Hamiltonians and Lagrangians of non-autonomous one-dimensional mechanical systems

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It is shown that a given non-autonomous system of two first-order ordinary differential equations can be expressed in Hamiltonian form. The derivation presented here allows us to obtain previously known results such as the infinite number of Hamiltonians in the autonomous case and the Helmholtz condition for the existence of a Lagrangian.

Keywords: Non-autonomous systems; Hamilton equations; Lagrangians.

Se muestra que un sistema dado, no autónomo, de ecuaciones diferenciales ordinarias de primer orden puede expresarse en forma hamiltoniana. La deducción presentada aquí nos permite obtener resultados previamente conocidos tales como el número infinito de hamiltonianas en el caso autónomo y la condición de Helmholtz para la existencia de una lagrangiana.

Descriptores: Sistemas no autónomos; ecuaciones de Hamilton; lagrangianas.

1. Introduction

As is well known, it is very convenient to express a given system of ordinary differential equations (not necessarily related to classical mechanics) as the Euler–Lagrange equations associated with some Lagrangian, L, or as the Hamilton equations associated with some Hamiltonian, H (see, *e.g.*, Ref. 1). One of the advantages of such identifications is the possibility of finding constants of motion, which are related to symmetries of L or H. Also, the Hamiltonian of a classical system is usually regarded as an essential element to find a quantum version of the mechanical system.

In the simple case of a mechanical system with forces derivable from a potential (that may depend on the velocities), there is a straightforward procedure for finding a Lagrangian or a Hamiltonian. However, in the case of non-conservative mechanical systems or of systems not related to mechanics, the problem of finding a Lagrangian or a Hamiltonian is more involved. A given system of n second-order ordinary differential equations is equivalent to the Euler–Lagrange equations for some Lagrangian if and only if a set of conditions (known as the Helmholtz conditions) are fulfilled (see, *e.g.*, Refs. 2, 3, and the references cited therein).

The aim of this paper is to give a straightforward procedure to find a Hamiltonian for a given system of two first-order ordinary differential equations (which may not be equivalent to a second-order ordinary differential equation) that possibly involves the time in an explicit form. The results derived here contain the Helmholtz condition for $n = 1$ (in the case where the given system is equivalent to a second-order equation). In Sec. 2 the main results of this paper are established, demonstrating that a given system of first-order ordinary differential equations can be expressed in Hamiltonian form looking for an integrating factor of a differential form made out of the functions contained in the system and, in Sec. 3, several examples are presented. In Sec. 4 we show that, in the appropriate case, our results lead to the Helmholtz condition for the existence of a Lagrangian.

2. Hamiltonians and canonical variables

We shall consider a system of first-order ordinary differential equations of the form

$$\dot{x} = f(x, y, t), \qquad \dot{y} = g(x, y, t), \qquad (1)$$

where f and g are two given functions. A system of this class can be obtained from a second-order equation

$$\ddot{x} = F(x, \dot{x}, t),$$

by introducing the variable $y = \dot{x}$. We are initially interested in finding a Hamiltonian function, H, and canonical variables, q, p, such that the corresponding Hamilton's equations be equivalent to the system (1).

Assuming that there is an invertible relation between the variables x, y and a set of canonical coordinates q, p, $x = x(q, p, t)$, $y = y(q, p, t)$, in such a way that Eqs. (1) are equivalent to the Hamilton equations for q and p with a Hamiltonian H, making use of the chain rule, one finds that

$$-g\,\mathrm{d}x + f\,\mathrm{d}y = \frac{\partial(x, y)}{\partial(q, p)}\mathrm{d}H - \frac{\partial y}{\partial t}\mathrm{d}x + \frac{\partial x}{\partial t}\mathrm{d}y$$

$$+ \text{ terms proportional to } \mathrm{d}t. \quad (2)$$

Therefore, given system (1), we start by considering the differential form

$$-(g - \phi)\mathrm{d}x + (f - \psi)\mathrm{d}y, \qquad (3)$$

where

$$\phi(q, p, t) \equiv \frac{\partial y(q, p, t)}{\partial t}, \qquad \psi(q, p, t) \equiv \frac{\partial x(q, p, t)}{\partial t}$$

are functions unspecified by now (see Eq. (10) below). For a fixed value of t, the differential form (3) is always integrable (see any standard text on ordinary differential equations, *e.g.*, Ref. 4); that is, there exist (locally) functions σ and H, which may depend parametrically on t, such that

$$-(g - \phi)\mathrm{d}x + (f - \psi)\mathrm{d}y = \sigma \mathrm{d}H. \qquad (4)$$

Now, for simplicity, without any loss of generality (since once we have found a set of canonical coordinates, we have the choice of making any canonical transformation afterwards), we choose $q \equiv x$ (hence, $\psi = 0$) and, therefore,

$$\frac{\partial(x, y)}{\partial(q, p)} = \frac{\partial p}{\partial y}.$$

Then, by comparing Eqs. (2) and (4), the canonical momentum, p, must be such that

$$\frac{\partial p(x, y, t)}{\partial y} = \frac{1}{\sigma(x, y, t)}. \qquad (5)$$

Hence

$$\mathrm{d}p = \frac{\partial p}{\partial x}\mathrm{d}x + \frac{1}{\sigma}\mathrm{d}y + \frac{\partial p}{\partial t}\mathrm{d}t \qquad (6)$$

or, equivalently,

$$\mathrm{d}y = -\sigma\frac{\partial p}{\partial x}\mathrm{d}x + \sigma \mathrm{d}p - \sigma\frac{\partial p}{\partial t}\mathrm{d}t; \qquad (7)$$

thus, recalling that $x = q$, this last expression shows that

$$\phi = -\sigma\frac{\partial p(x, y, t)}{\partial t} \qquad (8)$$

and we can also write Eq. (6) in the form

$$\mathrm{d}p = \frac{\partial p}{\partial x}\mathrm{d}x + \frac{1}{\sigma}\mathrm{d}y - \frac{\phi}{\sigma}\mathrm{d}t. \qquad (9)$$

Since this is an exact differential, we have

$$\frac{\partial \sigma^{-1}}{\partial t} = \frac{\partial}{\partial y}(-\sigma^{-1}\phi) = -\sigma^{-1}\frac{\partial \phi}{\partial y} - \phi\frac{\partial \sigma^{-1}}{\partial y}. \qquad (10)$$

This equation establishes a relation between the integrating factor and the function ϕ (see examples below).

From Eqs. (4), with $\psi = 0$, and (9) we have

$$\begin{aligned}
\mathrm{d}H &= -\frac{1}{\sigma}(g - \phi)\mathrm{d}x + \frac{1}{\sigma}f\mathrm{d}y + \frac{\partial H}{\partial t}\mathrm{d}t \\
&= -\frac{1}{\sigma}(g - \phi)\mathrm{d}x + f\left(\mathrm{d}p - \frac{\partial p}{\partial x}\mathrm{d}x + \frac{\phi}{\sigma}\mathrm{d}t\right) + \frac{\partial H}{\partial t}\mathrm{d}t \\
&= -\left(\frac{g}{\sigma} - \frac{\phi}{\sigma} + f\frac{\partial p}{\partial x}\right)\mathrm{d}q + f\mathrm{d}p + \left(\frac{\partial H}{\partial t} + f\frac{\phi}{\sigma}\right)\mathrm{d}t.
\end{aligned}$$

Hence, considering H as a function of q, p, and t,

$$\frac{\partial H}{\partial p} = f = \dot{q} \qquad (11)$$

[see Eqs. (1)] and

$$-\frac{\partial H}{\partial q} = \frac{g}{\sigma} - \frac{\phi}{\sigma} + f\frac{\partial p}{\partial x} = \dot{p}, \qquad (12)$$

since, according to Eqs. (9) and (1),

$$\dot{p} = \frac{\partial p}{\partial x}\dot{x} + \frac{\dot{y}}{\sigma} - \frac{\phi}{\sigma} = \frac{\partial p}{\partial x}f + \frac{g}{\sigma} - \frac{\phi}{\sigma}.$$

Equations (11) and (12) are equivalent to the original system (1) and have the desired Hamiltonian form.

Summarizing, the system of equations (1) can be written in the form of the Hamilton equations, with the Hamiltonian determined by Eq. (4) and the canonical momentum defined by Eq. (9).

The fact that the left-hand side of Eq. (4) multiplied by σ^{-1} is an exact differential yields (when $\psi = 0$)

$$\frac{\partial}{\partial y}[-\sigma^{-1}(g - \phi)] = \frac{\partial}{\partial x}(\sigma^{-1}f),$$

which amounts to

$$(g-\phi)\frac{\partial \sigma^{-1}}{\partial y} + \sigma^{-1}\frac{\partial}{\partial y}(g-\phi) + f\frac{\partial \sigma^{-1}}{\partial x} + \sigma^{-1}\frac{\partial f}{\partial x} = 0. \qquad (13)$$

Hence, making use of Eqs. (1), (13), and (10), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\sigma^{-1} &= \frac{\partial \sigma^{-1}}{\partial x}\dot{x} + \frac{\partial \sigma^{-1}}{\partial y}\dot{y} + \frac{\partial \sigma^{-1}}{\partial t} \\
&= f\frac{\partial \sigma^{-1}}{\partial x} + g\frac{\partial \sigma^{-1}}{\partial y} + \frac{\partial \sigma^{-1}}{\partial t} \\
&= \phi\frac{\partial \sigma^{-1}}{\partial y} - \sigma^{-1}\frac{\partial}{\partial y}(g - \phi) - \sigma^{-1}\frac{\partial f}{\partial x} + \frac{\partial \sigma^{-1}}{\partial t} \\
&= -\sigma^{-1}\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right).
\end{aligned} \qquad (14)$$

(Note the cancelation of ϕ.)

Equation (14) shows that the function σ is determined up to a factor that is a constant of motion and, therefore, there exists an infinite number of Hamiltonians (and, correspondingly, of expressions for p). It may be noticed that Eq. (14) is just Liouville's theorem.

3. Examples

A first example is provided by the equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \eta(t),$$

where γ and ω_0 are constants, and $\eta(t)$ is an arbitrary function, which corresponds to a forced damped harmonic oscillator. Taking $y=\dot{x}$, we have $\dot{y} = -\gamma y - \omega_0^2 x + \eta(t)$, which is of the form (1) with $f(x,y,t)=y$, and $g(x,y,t) = -\gamma y - \omega_0^2 x + \eta(t)$. Then Eq. (14) reduces to

$$\frac{d}{dt}\sigma^{-1} = \gamma\sigma^{-1}$$

and we can take $\sigma = e^{-\gamma t}$ (any other choice would require the knowledge of the explicit form of η) then from Eq. (10) we see that

$$\frac{\partial\phi}{\partial y} = -\gamma,$$

which is satisfied with $\phi = -\gamma y$. Substituting all these expressions into Eq. (4) we have (with t treated as a constant)

$$\left(\omega_0^2 x - \eta(t)\right)dx + ydy = e^{-\gamma t}dH$$

and, therefore, we can take $H=e^{\gamma t}(y^2/2+\omega_0^2 x^2/2-\eta(t)x)$. Finally, from Eq. (9) we find that p can be chosen as $p = e^{\gamma t}y$. The corresponding Lagrangian can be calculated in the usual way, by means of the Legendre transformation.

The results of the previous section allow us to readily derive those of Ref. 5, corresponding to the autonomous case. In fact, when the functions f and g, appearing in Eqs. (1), do not depend explicitly on the time, from Eqs. (4) and (1), taking $\phi = 0 = \psi$, we have $\sigma\dot{H} = -g\dot{x} + f\dot{y} = -gf + fg = 0$. This means that H is *some* constant of motion, which is not unique; we can replace it by $H' = G(H)$, with G being an arbitrary function. H' is also a constant of motion and σ will not depend explicitly on t [see Eq. (10)], no matter what (time-independent) Hamiltonian we choose.

The expressions given above allow us to find H, which need not be related to the total energy. In the example considered in the appendix of Ref. 5, $f(x,y) = y, g(x,y) = -ky$, where k is a constant (*i.e.*, $\ddot{x}=-k\dot{x}$). Then, $-gdx + fdy = kydx + ydy = yd(kx + y)$ and, therefore, we can take $\sigma = y$ and $H = kx + y$.

We end this section by considering the problem studied in Ref. 6 (which corresponds approximately to a relativistic particle subjected to a constant force, λ, and a force of friction proportional to the square of the velocity), namely (with the appropriate changes in notation)

$$m\dot{y} = (\lambda - \gamma y^2)(1 - \alpha^2 y^2),$$

where m represents a mass, λ, γ, and α are constants. Thus, $f(x,y) = y, g(x,y) = (\lambda - \gamma y^2)(1 - \alpha^2 y^2)/m$, and

$$-gdx + fdy$$

$$= -\frac{1}{m}(\lambda - \gamma y^2)(1 - \alpha^2 y^2)dx + ydy$$

$$= (\lambda - \gamma y^2)(1 - \alpha^2 y^2)\left[-\frac{dx}{m} + \frac{ydy}{(\lambda - \gamma y^2)(1 - \alpha^2 y^2)}\right].$$

Comparing with Eq. (4) (with $\phi = 0 = \psi$) we immediately see that we can take

$$\sigma = (\lambda - \gamma y^2)(1 - \alpha^2 y^2)$$

and

$$H = -\frac{x}{m} + \int \frac{ydy}{(\lambda - \gamma y^2)(1 - \alpha^2 y^2)}$$

$$= -\frac{x}{m} + \frac{1}{2(\lambda\alpha^2 - \gamma)}\ln\left|\frac{\lambda - \gamma y^2}{1 - \alpha^2 y^2}\right|.$$

According to Eq. (9), the canonical momentum p can be taken as

$$p = \int \frac{dy}{(\lambda - \gamma y^2)(1 - \alpha^2 y^2)}.$$

Despite the huge difference with the expressions given in Ref. 6, one can show that the Hamiltonian obtained in that reference is essentially the exponential of our H. (See Eqs. (23) and (26) of Ref. 6.)

4. The Helmholtz condition

The case in which one starts with a second-order equation of the form

$$\ddot{x} = F(x, \dot{x}, t) \tag{15}$$

(considered in Refs. 2, 3) is a particular case of the treatment above if one defines, *e.g.*, $y \equiv \dot{x}$, which transforms Eq. (15) into the system

$$\dot{x} = y, \qquad \dot{y} = F(x, y, t),$$

which is of the form (1) with $f(x,y,t)=y$ and $g(x,y,t) = F(x,y,t)$. Then Eq. (14) reduces to

$$\frac{d}{dt}\sigma^{-1} = -\sigma^{-1}\frac{\partial F}{\partial y}, \tag{16}$$

which is the Helmholtz condition when there is one degree of freedom (see, *e.g.*, Ref. 2 and the references cited therein; note that $\sigma^{-1} = \partial p/\partial y = \partial p/\partial \dot{x} = \partial^2 L/\partial \dot{x}^2$ is the integrating factor w_{11} employed in these references).

On the other hand, not every system of equations of the form (1) comes from a second-order equation $\ddot{x} = F(x, \dot{x}, t)$. An example is given by

$$\dot{x} = f(x, t), \qquad \dot{y} = g(y, t),$$

where there is no coupling between the variables x, y. Here (choosing $\phi = 0 = \psi$)

$$-gdx + fdy = fg\left(-\frac{dx}{f} + \frac{dy}{g}\right).$$

Therefore, if we assume that $\sigma = fg$ does not depend explicitly on t [see Eq. (10)], we can take

$$H = -\int \frac{\mathrm{d}x}{f} + \int \frac{\mathrm{d}y}{g}$$

and, from Eq. (5),

$$p = \int \frac{\mathrm{d}y}{\sigma} = \frac{1}{f} \int \frac{\mathrm{d}y}{g}.$$

Thus, $H = pf - \int f^{-1}\mathrm{d}x$ and with the Hamiltonian being a linear function of p, the Legendre transformation is not defined nor the Lagrangian.

5. Concluding remarks

As we have shown, at least in the case of a system of two first-order ordinary differential equations, finding a Hamiltonian is essentially equivalent to finding an integrating factor for a linear differential form in two variables. The integrating factor also determines the expression for the canonical momentum. Equation (14) is analogous to the Helmholtz condition, but, in the present approach, it leads directly to the Hamiltonian (in the standard approach, finding a solution to the Helmholtz conditions, only gives the second partial derivatives $\partial^2 L/\partial \dot{x}_i \partial \dot{x}_j$). When the system is non-autonomous, it is convenient to find the integrating factor using Eq. (14), while in the autonomous case, it may be more simply obtained from the linear differential form itself. Finally, as shown in Sec. 4, there are systems of equations for which a Lagrangian does not exist, but a Hamiltonian description can be given.

Acknowledgment

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