Torques on quadrupoles

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Making use of the fact that a 2^l-pole can be represented by means of l vectors of the same magnitude, the torque on a quadrupole in an inhomogeneous external field is expressed in terms of the vectors that represent the quadrupole and the gradient of the external field. The conditions for rotational equilibrium are also expressed in terms of these vectors.

Keywords: Multipoles; torque.

Haciendo uso de que un multipolo de orden 2^l puede representarse mediante l vectores de la misma magnitud, la torca sobre un cuadripolo en un campo externo inhomogéneo se expresa en términos de los vectores que representan el cuadripolo y el gradiente del campo externo. Las condiciones de equilibrio rotacional se expresan también en términos de estos vectores.

Descriptores: Multipolos; torca.

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1. Introduction

As is well known, the torque on an electric dipole placed in an external electric field, $\mathbf{E}$, is given by

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}, \tag{1}$$

where $\mathbf{p}$ is the dipole moment and, therefore, the torque on the dipole vanishes when $\mathbf{p}$ is collinear with $\mathbf{E}$. In its elementary form, an electric dipole is formed by two electric charges, q and $-q$ (so that the total charge is equal to zero), separated by a small distance. Similarly, an electric quadrupole can be made out of two electric dipoles, with moments $\mathbf{p}$ and $-\mathbf{p}$ (so that the total charge and the total dipole moment are equal to zero), separated by a small distance, and we can go on constructing higher multipoles in this manner.

The field of any bounded charge distribution is the superposition of the fields produced by a point charge, a point dipole, a point quadrupole and so on. In the presence of an external inhomogeneous electric field, each multipole moment contributes to the force and the torque on the charge distribution. The torque on a point 2^l-pole depends on the $(l-1)$-th derivatives of the electric field (that is, on the l-th partial derivatives of the electrostatic potential); in particular, the torque on a point dipole, for which $l = 1$, involves the value of the electric field itself at the location of the dipole [see Eq. (1)].

The 2^l-pole moment of a charge distribution is given by an l-index tensor (a vector in the case of the dipole moment, a two-index tensor in the case of the quadrupole moment, and so on) which, being totally symmetric and traceless, can be algebraically expressed and geometrically represented by means of l vectors of the same magnitude [1,2] (this result is equivalent to the so-called Sylvester's theorem [3,4]). Furthermore, since the electrostatic potential of the external field satisfies the Laplace equation, the l-th partial derivatives of the electrostatic potential with respect to Cartesian coordinates also correspond to a totally symmetric, traceless l-index tensor that can be expressed and represented by a second set of l vectors of the same magnitude.

Thus, the quadrupole moment can be represented by two vectors, $\mathbf{c}$ and $\mathbf{d}$ (say), and, just as a dipole moment $\mathbf{p}$ can be associated with two charges $\pm q$ placed at the endpoints of the vector $\mathbf{p}/q$, the quadrupole moment represented by $\mathbf{c}$ and $\mathbf{d}$ can be associated with four charges, $-q$, q, $-q$, q, placed at the vertices of a parallelogram with sides $\mathbf{c}/\sqrt{6q}$ and $\mathbf{d}/\sqrt{6q}$ (see Sec. 2 below). In a similar way, any 2^l-pole can be represented by means of 2^l point charges.

The energy and the torque on a 2^l-pole in an external field, written in terms of the Cartesian components of the 2^l-pole moment, can be obtained in a straightforward manner (see, for example, Ref. 5), but the resulting expression is not very convenient since only $2l+1$ out of the 3^l Cartesian components of the 2^l-pole moment are independent. In this paper we show how to find expressions for the electrostatic energy and the torque on a 2^l-pole in terms of the l vectors that represent the 2^l-pole and the l vectors that represent the appropriate derivatives of the external field, by considering in detail the case of a quadrupole. Then we obtain the equilibrium orientations of the quadrupole, identifying the stable ones.

2. Torque on a quadrupole. Equilibrium orientations

Starting from the expression

$$U = \int \rho(\mathbf{r})\varphi(\mathbf{r})\mathrm{d}v$$

for the energy of a charge distribution with charge density ρ in an external field corresponding to the potential φ (see, for example, Refs. 5 and 6), by means of a Taylor expansion, one obtains

$$\begin{aligned} U &= \int \rho(\mathbf{r}) \Bigg[\varphi(\mathbf{0}) + \sum_{i=1}^{3} x_i \frac{\partial \varphi}{\partial x_i}(\mathbf{0}) \\ &\qquad + \frac{1}{2}\sum_{i,j=1}^{3} x_i x_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(\mathbf{0}) + \cdots \Bigg] \mathrm{d}v \\ &= Q\varphi(\mathbf{0}) + \sum_{i=1}^{3} p_i \frac{\partial \varphi}{\partial x_i}(\mathbf{0}) + \frac{1}{6}\sum_{i,j=1}^{3} Q_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(\mathbf{0}) \\ &\qquad + \frac{1}{30}\sum_{i,j,k=1}^{3} M_{ijk} \frac{\partial^3 \varphi}{\partial x_i \partial x_j \partial x_k}(\mathbf{0}) + \cdots, \end{aligned} \tag{2}$$

where the x_i are the Cartesian components of $\mathbf{r}$ and we have made use of the equation $\nabla^2\varphi = 0$ and the definitions

$$\begin{aligned} Q &= \int \rho(\mathbf{r})\mathrm{d}v, \qquad p_i = \int \rho(\mathbf{r}) x_i \mathrm{d}v, \\ Q_{ij} &= \int \rho(\mathbf{r})(3x_i x_j - r^2\delta_{ij})\,\mathrm{d}v, \qquad (3) \\ M_{ijk} &= \int \rho(\mathbf{r})(5x_i x_j x_k - r^2 x_i \delta_{jk} \\ &\qquad - r^2 x_j \delta_{ki} - r^2 x_k \delta_{ij})\,\mathrm{d}v, \end{aligned}$$

of the total charge, dipole, quadrupole, and octopole moments, respectively.

Recalling that, to first order in the angle $\delta\theta$, the change of an arbitrary vector **A** under a rotation about the axis defined by the unit vector **n** through the angle $\delta\theta$ is given by

$$\delta\mathbf{A} = \mathbf{n} \times \mathbf{A}\,\delta\theta = \delta\boldsymbol{\theta} \times \mathbf{A}, \tag{4}$$

where $\delta\boldsymbol{\theta} \equiv \mathbf{n}\,\delta\theta$, the corresponding change of the energy of a charge element under the rotation represented by $\delta\boldsymbol{\theta}$ is

$$\delta U = -\mathbf{F}\cdot\delta\mathbf{r} = -\mathbf{F}\cdot\delta\boldsymbol{\theta}\times\mathbf{r} = -\delta\boldsymbol{\theta}\cdot\mathbf{r}\times\mathbf{F} = -\boldsymbol{\tau}\cdot\delta\boldsymbol{\theta}, \tag{5}$$

where **F** and $\boldsymbol{\tau}$ denote the force and torque produced by the external electric field on the charge element. By integrating over the charge distribution one finds that the same relation (5) holds for the total energy and torque of the charge distribution. In the case of a point dipole, for instance, using the fact that $\mathbf{E} = -\nabla\varphi$, from Eq. (2) we have $U = -\mathbf{p}\cdot\mathbf{E}$ and, therefore, making use of Eq. (4),

$$\delta U = -\delta\mathbf{p}\cdot\mathbf{E} = -\delta\boldsymbol{\theta}\times\mathbf{p}\cdot\mathbf{E} = -\mathbf{p}\times\mathbf{E}\cdot\delta\boldsymbol{\theta},$$

and by comparing with Eq. (5) one obtains the usual expression (1) for the torque on a dipole in an external electric field **E**.

According to Eq. (2), the energy of a point quadrupole in an external electric field is

$$U = -\frac{1}{6}\sum_{i,j=1}^{3} Q_{ij} G_{ij}, \tag{6}$$

where we have defined

$$G_{ij} \equiv -\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(\mathbf{0}). \tag{7}$$

(Note that G_{ij} represents the gradient of the electric field, $G_{ij} = \partial E_i/\partial x_j$.) Both Q_{ij} and G_{ij} are symmetric traceless tensors and therefore there exist four vectors, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{a}$, and $\mathbf{b}$, with $|\mathbf{v}| = |\mathbf{w}|$, $|\mathbf{a}| = |\mathbf{b}|$, such that [1,2]

$$Q_{ij} = \tfrac{1}{2}(v_i w_j + v_j w_i) - \tfrac{1}{3}(\mathbf{v}\cdot\mathbf{w})\delta_{ij}, \tag{8}$$

$$G_{ij} = \tfrac{1}{2}(a_i b_j + a_j b_i) - \tfrac{1}{3}(\mathbf{a}\cdot\mathbf{b})\delta_{ij}; \tag{9}$$

hence,

$$\begin{aligned} U = &-\tfrac{1}{12}\big[(\mathbf{a}\cdot\mathbf{v})(\mathbf{b}\cdot\mathbf{w}) + (\mathbf{a}\cdot\mathbf{w})(\mathbf{b}\cdot\mathbf{v}) \\ &- \tfrac{2}{3}(\mathbf{a}\cdot\mathbf{b})(\mathbf{v}\cdot\mathbf{w})\big]. \end{aligned} \tag{10}$$

In fact, given a symmetric tensor Q_{ij}, one can always find three mutually orthogonal unit vectors, **X**, **Y**, **Z**, which are eigenvectors of Q_{ij}, satisfying

$$Q_{ij} = \lambda X_i X_j + \mu Y_i Y_j + \nu Z_i Z_j, \tag{11}$$

where λ, μ, and ν are the corresponding eigenvalues, which are all real. Since **X**, **Y**, and **Z**, are orthogonal unit vectors,

$$\delta_{ij} = X_i X_j + Y_i Y_j + Z_i Z_j, \tag{12}$$

and Q_{ij} is traceless if and only if $\lambda + \mu + \nu = 0$; hence, the eigenvector **Z** and its eigenvalue can be eliminated from Eq. (11) and we can write

$$\begin{aligned} Q_{ij} &= \lambda X_i X_j + \mu Y_i Y_j + (-\lambda - \mu)(\delta_{ij} - X_i X_j - Y_i Y_j) \\ &= (2\lambda + \mu) X_i X_j + (2\mu + \lambda) Y_i Y_j - (\lambda + \mu)\delta_{ij}. \end{aligned} \tag{13}$$

Assuming that λ and μ are the greatest and the smallest eigenvalue of Q_{ij}, respectively, one finds that $2\lambda + \mu$, and $-2\mu - \lambda$ are non-negative (indeed, $2\lambda + \mu \geqslant \lambda + \nu + \mu = 0$, and $2\mu + \lambda \leqslant \mu + \nu + \lambda = 0$). Thus, letting [2]

$$\begin{aligned} \mathbf{v} &\equiv \sqrt{2\lambda + \mu}\,\mathbf{X} + \sqrt{-2\mu - \lambda}\,\mathbf{Y}, \\ \mathbf{w} &\equiv \sqrt{2\lambda + \mu}\,\mathbf{X} - \sqrt{-2\mu - \lambda}\,\mathbf{Y}, \end{aligned} \tag{14}$$

we have $|\mathbf{v}| = |\mathbf{w}|$ and $\mathbf{v} \cdot \mathbf{w} = 3(\lambda + \mu)$ and from Eq. (13) we obtain

$$Q_{ij} = \tfrac{1}{2}(v_i w_j + v_j w_i) - \tfrac{1}{3}(\mathbf{v} \cdot \mathbf{w})\delta_{ij}, \qquad (15)$$

which is Eq. (8). Making use of the two-component spinor formalism, Eq. (15) follows from the fundamental theorem of algebra [1,7].

Remark 1. While we do not have an analog of Eq. (11) for symmetric tensors with three or more indices, using the spinor formalism, one can show that an analog of Eq. (15) holds for any *traceless* symmetric tensor [1,7].

Remark 2. Even in the case of a two-index tensor, Eq. (15), which involves two vectors only, is preferable to the well-known expression (11), which involves three eigenvectors. Furthermore, when there is degeneracy (*i.e.*, the eigenvalue ν coincides with λ or with μ), the two degenerate eigenvectors are not uniquely defined; they span a two-dimensional plane and any pair of mutually orthogonal vectors in this plane can be chosen as part of the basis $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$. On the other hand, this ambiguity does not arise in connection with the vectors $\mathbf{v}$ and $\mathbf{w}$ appearing in Eq. (15); when $\nu = \lambda$ or $\nu = \mu$, we have $2\lambda + \mu = 0$ or $2\mu + \lambda = 0$, respectively, which amounts to $\mathbf{v} = -\mathbf{w}$ or $\mathbf{v} = \mathbf{w}$, respectively. (Note that since $\lambda + \mu + \nu = 0$, the only case in which a triple degeneracy can occur is the trivial one with $\lambda = \mu = \nu = 0$.)

Remark 3. The vectors $\mathbf{v}$ and $\mathbf{w}$ appearing in Eq. (15), being of the same magnitude, can be viewed as the sides of a rhombus whose diagonals, $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$, are orthogonal to each other. In fact, from Eq. (14) we see that

$$\mathbf{v} + \mathbf{w} = 2\sqrt{2\lambda + \mu}\,\mathbf{X}, \qquad \mathbf{v} - \mathbf{w} = 2\sqrt{-2\mu - \lambda}\,\mathbf{Y}, \qquad (16)$$

that is, the eigenvectors of Q_{ij} with the greatest and the smallest eigenvalue lie on the plane of this rhombus pointing along its diagonals. (Hence, the eigenvector of Q_{ij} with the intermediate eigenvalue is orthogonal to the rhombus.)

Remark 4. The usual examples of quadrupoles considered in the textbooks consist of sets of four point charges, $-q$, q, $-q$, q, placed at the vertices of a parallelogram. If $\mathbf{c}$ and $\mathbf{d}$ are the vectors that go from one of the negative charges to the two positive ones then, from Eq. (3), one readily finds that

$$Q_{ij} = 3q(c_i d_j + c_j d_i) - 2q(\mathbf{c} \cdot \mathbf{d})\delta_{ij},$$

which is of the form (15) with, for example,

$$\mathbf{v} = \left(6q\frac{|\mathbf{d}|}{|\mathbf{c}|}\right)^{1/2}\mathbf{c}, \qquad \mathbf{w} = \left(6q\frac{|\mathbf{c}|}{|\mathbf{d}|}\right)^{1/2}\mathbf{d}.$$

This means that the vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel to the vectors joining one of the negative charges with the two positive ones. By contrast, the eigenvectors of Q_{ij} do not have a direct relationship with the geometry of the charge distribution. (However, as pointed out above, the eigenvector of Q_{ij} with the intermediate eigenvalue is perpendicular to the plane containing the charges.)

Making use of Eqs. (4) and (10), and the properties of the triple scalar product we find that the change of U under a rotation of the quadrupole given by $\delta\boldsymbol{\theta}$ is

$$\begin{aligned} \delta U &= -\tfrac{1}{12}[(\mathbf{a} \cdot \delta\boldsymbol{\theta} \times \mathbf{v})(\mathbf{b} \cdot \mathbf{w}) + (\mathbf{a} \cdot \mathbf{v})(\mathbf{b} \cdot \delta\boldsymbol{\theta} \times \mathbf{w}) \\ &\quad + (\mathbf{a} \cdot \delta\boldsymbol{\theta} \times \mathbf{w})(\mathbf{b} \cdot \mathbf{v}) + (\mathbf{a} \cdot \mathbf{w})(\mathbf{b} \cdot \delta\boldsymbol{\theta} \times \mathbf{v})] \\ &= -\tfrac{1}{12}[(\delta\boldsymbol{\theta} \cdot \mathbf{v} \times \mathbf{a})(\mathbf{b} \cdot \mathbf{w}) + (\mathbf{a} \cdot \mathbf{v})(\delta\boldsymbol{\theta} \cdot \mathbf{w} \times \mathbf{b}) \\ &\quad + (\delta\boldsymbol{\theta} \cdot \mathbf{w} \times \mathbf{a})(\mathbf{b} \cdot \mathbf{v}) + (\mathbf{a} \cdot \mathbf{w})(\delta\boldsymbol{\theta} \cdot \mathbf{v} \times \mathbf{b})] \end{aligned}$$

and therefore the torque on the quadrupole is given by

$$\begin{aligned} \boldsymbol{\tau} = \tfrac{1}{12}[&(\mathbf{b} \cdot \mathbf{w})\mathbf{v} \times \mathbf{a} + (\mathbf{a} \cdot \mathbf{v})\mathbf{w} \times \mathbf{b} \\ &+ (\mathbf{b} \cdot \mathbf{v})\mathbf{w} \times \mathbf{a} + (\mathbf{a} \cdot \mathbf{w})\mathbf{v} \times \mathbf{b}] \end{aligned} \qquad (17)$$

[*cf.* Eq. (1)]. Hence, making use of Eqs. (8) and (9), we see that the Cartesian components of the torque on the quadrupole can also be expressed in the form

$$\begin{aligned} \tau_i &= \frac{1}{12}\sum_{j,k,l=1}^{3} \varepsilon_{ijk}(b_l w_l v_j a_k + a_l v_l w_j b_k \\ &\quad + b_l v_l w_j a_k + a_l w_l v_j b_k) \\ &= \frac{1}{12}\sum_{j,k,l=1}^{3} \varepsilon_{ijk}(v_j w_l + v_l w_j)(a_k b_l + a_l b_k) \\ &= \frac{1}{3}\sum_{j,k,l=1}^{3} \varepsilon_{ijk} Q_{jl} G_{kl}, \end{aligned} \qquad (18)$$

where ε_{ijk} is the usual Levi-Civita symbol [*cf.* Ref. 5, Eq. (1.31)].

The Cartesian components of the quadrupole moment, Q_{ij}, are also equivalent to a set of spherical components, q_{2m}, $m = \pm 2, \pm 1, 0$ (see, for example, Refs. 8,1) and, in an entirely similar manner, one can define a set of spherical components, g_{2m} (say), corresponding to G_{ij}. Then both U and the spherical components of the torque can be written in terms of these spherical components. For instance, making use of Eqs. (4.6) of Ref. 8 and analogous definitions for g_{2m} one finds that

$$U = -\frac{4\pi}{5}(q_{22}g_{2,-2} - q_{21}g_{2,-1} + q_{20}g_{20} - q_{2,-1}g_{21} + q_{2,-2}g_{22}).$$

Now, making use of Eqs. (10) and (17), we shall find the orientations of a quadrupole, with respect to the gradient of the external electric field, for which the torque vanishes. To this end, it is convenient to introduce the vectors

$$\mathbf{V} \equiv \tfrac{1}{2}(\mathbf{v} + \mathbf{w}), \qquad \mathbf{W} \equiv \tfrac{1}{2}(\mathbf{v} - \mathbf{w}). \qquad (19)$$

Since $|\mathbf{v}| = |\mathbf{w}|$, $\mathbf{V}$ is orthogonal to $\mathbf{W}$; in fact, according to Eq. (16), $\mathbf{V}$ and $\mathbf{W}$ point along the principal axes of Q_{ij}

with the greatest and the smallest eigenvalue, respectively. In a similar manner, letting

$$\mathbf{A} \equiv \tfrac{1}{2}(\mathbf{a}+\mathbf{b}), \qquad \mathbf{B} \equiv \tfrac{1}{2}(\mathbf{a}-\mathbf{b}), \tag{20}$$

one finds that **A** and **B** point along the principal axes of G_{ij} with the greatest and the smallest eigenvalue, respectively, provided that these vectors are different from zero.

In terms of **A**, **B**, **V**, and **W**, the torque on the quadrupole (17) is given by

$$\begin{aligned}\boldsymbol{\tau} \;=\; & \tfrac{1}{3}\,[(\mathbf{V}\cdot\mathbf{A})\,\mathbf{V}\times\mathbf{A} - (\mathbf{V}\cdot\mathbf{B})\,\mathbf{V}\times\mathbf{B} \\ & - (\mathbf{W}\cdot\mathbf{A})\,\mathbf{W}\times\mathbf{A} + (\mathbf{W}\cdot\mathbf{B})\,\mathbf{W}\times\mathbf{B}]\end{aligned} \tag{21}$$

and, assuming that all vectors **A**, **B**, **V**, and **W** are different from zero, from Eq. (21) one finds that there exist *six* equilibrium orientations for the quadrupole, namely

(i) **A** collinear with **V** and **B** collinear with **W**,

(ii) **A** collinear with **W** and **B** collinear with **V**,

(iii) **A** orthogonal to **V** and **W** with **B** collinear with **V**,

(iv) **A** orthogonal to **V** and **W** with **B** collinear with **W**,

(v) **B** orthogonal to **V** and **W** with **A** collinear with **V**,

(vi) **B** orthogonal to **V** and **W** with **A** collinear with **W**.

The energy of the quadrupole can be expressed as

$$\begin{aligned}U \;=\; & -\frac{1}{18}[3(\mathbf{A}\cdot\mathbf{V})^2 + 3(\mathbf{B}\cdot\mathbf{W})^2 - 3(\mathbf{A}\cdot\mathbf{W})^2 \\ & - 3(\mathbf{B}\cdot\mathbf{V})^2 - |\mathbf{A}|^2|\mathbf{V}|^2 - |\mathbf{B}|^2|\mathbf{W}|^2 \\ & + |\mathbf{A}|^2|\mathbf{W}|^2 + |\mathbf{B}|^2|\mathbf{V}|^2]\end{aligned} \tag{22}$$

and, therefore, for each equilibrium orientation listed above, the energy of the quadrupole is

$$\begin{aligned}
& -\frac{1}{18}(2|\mathbf{A}|^2|\mathbf{V}|^2 + 2|\mathbf{B}|^2|\mathbf{W}|^2 + |\mathbf{A}|^2|\mathbf{W}|^2 + |\mathbf{B}|^2|\mathbf{V}|^2) && (\mathbf{A}\parallel\mathbf{V},\ \mathbf{B}\parallel\mathbf{W}),\\
& -\frac{1}{18}(-2|\mathbf{A}|^2|\mathbf{W}|^2 - 2|\mathbf{B}|^2|\mathbf{V}|^2 - |\mathbf{A}|^2|\mathbf{V}|^2 - |\mathbf{B}|^2|\mathbf{W}|^2) && (\mathbf{A}\parallel\mathbf{W},\ \mathbf{B}\parallel\mathbf{V}),\\
& -\frac{1}{18}(-2|\mathbf{B}|^2|\mathbf{V}|^2 - |\mathbf{A}|^2|\mathbf{V}|^2 - |\mathbf{B}|^2|\mathbf{W}|^2 + |\mathbf{A}|^2|\mathbf{W}|^2) && (\mathbf{A}\perp\mathbf{V},\mathbf{W}\text{ with }\mathbf{B}\parallel\mathbf{V}),\\
& -\frac{1}{18}(2|\mathbf{B}|^2|\mathbf{W}|^2 - |\mathbf{A}|^2|\mathbf{V}|^2 + |\mathbf{A}|^2|\mathbf{W}|^2 + |\mathbf{B}|^2|\mathbf{V}|^2) && (\mathbf{A}\perp\mathbf{V},\mathbf{W}\text{ with }\mathbf{B}\parallel\mathbf{W}),\\
& -\frac{1}{18}(2|\mathbf{A}|^2|\mathbf{V}|^2 - |\mathbf{B}|^2|\mathbf{W}|^2 + |\mathbf{A}|^2|\mathbf{W}|^2 + |\mathbf{B}|^2|\mathbf{V}|^2) && (\mathbf{B}\perp\mathbf{V},\mathbf{W}\text{ with }\mathbf{A}\parallel\mathbf{V}),\\
& -\frac{1}{18}(-2|\mathbf{A}|^2|\mathbf{W}|^2 - |\mathbf{A}|^2|\mathbf{V}|^2 - |\mathbf{B}|^2|\mathbf{W}|^2 + |\mathbf{B}|^2|\mathbf{V}|^2) && (\mathbf{B}\perp\mathbf{V},\mathbf{W}\text{ with }\mathbf{A}\parallel\mathbf{W}).
\end{aligned} \tag{23}$$

Thus, the stable equilibrium orientation corresponds to **A** collinear with **V** and **B** collinear with **W**, that is, the rhombus with sides **a**, **b** is coplanar with the rhombus with sides **u**, **v**, and the bisector of **a** and **b** is collinear with the bisector of **u** and **v**.

In other words, a quadrupole is in one of its six equilibrium orientations when each of the principal axes of the quadrupole moment Q_{ij} coincides with one of the principal axes of the gradient of the electric field G_{ij}. The stable equilibrium orientations occur when the principal axis of Q_{ij} with the greatest eigenvalue coincides with the principal axis of G_{ij} with the greatest eigenvalue and, simultaneously, the principal axis of Q_{ij} with the smallest eigenvalue coincides with the principal axis of G_{ij} with the smallest eigenvalue. (From Eqs. (10) we see that the energy of a quadrupole is invariant under the substitution of **v** and **w** by their negatives and therefore, given a stable equilibrium orientation of the quadrupole, another stable equilibrium orientation is obtained by rotating the quadrupole through 180° about the normal to the rhombus with sides **v** and **w**.) Thus, the equilibrium orientations can be succinctly characterized in terms of the principal axes of Q_{ij} and G_{ij}.

Substituting Eq. (11) and an analogous expression for G_{ij} into Eq. (18), one finds the torque on the quadrupole in terms of the eigenvectors and eigenvalues of Q_{ij} and G_{ij}; the resulting expression readily shows that, when each eigenvector of Q_{ij} is collinear with an eigenvector of G_{ij}, the torque vanishes. However, such an expression does not seem useful in proving that these are all the equilibrium orientations, since the torque is a linear combination of the nine cross-products that can be formed by multiplying each eigenvector of Q_{ij} by each eigenvector of G_{ij}. By contrast, in Eqs. (17) and (21) the torque is expressed in terms of four cross products only, which simplifies the search for the equilibrium orientations.

Equation (6) shows that, apart from the factor $-1/6$, the energy of the quadrupole for each equilibrium orientation is a sum of products of the eigenvalues of Q_{ij} by the eigenvalues of G_{ij}. In fact, from Eqs. (16) and (19) we see that $|\mathbf{V}|^2 = 2\lambda+\mu$ and $|\mathbf{W}|^2 = -2\mu-\lambda$, with analogous expressions for $|\mathbf{A}|^2$ and $|\mathbf{B}|^2$, in terms of the eigenvalues of G_{ij}; hence the values of the energy given in Eq. (23) are equivalent to

$$-\tfrac{1}{6}(\lambda\tilde{\lambda}+\mu\tilde{\mu}+\nu\tilde{\nu}), \quad -\tfrac{1}{6}(\lambda\tilde{\mu}+\mu\tilde{\lambda}+\nu\tilde{\nu}),$$
$$-\tfrac{1}{6}(\lambda\tilde{\mu}+\mu\tilde{\nu}+\nu\tilde{\lambda}), \quad -\tfrac{1}{6}(\lambda\tilde{\nu}+\mu\tilde{\mu}+\nu\tilde{\lambda}),$$
$$-\tfrac{1}{6}(\lambda\tilde{\lambda}+\mu\tilde{\nu}+\nu\tilde{\mu}), \quad -\tfrac{1}{6}(\lambda\tilde{\nu}+\mu\tilde{\lambda}+\nu\tilde{\mu}),$$

respectively, where $\tilde{\lambda}$, $\tilde{\mu}$, and $\tilde{\nu}$ are the eigenvalues of G_{ij} with $\tilde{\lambda} \geqslant \tilde{\nu} \geqslant \tilde{\mu}$.

A simple alternative proof that the equilibrium orientations of the quadrupole correspond to the coincidence of the eigenvectors of Q_{ij} and G_{ij} is obtained as follows. If the matrix (M_{ij}) is the product of the matrices (Q_{ij}) and (G_{ij}), we have

$$M_{jk} = \sum_{l=1}^{3} Q_{jl}G_{lk} = \sum_{l=1}^{3} Q_{jl}G_{kl}$$

(using the fact that (G_{ij}) is symmetric) and, according to Eq. (18),

$$\tau_i = \frac{1}{3}\sum_{j,k=1}^{3} \varepsilon_{ijk}M_{jk},$$

which is equal to zero if and only if (M_{ij}) is symmetric. But the product of two symmetric matrices is symmetric if and only if the matrices commute, which happens if and only if there is a basis formed by common eigenvectors of the two matrices. (This is the finite-dimensional version of the well-known proposition employed in quantum mechanics which asserts that two observables commute if and only if there exists a set of common eigenstates.) It should be noticed, however, that these arguments cannot be applied in the case of higher multipoles, since they are represented by objects with three or more indices.

We end this section with some remarks. All the results of this section also apply in the case of a magnetic multipole in an external magnetic field. Making use of Eq. (2) we can also find the force on an electric 2^l-pole; in the case of a dipole, one obtains $\mathbf{F} = \mathbf{p}\cdot\nabla\mathbf{E}$, which, by virtue of Eq. (9), is equivalent to $\mathbf{F} = 1/2[(\mathbf{p}\cdot\mathbf{a})\mathbf{b} + (\mathbf{p}\cdot\mathbf{b})\mathbf{a}] - 1/3(\mathbf{a}\cdot\mathbf{b})\mathbf{p}$. The torque on a 2^l-pole can be expressed in terms of the l vectors representing the multipole and the l vectors representing the $(l-1)$-th derivatives of the external field. On the other hand, the force on a 2^l-pole can be expressed in terms of the l vectors representing the multipole and the $l+1$ vectors representing the l-th derivatives of the external field.

3. Example

As an example of the foregoing results we shall consider the field produced by a point charge Q at the origin. The corresponding electrostatic potential is given by

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r},$$

and a straightforward computation yields [see Eq. (7)]

$$G_{ij} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{3x_ix_j}{r^5} - \frac{\delta_{ij}}{r^3}\right). \tag{24}$$

(Note that we are not evaluating the derivatives at the origin as in Eq. (7), but at an arbitrary point (x, y, z).) Owing to the spherical symmetry of the field, we expect one of the eigenvectors of G_{ij} to be in the radial direction, with two degenerate eigenvectors tangential to the sphere $r = \text{const}$. As shown above, this degeneracy implies that **a** is collinear with **b** (see Remark 2). In fact, comparison of Eq. (24) with Eq. (9) shows that when Q is positive **a** is antiparallel to **b**, while if Q is negative **a** is parallel to **b**; thus, in the first case, the eigenvector of G_{ij} with the smallest eigenvalue points radially and, in the second case, the eigenvector of G_{ij} with the greatest eigenvalue points radially.

We now consider a quadrupole formed by four point charges, $-q$, q, $-q$, q (with $q > 0$), at the vertices of a parallelogram. According to the preceding discussion, the eigenvector of the quadrupole moment Q_{ij} with the greatest (respectively, smallest) eigenvalue bisects the angle of the parallelogram at the vertex occupied by one of the negative (respectively, positive) charges. Taking into account the results of the preceding paragraph, the equilibrium orientations of this quadrupole in the field of a point charge are those for which the radial direction bisects one of the angles of the parallelogram or is orthogonal to the plane of the parallelogram. In the stable equilibrium orientation, the radial direction bisects the angle of the parallelogram at the vertex occupied by one of the positive charges if $Q > 0$ or by one of the negative charges if $Q < 0$.

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