Some statistical mechanical properties of photon black holes

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We show that if the total internal energy of a black hole is constructed as the sum of N photons all having a fixed wavelength chosen to scale with the Schwarzschild radius as $\lambda = \alpha R_s$, then N will scale with R_s^2. A statistical mechanical calculation of the configuration proposed yields $\alpha = 4\pi^2/\ln(2)$ and a total entropy of the system $S = k_B N \ln(2)$, in agreement with the Bekenstein entropy of a black hole. It is shown that the critical temperature for Bose-Einstein condensation for relativistic particles of $\lambda = \alpha R_s$ is always well below the Hawking temperature of a black hole, in support of the proposed internal configuration. We then examine our results from the point of view of recent loop quantum gravity ideas and find that a natural consistency of both approaches appears. We show that the Jeans criterion for gravitational instability can be generalised to the special and general relativistic regimes and holds for any type of mass-energy distribution.

Keywords: Physics of black holes; classical black holes; quantum aspects of black holes; evaporation; thermodynamics.

En este artículo estudiamos la relación entre la energía y la entropía de un gas de fotones tipo cuerpo negro, contenido dentro de un recinto adiabático de radio R, cuando es comprimido hacia un régimen auto–gravitacional. Mostramos que este régimen coincide aproximadamente con el régimen de un agujero negro para el sistema, *i.e.*, $R \sim R_s$, donde R_s representa al radio de Schwarzschild del sistema. La entropía del sistema resulta estar siempre por debajo de la cota Holográfica, incluso cuando $R \to R_s$. Una posible configuración cuántica para el gas de fotones a $R \to R_s$ se sugiere, la cual satisface todas las condiciones de agujero negro para la energía, entropía y temperatura. Finalmente, examinamos nuestros resultados desde el punto de vista de algunas ideas recientes de Loop Quantum Gravity.

Descriptores: Física de agujeros negros; agujeros negros classicos; aspectos cuánticos de agujeros negros; evaporación; termodinámica.

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1. Introduction

In general, black holes are defined uniquely by their mass, angular momentum and charge (cf. [1]). In this paper we shall deal exclusively with Schwarzschild black holes, where the angular momentum and charge are both zero. These black holes of mass M are understood as systems defined uniquely by the condition $R \leq R_s$ and have the following properties:

$$R_s = \frac{2GM}{c^2}, \tag{1}$$

$$\frac{S}{k_B} = \frac{A_{BH}}{4A_P}, \tag{2}$$

$$T_{BH} = \frac{hc}{8\pi^2 k_B R_s}. \tag{3}$$

Equation (1) defines the Schwarzschild radius in terms of the mass. Relation (2) establishes the entropy S of the system as $1/4$ of the horizon area $A_{BH} = 4\pi R_s^2$, in units of the Planck area $A_P = \hbar G/c^3$. Equation (3) states that the black hole radiates as a black body of the given temperature T_{BH}. This emission process implies a loss of energy for the system, which results in an evaporation rate for the black hole given by: $dM/dt \propto (\hbar c^4/G^2)M^{-2}$.

Notice that $dS/dM = (c^2 T)^{-1}$, fixing the internal energy of the black hole as $U = Mc^2$. Equation (1) has been about in speculative form since the 18th century, and was given a firm theoretical footing within the framework of general relativity during the 1930s. Equations (2)-(3) are the result of the vigorous development in black hole thermodynamics of the 1960s and 1970s by various authors, notably Bekenstein and Hawking (cf. [2] and references therein).

It has been suggested (*e.g.* [3-5] for a review) that for any physical system, Eq. (2) should hold always, with $\leq$ replacing the equality, which in turn should hold only in the black hole regime. This is termed the holographic principle, from the fact that the information content of an object would be limited not by its 3D volume, but by its 2D bounding surface. The interesting connection implied between quantum mechanics through A_P and gravity through the particle horizon, has raised the hope that the validity of the holographic principle would yield important clues regarding quantum theories of gravity. In this sense, even a heuristic study as to the possible origin of this principle should prove valuable.

We study the behaviour of a classical black body photon gas as it is compressed into a black hole, and propose a simple model for such a system using only photons confined to the Schwarzschild radius at their lowest possible momentum level. Two parameters determine the model, with a restriction only on the product of both of them. A formal statistical mechanical calculation is given through which these parameters are determined, leading to agreement with all black hole structural properties of Eqs. (1)-(3). The study of a self–gravitating photon sphere (a geon) was first introduced by Wheeler [6] in 1955. Considerable development in this area has taken place since then related to different physical properties of boson stars (see *e.g.* [7, 8]) and their stability.

Given recent proposals of a physical model for a black hole interior within the framework of loop quantum gravity [9], we analyse our model in this context. Taking the view that the Bekenstein entropy has a statistical mechanical origin in terms of counting states on the surface defined by the Schwarzschild radius of a black hole, canonical quantum gravity has yielded scenarios in which this entropy can be derived from first principles. We find no inconsistencies with the quantum gravity approach, which in fact allows us to explicitly and independently re-evaluate the parameters introduced in the quantum simple model, obtaining the same results.

2. Classical Limit

Most of the material in this section can be found elsewhere, *e.g.* [10]; it is reproduced here for context. For a photon gas having a black body spectrum the following well known relations define the total electromagnetic energy E_{EM} and entropy S in terms of the volume and temperature:

$$E_{EM} = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3}\frac{4\pi R^3}{3}, \tag{4}$$

$$\frac{S}{k_B} = \frac{4\pi^2}{45}\frac{(k_B T)^3}{(\hbar c)^3}\frac{4\pi R^3}{3}, \tag{5}$$

for a spherical region of radius R.

If we think of an ideal adiabatic wall enclosing this spherical region, we immediately obtain the well known scaling of $T \propto R^{-1}$, and we can eliminate $(k_B T)$ from Eq. (4) in favour of S and R to obtain

$$E = C\frac{\hbar c}{R}\left(\frac{S}{k_B}\right)^{4/3}, \tag{6}$$

where C is a numerical constant of order unity. If we think of the contraction as proceeding into the black hole regime, we can think of the radius as reaching R_s, which in this case would yield

$$R_s = \frac{2GE_{EM}}{c^4}. \tag{7}$$

To obtain Eq. (7) we have used Eq. (1), replacing M by E_{EM}/c^2. Substitution of E_{EM} from Eq. (6) into Eq. (7) leads to

$$\left(\frac{S}{k_B}\right)^{4/3} = C'\frac{A_{BH}}{A_p}, \tag{8}$$

where C' is a numerical factor of order unity. Two interesting conclusions are immediately evident from this last equation. Firstly, it is obvious that for any black body photon gas having $R_s > R_p$, a Schwarzschild radius larger than the Plank length, the holographic principle will be valid throughout the contraction process, as the horizon area will always be greater than A_{BH}. Second, that the classical equations for the diluted photon gas, Eqs. (4) and (5), cannot be valid in the black hole regime, since the required relationship for the entropy of such an object is Eq. (2) and the exponent of Eq. (8) is $4/3$ and not 1. The inconsistency of Eq. (8) with Eq. (2) indicates that physical processes which the system being modelled surely experiences, such as pair creation at high temperatures and quantum effects related to the dimensions of the system being comparable to the typical de Broglie wavelengths of the photons, are not been taken into account by the classical description of the photon gas through Eqs. (4) and (5). So far, we have assumed that the photon gas was being compressed by some external agency; however, if it is to form a self gravitating object, an equilibrium configuration should exist, and possibly a collapse beyond this. This point can be estimated by evaluating the Jeans length R_J of the problem for a speed of sound $v_s = c/\sqrt{3}$,

$$R_J = \frac{c}{(3G\rho)^{1/2}}. \tag{9}$$

In this context, the mass density ρ is equivalent to E_{EM}/V. Notice that Eq. (9) is derived directly from Einstein's equations, where the pressure term is directly $\mathrm{d}E/\mathrm{d}V$; hence no assumption of particle interactions is being made (see the appendix for details on this). From Eq. (9) we see that since ρ scales with T^4, R_J will scale with T^{-2}, which is interesting given that under adiabatic conditions the radius of the system will scale with T^{-1}. This means that gravitational instability will occur, i.e. $R > R_J$ above a certain critical temperature, below a certain critical equilibrium radius R_c. The situation becomes increasingly unstable in going towards smaller radii and larger temperatures. In general, for a fluid of mass-energy M, radius R and sound speed v_s,

$$R_J = v_s\left(\frac{4\pi R^3}{3GM}\right)^{1/2}, \tag{10}$$

which expressing M in terms of the Schwarzschild radius through Eq. (1) reads

$$R_J = \left(\frac{8\pi}{3}\right)^{1/2}\frac{R v_s}{c}\left(\frac{R}{R_s}\right)^{1/2}.$$

If we take the critical condition $R = R_J$, the previous relation gives

$$\frac{R_J}{R_s} = \left(\frac{3}{8\pi}\right)\left(\frac{c}{v_s}\right)^2. \tag{11}$$

Equation (11) shows that, since for all non–relativistic systems $v_s \ll c$, we should expect $R_J \gg R_s$. Indeed, for most astrophysical applications, the Jeans radius of a system is many orders of magnitude larger than the Schwarzschild radius. However, in going to a relativistic fluid, the condition $v_s \simeq c$ will apply, leading to $R_J \simeq R_s$. In other words, the self-gravitating regime will appear only close to the black hole regime. For the adiabatic photon gas we have studied, taking $v_s = c/\sqrt{3}$ and Eqs. (4), (5), and (9), in correspondence with Eq. (11) we obtain also $R_J \simeq R_s$. This last result shows that the self-gravitating regime for the photon gas we have studied does not appear until one is very close to the black hole regime, at scales where the analysis leading to Eq. (8) already showed that the structure equations for the gas (4) and (5) are no longer valid. In any case, the analysis following Eq. (9) together with Eq. (11), strongly suggests that any self-gravitating photon gas will be very close to catastrophic collapse and black hole formation.

The above results are in fact valid in the regime where the self-gravity of the radiation field is important, as shown by [10]; instability sets in for $R < 2R_s$, but equilibrium maximum entropy configuration exists above this radius, which however also show the scaling of Eq. (8), [*c.f.* their results following their Eq. (41)].

3. Quantum limit

The gravitational collapse and transition between the classical regime of Sec. 2 and a black hole will not be treated explicitly. Advances in that direction can be found for example in Refs. 10 to 12.

Being subject to the extreme gravitational regime of $R \to R_s$, it is reasonable to expect that the photons will be highly limited in momentum space. At this point we introduce as a hypothesis that all photons will have a wavelength $\lambda = \alpha R_s$, with α a numerical constant. We can evaluate the internal energy of the system as:

$$E_{EM} = \frac{Nhc}{\alpha R_s}, \tag{12}$$

where N is the total number of photons. Establishing a correspondence between E_{EM} and the internal energy of a black hole as $E_{EM} = Mc^2$, and using Eq. (1) to express M in terms of R_s, the above expression yields:

$$N = \frac{\alpha}{16\pi^2}\frac{A_{BH}}{A_p}. \tag{13}$$

If we think of a correspondence between the total entropy of the system and the total photon number given by $S/k_B = \beta N$, with β a proportionality constant expected to be of order unity, we find by comparison with Eq. (2) that the configuration we propose will satisfy all required black hole properties if the condition $\alpha\beta = 4\pi^2$ is satisfied.

We can compute β directly by calculating the entropy of the proposed system from first principles, through the thermodynamic potential Ω given by:

$$\Omega = k_B T \sum_k \ln\left(1 - e^{[\mu-\epsilon_k]/k_B T}\right), \tag{14}$$

where the summation is over quantum states, μ is the chemical potential, and ϵ_k is the energy of the $k-th$ state. Since the total number of components of the system is given by:

$$N = \sum_k \left(\frac{1}{e^{[\epsilon_k-\mu]/k_B T} - 1}\right), \tag{15}$$

and given that in the system proposed all photons have the same energy, we can write

$$N = \frac{N}{e^{[\epsilon-\mu]/k_B T} - 1}. \tag{16}$$

Note that each photon is assumed to be in a distinct detailed quantum level, and hence the analogy with a condensate is not complete. Now,

$$[\epsilon - \mu]/k_B T = \ln(2), \tag{17}$$

which when substituted back into Eq. (14) gives:

$$\Omega = -k_B T N \ln(2). \tag{18}$$

This last result now yields the entropy for the system through $S = -\partial\Omega/\partial(k_B T)|_{V,N}$ as $S = N\ln(2)$, providing a justification for the assumption of $S/k_B = \beta N$ made above, in terms of simple statistical physics arguments. We hence obtain $\beta = \ln(2)$ and $\alpha = 4\pi^2/\ln(2)$. Note that the previous results are of general validity for bosons; the case of photons is obtained with $\mu = 0$.

We also note that given the restriction of a fixed total internal energy, according to the hypothesis that this is to be the sum of the energies of N photons, the maximum entropy state will be the one with the most photons. In that case, all photons are at their lowest possible energy $\lambda = \alpha R_s$. Hence, the configuration proposed is also a maximum entropy state and is suggestive of a micro–physical origin for black hole entropy.

It has been argued [13] that the shortest scale that can enter into any physical theory is the Planck length. Although so far we have been working under the assumption of macroscopic black holes, we can extrapolate to the very small scales as follows. In the context of the ideas presented here a natural lower limit for the Schwarzschild radius of order the Plank length appears by setting $N = 1$ in Eq. (13), a single photon black hole. The energy e associated to this single photon is $e \approx \hbar c/(Gh/c^3)^{1/2} \approx 10^{28}$ eV.

Note that using Eq. (1) to substitute M for R_s in Eq. (13) gives the following quantisation for the mass M_N of a black hole in units of Planck mass:

$$\frac{M_N}{m_p} = \left(\frac{\pi}{\alpha}\right)^{1/2} N^{1/2}, \tag{19}$$

where $m_P = (\hbar c/G)^{1/2}$, a quantisation suggested already by Bekenstein's entropy equation (2). For sufficiently small black hole masses, this equation suggests a discrete spectrum associated with the transitions $N = 1 \to 0$, $N = 2 \to 1$, etc. The change in mass ΔM_N corresponding to a $\Delta N = 1$ transition is given by

$$\Delta M_N = M_N \left\{ \left[1 + \frac{\pi}{\alpha} \left(\frac{m_P}{M_N} \right)^2 \right]^{1/2} - 1 \right\}. \tag{20}$$

In the limit of a macroscopic black hole, where $M_N \gg m_p$, the above equation implies that

$$\frac{\Delta M_N}{m_P} = \frac{\pi}{2\alpha} \frac{m_P}{M_N}. \tag{21}$$

It is interesting to note that $c^2 \Delta M_N$ approximately corresponds to $k_B T_{BH}$ for a black hole mass M_N.

If we now identify the time scale Δt for the mass loss ΔM_N to take place with the limit of the Heisenberg uncertainty principle, we can set up $\Delta t \sim \hbar / c^2 \Delta M_N$. Under the above considerations, the mass evaporation rate for a black hole is

$$\frac{\Delta M_N}{\Delta t} \propto \frac{\hbar c^4}{G^2} \frac{1}{M_N^2}, \tag{22}$$

which is within a numerical constant of the standard evaporation rate for a black hole [2], seen here as the macroscopic limit of an intrinsically quantum process.

If the structure of the photon configuration we are describing is in any way related to quantum phenomena akin to Bose-Einstein condensation, we should expect the temperature to lie well below the critical temperature T_c for Bose-Einstein condensation. This, for relativistic particles, can be calculated in an analogous way to well-known Bose-Einstein critical temperatures for non-relativistic particles (*e.g.* [14]), by integrating the expression

$$dN = \frac{g V p^2 dp}{2\pi^2 \hbar^3 \left[e^{(\epsilon - \mu)/k_B T} \right]}, \tag{23}$$

for $\mu = 0$, $g = 2$ (photons) and in this case $\epsilon = pc$, giving

$$N = \frac{V (T_c k_B)^3}{\pi^2 \hbar^3 c^3} \int_0^\infty \frac{z^2 dz}{e^z - 1},$$

where $z = \epsilon/(k_B T)$. Evaluation of the above integral yields 2.202 and so, using $V = (4\pi/3) R_s^3$, leads to

$$N = \left(\frac{8.808}{3\pi} \right) \frac{R_s^3 (T_c k_B)^3}{(\hbar c)^3}. \tag{24}$$

If we now use the expression for N in Eq. (13), and writing T_c in units of T_{BH} for a black hole of radius equal to the Planck length R_p, which correspond to a temperature T_{BHp}, we get

$$\left(\frac{T_c}{T_{BHp}} \right)^3 = \left(\frac{6\pi^3}{1.101} \right) \frac{\alpha R_p}{R_s}, \tag{25}$$

which for $\alpha = 4\pi^2 / \ln(2)$, as determined through the statistical mechanical calculation shown above, yields,

$$\left(\frac{T_c}{T_{BHp}} \right) = 21.3 \left(\frac{R_p}{R_s} \right)^{1/3}. \tag{26}$$

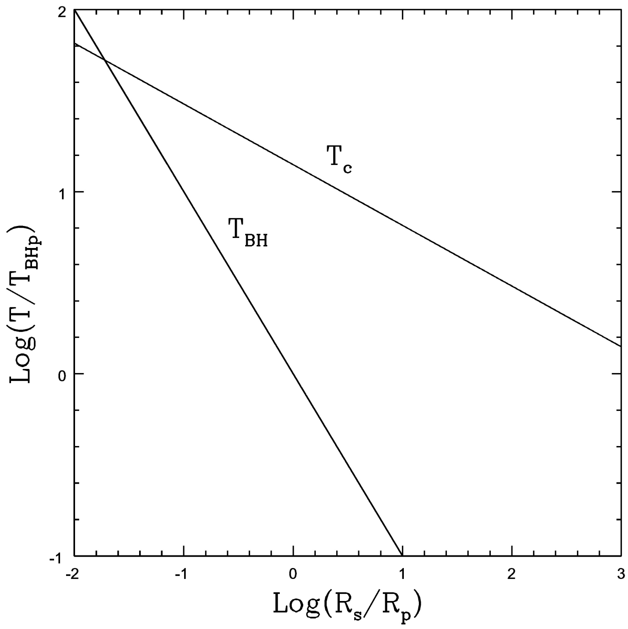

FIGURE 1. The solid line shows the black hole temperature, in units of this quantity for a black hole having a Schwarzschild radius equal to the Planck length, as a function of the Schwarzschild radius, in units of the Planck length. The dotted line gives the critical temperature for Bose-Einstein condensation of photons in a black hole, Eq. (26), in the same units as the solid curve, as a function of the same quantity.

As T_c scales with $R_s^{-1/3}$ and T_{BH} scales with R_s^{-1}, it is clear that for all black holes larger than a certain limit, the condition $T_{BH} \ll T_c$ will be satisfied. A comparison of both temperatures is shown in Fig. 1, as a function of R_s, from which we see that T_c is already over an order of magnitude greater than T_{BH} at $R_s = R_p$. Any realistic black hole will be at a temperature much lower than the critical temperature for Bose-Einstein condensation for photons, showing the internal consistency of the model.

The physics described so far is clearly highly idealised, however in a core collapse process within a massive star, as for the central region $R \to R_s$, the typical speeds and v_s of the constituent particles must necessarily tend to c, with de Broglie wavelengths not greater than the Schwarzschild radius.

At this point, quantum effects similar to Bose-Einstein condensation could take place, packing all (or most) photons into the lowest energy state. In this sense, typical wavelengths of the order of the Schwarzschild radius would be expected, as longer wavelengths would be prohibited by the containment imposed by gravity, and shorter wavelengths would imply an expansion in momentum space. In this sense, the identification we have maintained of the constituent particles as photons is shown to be largely arbitrary, and any relativistic bosons will yield essentially identical conclusions.

4. Loop quantum gravity approach

We now explore the ideas of the previous sections within the framework of loop gravity. In particular, we shall see that this allows us an independent re-evaluation of the constants α and β established in the last section, with the aid of some general results from loop quantum gravity.

The loop quantisation of $3+1$ general relativity is described in terms of a set of spin network states which span the Hilbert space on which the theory is based. These spin network states are labelled by closed abstract graphs with spins assigned to each link and intertwining operators assigned to each vertex.

A recent result that follows from the theory is that if a surface Σ is intersected by a link ℓ_i of a spin network carrying the label j_i, it acquires an area [15, 16]

$$A_\Sigma(j_i) = 8\pi A_p \gamma \sqrt{j_i(j_i+1)}, \tag{27}$$

where γ is the Immirzi parameter.

Let us now consider for our purposes that the horizon Σ is intersected by a large number N_ℓ of links. Each intersection with Σ represents a puncture. In the limit of large N_ℓ, one can say that each puncture is equipped with an internal space H_j (the space of all flat $U(1)$ connections on the punctured sphere) of dimension [17]

$$\dim H_j = 2j+1. \tag{28}$$

Each puncture of an edge with spin j increases the dimension of the boundary Hilbert space by a factor of $2j+1$. Under these considerations, it follows that the entropy can be calculated by

$$S(j_p) = \ln\left(\prod_p \dim H_{j_p}\right). \tag{29}$$

Statistically, the most important contribution comes from those configurations in which the lowest possible spin j_{min} dominates, so we can write the entropy (29) as

$$S(j_{min}) = N_\ell \ln(2j_{min}+1), \tag{30}$$

where N_ℓ is given by

$$N_\ell = \frac{A_{BH}}{A_\Sigma(j_{min})}. \tag{31}$$

Due to the fact that the assumed gauge group of loop quantum gravity is SU(2), then it follows that $j_{\mathrm{min}} = 1/2$, and so the Immirzi parameter is given by [16]

$$\gamma = \frac{\ln 2}{\pi\sqrt{3}}, \tag{32}$$

and (31) becomes

$$N_\ell = \frac{1}{4\ln 2}\frac{A_{BH}}{A_p}. \tag{33}$$

The number of links N_ℓ associated with the particles we are dealing with in this article must be proportional to the number of particles N. The simplest possible configuration is the one in which the proportionality factor is equal to unity, and so

$$N_\ell = N. \tag{34}$$

Using this relation and Eqs. (13) and (33), we can evaluate α to obtain

$$\alpha = \frac{4\pi^2}{\ln 2}. \tag{35}$$

The parameter β previously defined through the relation $S = \beta N$ can now be re-derived independently through Eq. (30) to yield:

$$\beta = \frac{N_\ell}{N}\ln(2j+1)_{j=1/2} = \ln 2. \tag{36}$$

From (35) and (36) we can see that the product $\alpha\beta = 4\pi^2$, as required by the considerations on Sec. 3. It is interesting that the model proposed in the previous sections is seen not to be in conflict with a loop gravity approach, which indeed allows us to re-evaluate the scaling parameters introduced earlier independently, and in accordance with the expectations of the physics discussed above.

5. Conclusions

A photon gas contained within an adiabatic enclosure will satisfy the holographic principle, at least until just before reaching the black hole regime, which approximately coincides with the self-gravitating condition and where the classical description is no longer valid.

A configuration where all photons have the same energy within $R = R_S$ can be constructed to satisfy all black hole conditions.

This configuration gives rise to a discrete evaporation spectrum for R_S close to the Planck length, and in the macroscopic limit permits a re-derivation of the standard black hole evaporation rate.

Our results are consistent with the loop quantum gravity scheme, satisfying the constraints required to be in agreement with the Bekenstein-Hawking entropy for a black hole.

A comparison of both regimes studied is highly suggestive of a heuristic proof of the holographic principle, as any real system would require an increase in entropy (at fixed energy and volume) to be turned into the photon gas we have studied.

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A Relativistic Jeans criterion for gravitational instability

In order to show that the Jeans gravitational instability limit is valid in the general relativistic regime, let us proceed as follows. The condition of hydrostatic equilibrium in general relativistic fluid mechanics is obtained by using the fact that the field is static. This means that one can describe the problem in a frame of reference in which the fluid is at rest, with all hydrodynamical quantities independent of time. This also implies that the mixed space and time components of the metric tensor are null. Under these assumptions, the equation of hydrostatic equilibrium is then given by [18]

$$\frac{1}{w}\frac{\partial p}{\partial r} = -\frac{1}{2}\frac{\partial}{\partial r}\log g_{00}, \tag{A.1}$$

where g_{00} is the time component of the metric tensor, $w{=}e{+}p$ is the enthalpy per unit volume, e the internal energy density and p the pressure. Oppenheimer & Volkoff [19] showed that Eq. (A.1) can lead to the form [20]

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{(e+p)}{r\left(r-\frac{2GM(r)}{c^2}\right)}\left\{\frac{GM(r)}{c^2}+\frac{4\pi}{c^2}Gr^3\rho\right\}, \tag{A.2}$$

where the mass-energy $M(r)$ within a radius r is given by

$$M(r) = 4\pi\int_0^r \rho\, r^2\,\mathrm{d}r, \tag{A.3}$$

and $\rho(r) := e/c^2$ is the mass–energy density of the fluid. Note that for the case of relativistic and non–relativistic dust particles, $M(r)$ represents the mass of particles within radius r. For the case of a photon gas, $M(r)$ is the mass corresponding to the internal energy.

We now assume that the plasma obeys a Bondi-Wheeler equation of state

$$p = (\kappa-1)\,e, \tag{A.4}$$

with constant index κ. This means that the sound velocity v_s is given by the relation $v_\mathrm{s}^2 = c^2(\kappa-1)$, and so, the left hand side of Eq. (A.2) can be written as $\left(v_\mathrm{s}^2/c^2\right)\,\mathrm{d}e/\mathrm{d}r$. Seen in this way, the left hand side of Eq. (A.2) no longer represents gradients of pressure which are in balance with self–gravitational forces related to the plasma. Indeed, the balance with the gravitational forces produced by the plasma is now related to the gradients of its proper internal energy density e by

$$(\kappa-1)\frac{\mathrm{d}e}{\mathrm{d}r} = -\frac{ke}{r\left(r-\frac{2GM(r)}{c^2}\right)}\left\{\frac{GM(r)}{c^2}+\frac{4\pi}{c^2}Gr^3\rho\right\}. \tag{A.5}$$

Let us now take the absolute value on both sides of Eq. (A.4), to the order of magnitude $\mathrm{d}e/\mathrm{d}r \approx e/r$ and $M(r) \approx (4/3)\,\pi\, r^3\rho$. A gravitational instability occurs when the absolute value of the left hand side of Eq. (A.2) [or equivalently Eq. (A.5)] is less than the absolute value of its right hand side. Using all the above statements, it follows that this instability occurs when the radial coordinate r is such that

$$r \gtrsim \sqrt{\frac{3}{8\pi\,(3\kappa-1)}}\,\frac{v_\mathrm{s}}{\sqrt{G\rho}} := \Lambda_\mathrm{J}. \tag{A.6}$$

The quantity Λ_J on the right hand side of Eq. (A.6) is of the same order of magnitude as the standard Jeans length used in non-relativistic fluid dynamics. In other words, the criterion (A.6) means that the Jeans criterion for gravitational collapse is also valid in the relativistic regime as well.

For the particular case studied in this article, the constant $\kappa = 4/3$ for a photon gas and the Jeans criterion can be applied to it, giving $r \gtrsim \left(1/2\sqrt{2\pi}\right)v_\mathrm{s}/\sqrt{G\rho}$.

We see than the Jeans criterion can be generalised beyond an equilibrium between gas pressure and rest–mass self–gravity, to a very general equilibrium between energy–momentum flux and total self–gravity.

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