fue optimizado dos veces en tiempo finito y dimensión finita. Primero, optimizamos la potencia y la eficiencia en tiempo finito, manteniendo las conductancias térmicas fijas. Como el cociente de tiempos óptimo de la primera optimización es el mismo para potencia máxima y eficiencia máxima lo sustituimos, en las ecuaciones obtenidas, para que el modelo sea nuevamente optimizado pero ahora en dimensión finita. Entonces, la segunda optimización fue realizada, manteniendo el coeficiente global de transferencia de calor fijo. Obtuvimos, en ambas optimizaciones, expresiones analíticas para la eficiencia que maximiza la potencia y la máxima eficiencia. Al cambiar el orden en el cual las optimizaciones parciales fueron realizadas, se obtuvo una propiedad optimal notable: los recursos de tiempo total y área total de transferencia de calor resultan ser proporcionales.

Descriptores: Irreversibilidades internas y externas; máquinas térmicas; termodinamica de tiempo y dimensión finitas.

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1. Introduction

The first goal of classical thermodynamics was to evaluate how well heat engines perform and how well they might perform within ideal limits. The ideal limit is the reversible Carnot engine, with an efficiency given by:

$$\eta_C = 1 - \frac{T_L}{T_H},$$

where T_L and T_H are the temperatures of the hot and cold reservoirs between which the heat engine operates. The model with a reversible cycle is known as Carnot's heat engine. Since all processes in a Carnot engine are reversible the thermal efficiency can only be approached by infinitely slow process. As a result, it is not possible to obtain any amount of power (power is zero).

In practice, all thermodynamic processes take place in finite-size devices and in finite time. As a result, irreversibility conditions are present. Thus the Carnot cycle gives an upper unreachable bound for thermal efficiency but when it comes to obtaining maximum power, it is formally impossible. Recently, attention has turned to the study of bounds for other quantities, such as power, efficiency and so on. The seminal work for this broadening of scope was apparently Curzon and Ahlborn's paper [1]. Bejan [2] noted that this result had previously been discovered independently, in engineering, in 1957 by Novikov and Chambadal (CNCA-engine). Thus, a more realistic bound could be placed on the efficiency of a heat engine operating at its maximum power point, the so-called CNCA-efficiency (Chambadal, Novikov, Curzon and Ahlborn efficiency):

$$\eta_{CNCA} = 1 - \sqrt{\frac{T_L}{T_H}}. \tag{1}$$

Here the sole source of irreversibility in the cycle is a linear finite rate of heat transfer between the working fluid and its two heat reservoirs. We can take as characteristic parameters (variables) of a CNCA-engine: the internal temperature ratio (x) - for Otto and Brayton engines this parameter corresponds to compression or pressure relations - and the time ratio (y) - the contact time between the working fluid and hot-side reservoir and total contact time ratio.

The CNCA-engine proposed a new approach to thermodynamics: Finite Time Thermodynamics (FTT). This approach takes time into account of in the analysis of thermodynamic processes and emphasizes maximum power as an interesting bound. Inspired by this approach, Rubin [3] used it to study the endoreversible engine, which he defined as: *an engine such that during its operation its working fluid* *undergoes reversible thermodynamics.* Endoreversible models have, as a subfield, become an important paradigm of FTT [4]. In these models one can adopt the formalism of De Vos [5], which is based on only one characteristic parameter, for us x since, heat flows are considered.

The class of irreversible phenomena investigated in FTT has expanded to include friction, diffusion and chemical reactions. The models used have become increasingly complex, by taking multiple reservoirs, with heat leaks, finite capacities and diverse heat transfer laws [4, 6, 7]. Contrary to these research trends, our work goes back to combine the characteristics of the CNCA-engine, including two additional irreversibilities, with other model of engine appeared in the engineering field.

In the early stages of the field, Bejan [8] in 1982 proposed as a problem in a textbook a steady-state plant model (B-engine) where the irreversibility is due to three sources: the hot-side exchanger, the cold-side exchanger and the heat leaking through the plant to the ambient. Internally the plant is reversible. By a double optimization of the power, first with respect to the internal temperature ratio and afterwards with respect to the allocation ratio of the heat exchangers (z) - external conductances allocation ratio -, the internal temperature optimum ratio is the same as that found in FTT. In addition, an optimum balance between sizes of the hot-side and cold-side exchangers were obtained by him. Remarkable characteristics of the B-engine was: the fact that the efficiency at maximum power is less than the corresponding efficiency of the CNCA-engine (Eq. (1)); the hot-side and cold-side exchangers are worked as finite-size devices and they have the same allocation ratio; and that, if there is a heat leak through the plant or engine to the ambient, the behavior of the power and efficiency results in a loop-type curve [9].

Most of the real engines, as the Brayton or gas Turbine cycle, have the loop-shaped qualitative behavior illustrated in Fig. 2 (below), when work or dimensionless power are plotted against efficiency [10]. The differences between one engine type and another are size and the sources of irreversibility, which give work-efficiency curves with a loop shape [11, 12]. It is important to note that the difference between maximum efficiency and the efficiency that produces maximum power is small but always positive [13].

In many applications, such as power plants, it is more convenient to use the B-engine instead of a CNCA-engine. While heat exchanges are sequential processes in the CNCA-engine, they are simultaneous processes in the B-engine. The differences arising due to this have been considered previously by Wu *et al.* [14]. In conclusion, there are two models: the CNCA-engine and the B-engine.

In general, in additional to the irreversibilities described, for the internal dissipations of the working fluid a factor I, that makes the Claussius inequality an equality, can be introduced [9,15] (this factor I can correspond, for example, to the irreversibilities included in the adiabatic compression and expansion efficiencies [16]).

In this work, we consider the class of irreversible Carnot cycles that results from combining the characteristics of two models found in the literature: the model in finite time and the model in finite size. The performance of the resulting model, including three irreversibilities, was doubly-optimized in finite time and finite size. The first optimization of power and efficiency, maintaining the thermal conductances fixed, was performed in finite time. Since the optimum time ratio from the first optimization, being the same for both maximum power and maximum efficiency, this means that the model can be newly optimized but now in finite size. Then, the second optimization, maintaining the overall heat transfer coefficient as a constant, was performed. For both optimizations, analytical expressions for the efficiency that maximizes power and maximum efficiency were obtained. Changing the order in which partial optimizations were carried out, a remarkable optimal property was obtained: the resources of total contact time and the total area of heat transfer are proportional. Similar models including these three irreversibilities treated only in finite time can be found in Refs. 9, 15 and 17.

2. Irreversible Carnot engine

Considering the class of irreversible Carnot engines [4] shown in Fig. 1, which satisfy the following four conditions:

(i) There is thermal resistance between the working fluid and the heat reservoirs.

(ii) There is a heat loss Q_{leak} from the hot reservoir to the cold reservoir.

(iii) All heat transfer is assumed to be linear in temperature differences, that is, Newtonian.

(iv) Besides thermal resistance and heat loss, there are other irreversibilities in the engine, the internal irreversibilities. Specifically, there is a parameter that makes the Claussius an inequality equality:

$$\frac{Q_2}{T_2} - I\frac{Q_1}{T_1} = 0, \tag{2}$$

where $I = (\Delta S_2/\Delta S_1) \geq 1$ [9].

Thus, the irreversible Carnot engine operates with fixed time t allowed for each cycle. The heat leakage Q_{leak} is [8]:

$$Q_{leak} = K(T_H - T_L)t.$$

The heats Q_H, Q_L transferred from the hot-cold reservoirs are given by:

$$Q_H = Q_1 + Q_{leak} = \alpha(T_H - T_1)t_H + K(T_H - T_L)t \tag{3}$$

$$Q_L = Q_2 + Q_{leak} = \beta(T_2 - T_L)t_L + K(T_H - T_L)t \tag{4}$$

where α, β and K are the thermal conductances and t_H, t_L are the times for the heat transfer in the isothermal branches,

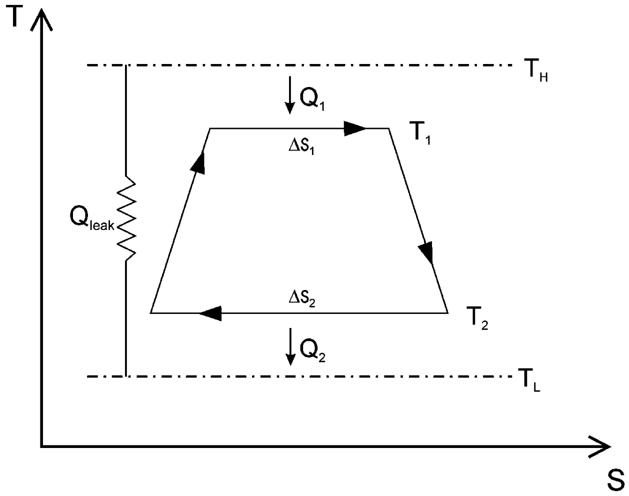

FIGURE 1. A Carnot cycle with heat leak, finite-rate heat transfer and internal dissipations of the working fluid.

respectively. The connecting adiabatic branches are often assumed to proceed in negligible time [10], so that the cycle contact total time t is:

$$t = t_H + t_L \tag{5}$$

By the first law, and combining Eqs. (3) and (2), we obtain :

$$W = Q_1(1 - Ix) = \frac{T_H(1 - Ix)\left(1 - \frac{\mu}{x}\right)}{\frac{1}{\alpha t_H} + \frac{I}{\beta t_L}}, \tag{6}$$

$$Q_H = Q_1 + Q_{leak} = \frac{W}{1 - Ix} + K(T_H - T_L)t, \tag{7}$$

where $x = T_2/T_1$; $\mu = T_L/T_H$.

3. Maximum power and efficiency in finite time

In this section as we analyze the CNCA operation the thermal conductances α, β are fixed. The Eq. (5) give us the total time of the cycle, so it can be parametrized as:

$$t_H = yt;\ t_L = (1 - y)t$$

Therefore, the dimensionless power output

$$p = W/(\alpha t T_H),$$

the dimensionless heat transfer per unit time

$$q_H = Q_H/(\alpha t T_H)$$

and the efficiency of the engine are [by Eqs. (6), (7)]:

$$p=\frac{(1-Ix)\left(1-\frac{\mu}{x}\right)}{\frac{1}{y}+\frac{I}{\gamma(1-y)}},\quad q_H=\frac{\left(1-\frac{\mu}{x}\right)}{\frac{1}{y}+\frac{I}{\gamma(1-y)}}+L(1-\mu) \tag{8}$$

$$\eta=\frac{y\gamma\,(1-Ix)\left(1-\frac{\mu}{x}\right)(1-y)}{y\left(1-\frac{\mu}{x}\right)\gamma\,(1-y)+L\,(1-\mu)\,(\gamma\,(1-y)+yI)} \tag{9}$$

where $\gamma = \beta/\alpha$ and $L = K/\alpha$ are fixed. Using the extremes conditions:

$$\frac{\partial p}{\partial x}|_{(x_{mp},y_{mp})} = 0; \quad \frac{\partial p}{\partial y}|_{(x_{mp},y_{mp})} = 0,$$

when the power reaches its maximum, x_{mp} and y_{mp} are given by:

$$x_{mp} = \sqrt{\frac{\mu}{I}},\quad y_{mp} = \frac{\sqrt{\gamma}}{\sqrt{I} + \sqrt{\gamma}} \tag{10}$$

Now, the efficiency that maximizes the power η_{mp} and the maximum power are given by [Eqs. (9) and (8)],

$$\eta_{mp} = \frac{\left(1 - \sqrt{I\mu}\right)^2}{\left(1 - \sqrt{I\mu}\right) + L\,(1 - \mu)\left(1 + \sqrt{\frac{I}{\gamma}}\right)} < \eta_{CNCA};$$

$$p_{\max} = \left(\frac{1 - \sqrt{I\mu}}{1 + \sqrt{\frac{I}{\gamma}}}\right)^2$$

Figure 2 shows clearly that, for $L = 0$, very large differences exist between η_{mp} and the maximum efficiency $\eta_{\max}$ and $p_{\max}$ and p_{me}; when there is a heat leak ($L > 0$), this difference decreases. If the efficiency is less than η_{mp}, the power output decreases as the efficiency decreases and if the power output is less than p_{me}, the efficiency also decreases as the power output decreases. Obviously, the working states $p < p_{me}$ and $\eta < \eta_{mp}$ are not the optimal operating states of the heat engine. In general, the heat engine should be operated between η_{mp} and $\eta_{\max}$ or $p_{\max}$ and p_{me}. However, the difference between η_{mp} and $\eta_{\max}$ is positive. We will obtain an analytical expression for $\eta_{\max}$ which shows that operation at maximum efficiency is better than operation at maximum power output (see Ref. 11 and Fig. 3).

The efficiency reaches its maximum when x_{me} and y_{me} are given by:

$$x_{me} = \frac{\sqrt{I}\gamma\mu + B}{\sqrt{I}\,(\gamma + \Omega)},\quad y_{me} = \frac{\sqrt{\gamma}}{\sqrt{I} + \sqrt{\gamma}}, \tag{11}$$

with

$$\Omega = L\,(1 - \mu)\left(\sqrt{I} + \sqrt{\gamma}\right)^2$$

and

$$B = \sqrt{\mu\Omega\,(\Omega + \gamma\,(1 - I\mu))}.$$

The power for maximum efficiency p_{me} and the maximum efficiency $\eta_{\max}$ are given by [Eqs. (9) and (8)]:

$$\eta_{\max} = \frac{\gamma\left(B - \sqrt{I}\mu\Omega\right)\left(\gamma + \Omega - \gamma I\mu - \sqrt{I}B\right)}{B\,(\gamma + \Omega)^2},$$

$$p_{me} = \frac{\gamma L\,(1 - \mu)\left(B - \sqrt{I}\mu\Omega\right)\left(\gamma + \Omega - \gamma I\mu - \sqrt{I}B\right)}{\Omega\,(\gamma + \Omega)\left(\sqrt{I}\gamma\mu + B\right)}$$

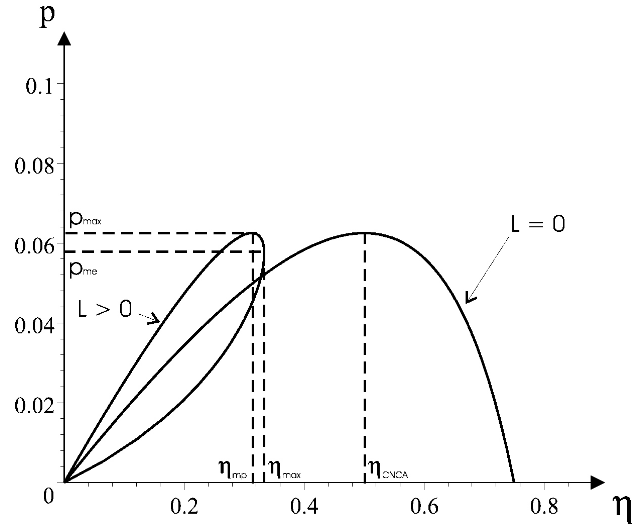

FIGURE 2. Behavior of the dimensionless power versus efficiency with respect to the parameter x, when $\gamma = I = 1, \mu = 0.25; L = 0$ and 0.1, and with $y_{mp} = 1/2$.

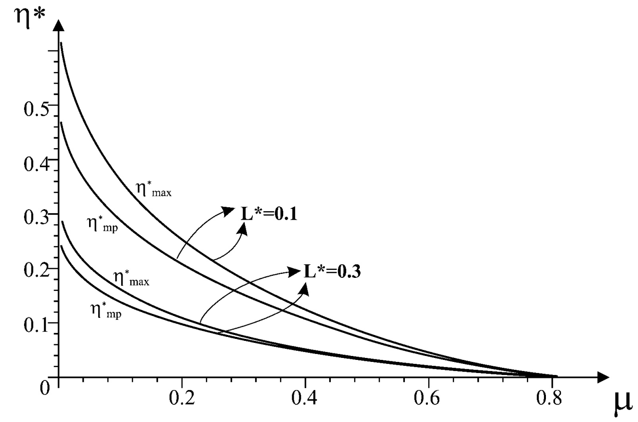

FIGURE 3. Behavior of the efficiencies $\eta^*_{\max}$and η^*_{mp} versus μ for values $L^* = 0.1$ and 0.3 and with I =1.235.

Equations (10) and (11) say that the optimum value for the time ratio is the same for both maximum power and efficiency. This means that

$$y_{mp} = y_{me} = \frac{\sqrt{\gamma}}{\sqrt{I} + \sqrt{\gamma}} = \frac{\sqrt{\beta\alpha}}{\sqrt{\beta\alpha} + \alpha\sqrt{I}}, \qquad (12)$$

that is, when the engine operates at maximum power and maximum efficiency.

Nevertheless, for the case of the B-engine, the heat transfer process are not simultaneous but they also take the same time. Equation (12) says that the times must be different. However, we can suppose that the irreversible Carnot engine is a B-engine, when the value of Eq. (12) is included in the equations of work and heat transfer [Eqs. (6) and (7)]. So, we can perform a second optimization now of finite size.

4. Maximum power and efficiency in finite size

As can be seen in the above section, the time ratio is the same for both maximum power and maximum efficiency [Eq. (12)]. Thus, this value could be included in the equations of work and heat transfer [Eqs. (6) and (7)] to proceed to a second optimization, with x as the first variable, and the second variable will be the allocation (sizes) of the heat exchangers. In so doing, it could be said that

$$\alpha = UA_H;\ \beta = UA_L,$$

where U is the overall heat transfer coefficient and A_H and A_L are the areas available for heat transfer. Then, an approach might be to suppose that U is fixed, the same on the hot side and the cold side exchangers, and that the total area $A = A_H + A_L$ can be allocated at will between both. The optimization problem is then to select, in addition to of the optimum temperature ratio (x), the best allocation ratio. To take UA as a fixed value can be justified in terms of the area purchased, the fixed running costs and capital costs that together determine the overall heat transfer coefficient (see Ref. 18). Thus, for the optimization we can take:

$$\frac{\alpha}{U} + \frac{\beta}{U} = A$$

and parametrize it as:

$$\alpha = zAU;\ \beta = (1-z)UA$$

We obtain, from Eqs. (6) and (12), the dimensionless power output ($p = W/(AUtT_H),$):

$$p^* = p(x,z) = \frac{\sqrt{I\left(z-z^2\right)^3}\,(1-Ix)\left(1-\frac{\mu}{x}\right)}{\left(\sqrt{I\left(z-z^2\right)}+Iz\right)\left((1-z)+\sqrt{I\left(z-z^2\right)}\right)} \qquad (13)$$

and the efficiency of the engine [Eqs. (6) and (7)] is:

$$\eta^* = \frac{\sqrt{I\left(z-z^2\right)^3}\,(1-Ix)\left(1-\frac{\mu}{x}\right)}{\sqrt{I\left(z-z^2\right)^3}\left(1-\frac{\mu}{x}\right)+L^*(1-\mu)\left(\sqrt{I\left(z-z^2\right)}+Iz\right)\left((1-z)+\sqrt{I\left(z-z^2\right)}\right)} \qquad (14)$$

where $L^* = K/AU$ is fixed.

By proceeding as in the above section, power p^* reaches its maximum when, x_{mp} and z_{mp} are given by:

$$x_{mp} = \sqrt{\frac{\mu}{I}}, \qquad z_{mp} = \frac{1}{\sqrt[3]{I}+1}. \tag{15}$$

And, the efficiency that maximizes the power η^*_{mp} and the maximum power are given by:

$$\eta^*_{mp} = \frac{\left(1-\sqrt{I\mu}\right)^2}{\left(1-\sqrt{I\mu}\right)+L^*\left(1-\mu\right)\left(\sqrt[3]{I}+1\right)^3},$$

$$p^*_{\max} = \frac{\left(1-\sqrt{I\mu}\right)^2}{\left(\sqrt[3]{I}+1\right)^3}. \tag{16}$$

The efficiency reaches its maximum if x_{me} and z_{me} are given by:

$$x_{me} = \frac{I\mu+2\Gamma}{I\left(C+1\right)}, \quad z_{me} = \frac{1}{I+1},$$

with

$$C = 4L^*\left(I+1\right)\left(1-\mu\right)$$

and

$$\Gamma = \sqrt{I\mu C\left(C+\left(1-\mu I\right)\right)}.$$

The maximum efficiency $\eta^*_{\max}$ and the power for maximum efficiency p^*_{me} are given by [see Eq. (14)]

$$\eta^*_{\max} = \frac{\left(C+1-I\mu-\Gamma\right)\left(\Gamma-I\mu C\right)}{4L^*\left(1-\mu\right)\left(C+1\right)\left(I+1\right)\left(I\mu+\Gamma\right)+\left(\Gamma-I\mu C\right)} \tag{17}$$

$$p^*_{me} = \frac{\left(C+1-I\mu-\Gamma\right)\left(\Gamma-I\mu C\right)}{4\left(I+1\right)\left(C+1\right)\left(I\mu+\Gamma\right)} \tag{18}$$

5. Discussion

The optimum value (12) was used conduced to perform a second optimization, in finite size, to obtain expressions for the efficiency η^*_{mp} and $\eta^*_{\max}$ [Eqs. (16) and (17)]. Figure 3 shows the behavior of those efficiencies for $I = 1.235$ and also that the maximum efficiency $\eta^*_{\max}$ is greater than the efficiency at maximum power η^*_{mp}.

Double optimization gives results that could be applied to the design of power plants. For instance, if the plant operates at maximum power, it was found that the relation from the heat transfer areas for the cold side to the hot side, is:

$$A_L = \sqrt[3]{I}A_H \geq A_H$$

when the plant operates at maximum power, but

$$A_L = IA_H \geq A_H$$

if the operation is at maximum efficiency. In both cases, a greater surface for the heat transfer is required for the cold side as compared with the hot side. Since $I \geq \sqrt[3]{I}$, the area is greater when the operation is at maximum efficiency; therefore, by satisfying this latter condition, one for the operation at maximum power is immediately satisfied.

In accordance with the definitions adopted for thermal conductance, the one for the cold side turns out be greater than the one for the hot side; then this results in:

$$t_L = \sqrt[3]{I}t_H \geq t_H$$

for maximum power, and

$$t_L = t_H$$

for maximum efficiency. Since $\sqrt[3]{I} \geq 1$, the time of heat transfer for the isothermal of the cold side is greater when the operation is at maximum power; then, satisfying this latter condition, the one for the operation at maximum efficiency is immediately satisfied. When $I = 1$, the time is the same for both operations.

Changing the order in which partial optimizations were carried out, we obtain:

$$t_L = \sqrt[3]{I}t_H \geq t_H$$

$$A_L = \sqrt[3]{I}A_H \geq A_H$$

when the plant operates at maximum power, but

$$t_L = It_H \geq t_H$$

$$A_L = A_H$$

for maximum efficiency.

Therefore, we have the following remarkable optimal property:

$$\frac{A}{t} = \frac{A_L}{t_L} = \frac{A_H}{t_H}$$

which is satisfied when the heat engine operates at maximum power and efficiency. Physically, for $I > 1$, the irreversibility produces an inverse relationship between the total area and the total contact time; that is, less time is needed to transfer the heat that the engine processes. This is due to the fact that less heat goes through the engine. Part of the heat is lost because of internal irreversibility. For $I = 1$, the relationship between area and contact time is inversely proportional; that is, if the area is augmented the time is reduced. This result does not depends explicitly on I and differs from one the presented in Ref. 9.

Also, for these operation conditions, we have:

$$t_L = \sqrt[3]{I}t_H \geq t_H$$

$$A_L = \sqrt[3]{I}A_H \geq A_H,$$

and as $\sqrt[3]{I} \geq 1$, a greater surface and greater time for the heat transfer are required for the cold side than for the hot side.

In general it has been supposed that $I \geq 1$; but some times it has been considered that $I = 1$. In this case, the internal irreversibilities can be physically interpreted as part of the engine's heat leak, which brings us to the engine modeled in Ref. 8. Thus, the results of the B-engine and new results given by Eqs. (16) and (17) are obtained ($L^* > 0$). Now, if $L^* = 0$, the results of the CNCA-engine are obtained and Eqs. (16), (18) and (17) depend only on the temperatures of the reservoirs.

Therefore, this is an indication that the methodology herein presented can be applied to other models of engines, for instance, those proposed by J. Chen [9] and Z. Yan and L. Chen [17].

One of the contributions of this work is the use of a methodology that could be extended to the double optimization of power and efficiency for another models of engines and to the field of thermoeconomics. Further work is underway.

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