

# Obtaining time-dependent invariants by the Sarlet-Bahar method for a non-linear equation

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Recibido el 4 de julio de 2006; aceptado el 19 de diciembre de 2006

Applying the Sarlet-Bahar method one obtains the invariant of equations of motion of the type  $\ddot{\rho} + \omega^2(t)\rho/2 = \alpha(t)F(\beta(t)\rho)$ . The corresponding auxiliary equation for the Ermakov system is also obtained, and the results obtained by other authors are generalized.

*Keywords:* Ermakov systems; Noether symmetries; Ermakov invariants; constants of motion.

Aplicando el método de Sarlet-Bahar, se obtiene el invariante para ecuaciones de movimiento del tipo  $\ddot{\rho} + \omega^2(t)\rho/2 = \alpha(t)F(\beta(t)\rho)$ . También se obtiene la ecuación auxiliar correspondiente al sistema de Ermakov y se generalizan los resultados obtenidos por otros autores.

*Descriptores:* Sistemas de Ermakov; simetrías de Noether; invariantes de Ermakov; constantes de movimiento.

PACS: 02.90.+p

## 1. Introduction

Finding a constant of motion is very important when one studies dynamical systems. To achieve this goal, Noether's theorem is a helpful formalism because it provides links between the symmetries of the physical laws inherent to the action functional and the corresponding constants of motion of the system. The importance of an integral of motion lies in the fact that it reduces the order of the equation of motion and provides information about the underlying physical system; of course, the existence of a second integral of motion would imply the complete integration of the differential equations which describe the physical system [1]. Sometimes, the Lagrangian does not possess any obvious symmetry but through Noether's theorem one can force the Lagrangian to admit a symmetry transformation [2]. Otherwise, applying Noether's theorem to a Lagrangian one can find the corresponding Ermakov system, which is generally defined as a system of two coupled differential equations (one of which is the equation of motion and the other one is known as the auxiliary equation) and an invariant, Ray *et al.* [3, 4]. In his treatment, Ermakov assumed that the frequency  $\omega(t)$  is a function of time only, and left open the problem of how to find the auxiliary equation [5]. With regard to this question, Sarlet *et al.* [6] presented a simple direct method for the construction of first integrals for second order Newtonian systems. Their method consists in trying to construct a first integral by multiplying the equation of motion by an appropriate integrating factor. Additionally, this procedure provides a beautiful mechanism for finding the auxiliary equation as well. In the present paper, we show an application of the Sarlet-Bahar method to a non-linear equation of motion and generalize the results obtained by Ray *et al.* [7].

The outline of the paper is as follows: in Sec. 2, using the Sarlet-Bahar method, the auxiliary equation and the invariant

are obtained. For that, we follow the Ray *et al.* procedure described in paper [7]. In Sec. 3 the results are discussed, and we point out our conclusions.

## 2. The Sarlet-Bahar Method

To illustrate the Sarlet-Bahar method we begin with the equation of motion

$$\ddot{\rho} + \frac{1}{2}\omega^2(t)\rho = \alpha(t)F(\beta(t)\rho), \quad (1)$$

where  $\omega^2(t)$  is an arbitrary function of  $t$ ,  $\alpha$ ,  $\beta$  and  $F$  are arbitrary functions of their arguments. This type of non-homogeneous and non-linear equation describes, for example, the behavior of forced oscillations with time dependent amplitude of the driving force, or parametrically excited systems.

The procedure consists in looking for an expression for each term of Eq. (1) as a total differential by adding and subtracting terms. According to this, we multiply Eq. (1) by  $\gamma\dot{\rho}$ , with  $\gamma$  an arbitrary function of time, and after some algebra one finds the following equation:

$$\begin{aligned} \frac{d}{dt} \left[ \frac{\gamma\dot{\rho}^2}{2} + \frac{\lambda\rho^2}{2} - \frac{\alpha\gamma}{\beta} \int^{\beta\rho} F(\eta)d\eta + \phi(\kappa\rho) \right] \\ - \frac{\dot{\gamma}\dot{\rho}^2}{2} + \frac{1}{2}\gamma\omega^2\rho\dot{\rho} - \lambda\rho\dot{\rho} - \frac{\dot{\lambda}\rho^2}{2} + \frac{\gamma\alpha\dot{\beta}\rho F(\beta\rho)}{\beta} \\ - \frac{1}{\xi} \frac{d}{d\xi} \left[ \kappa\dot{\rho}^2 + \kappa^2\rho\dot{\rho} \right] \\ + \left[ \frac{\dot{\gamma}\alpha}{\beta} + \frac{\gamma\dot{\alpha}}{\beta} - \frac{\alpha\gamma\dot{\beta}}{\beta^2} \right] \int^{\beta\rho} F(\eta)d\eta = 0, \quad (2) \end{aligned}$$

where  $\lambda(t)$ ,  $\kappa(t)$  and  $\phi(\varkappa\rho)$  are arbitrary functions, and  $\xi = \varkappa(t)\rho$ . These functions will help to write Eq. (1) as a total differential, and the goal of this method is to determine them.

A first step in requiring Eq. (2) to have an exact differential is that

$$\left[ \frac{\dot{\gamma}\alpha}{\beta} + \frac{\gamma\dot{\alpha}}{\beta} - \frac{\alpha\gamma\dot{\beta}}{\beta^2} \right] = 0,$$

for which it is necessary that

$$\beta = \alpha(t)\gamma(t). \quad (3)$$

Using this result in Eq. (2), it transforms into

$$\begin{aligned} & \frac{d}{dt} \left[ \frac{\gamma\rho^2}{2} + \frac{\lambda\rho^2}{2} - \int^{\beta\rho} F(\eta) d\eta + \phi(\kappa\rho) + a\rho\dot{\rho} \right] \\ & - \left[ \frac{\dot{\gamma}}{2} + a \right] \rho^2 + \left[ -\dot{a} + \frac{1}{2}\gamma\omega^2 - \lambda - \frac{\kappa^2}{\xi} \frac{d\phi}{d\xi} \right] \rho\dot{\rho} \\ & + \left[ \frac{\gamma\dot{\beta}}{\beta} - a \right] \rho\ddot{\rho} + \left[ -\frac{\dot{\lambda}}{2} + \frac{\gamma\dot{\beta}\omega^2}{\beta} - \frac{\kappa\dot{\kappa}}{\xi} \frac{d\phi}{d\xi} \right] \rho^2 = 0, \quad (4) \end{aligned}$$

where the equation of motion (1) and the identity

$$\frac{d}{dt} (a\rho\dot{\rho}) - \dot{a}\rho\dot{\rho} - a\dot{\rho}^2 - a\rho\ddot{\rho} = 0 \quad (5)$$

were taken into account. Equating the coefficients of  $\dot{\rho}^2$ ,  $\rho\dot{\rho}$ ,  $\rho\ddot{\rho}$ ,  $\rho^2$  with zero in Eq. (4), one obtains

$$a = -\frac{\dot{\gamma}}{2} \quad (6)$$

$$\dot{a} = \frac{1}{2}\gamma\omega^2 - \lambda - \frac{\kappa^2}{\xi} \frac{d\phi}{d\xi} \quad (7)$$

$$a = \frac{\gamma\dot{\beta}}{\beta} \quad (8)$$

$$\dot{\lambda} = \frac{\gamma\dot{\beta}\omega^2}{\beta} - \frac{2\kappa\dot{\kappa}}{\xi} \frac{d\phi}{d\xi}. \quad (9)$$

Equations (6) and (8) yield

$$\beta = \frac{1}{\sqrt{\gamma}} \quad (10)$$

whereas Eqs. (6) and (7) lead to

$$\lambda = \frac{\dot{\gamma}}{2} + \frac{1}{2}\gamma\omega^2 - \frac{\kappa^2}{\xi} \frac{d\phi}{d\xi} \quad (11)$$

and, with the aid of Eqs. (9) and (11), one obtains

$$\ddot{\gamma} + 2\dot{\gamma}\omega^2 + 2\gamma\omega\dot{\omega} = 2\kappa^2 \frac{d}{dt} \left( \frac{1}{\xi} \frac{d\phi}{d\xi} \right). \quad (12)$$

Multiplying the last equation by  $\gamma$ , it can be rewritten as

$$\frac{d}{dt} \left[ \gamma\ddot{\gamma} - \frac{1}{2}\dot{\gamma}^2 + \gamma^2\omega^2 \right] = 2\gamma\kappa^2 \frac{d}{dt} \left( \frac{1}{\xi} \frac{d\phi}{d\xi} \right). \quad (13)$$

Choosing

$$\kappa = \frac{1}{\sqrt{\gamma}} \quad \text{and} \quad \gamma = x^2, \quad (14)$$

Eq. (13) becomes

$$\ddot{x} + \omega^2(t)x = \frac{1}{x^2} \frac{d\phi}{d\xi}, \quad (15)$$

which plays the role of the auxiliary equation in the Ermakov formalism. Through the transformation given by Eq. (14), the expressions for  $\alpha$ ,  $\beta$ ,  $\lambda$ , and  $a$  take the form

$$\alpha = \frac{1}{x^3} \quad (16)$$

$$\beta = \frac{1}{x} \quad (17)$$

$$\lambda = \dot{x}^2 \quad (18)$$

$$a = -x\dot{x} \quad (19)$$

and by substituting  $\alpha$ ,  $\beta$ ,  $\lambda$  and  $a$  [Eqs. (16)-(19)] back into the first term of Eq. (4), we obtain the invariant

$$\mathcal{I} = \frac{1}{2} [x\dot{\rho} - \dot{x}\rho]^2 + \int^{\frac{\rho}{x}} f(\eta) d\eta + \varphi\left(\frac{\rho}{x}\right), \quad (20)$$

where  $\varphi\left(\frac{\rho}{x}\right)$  is an arbitrary function of its argument.

### 3. Application of the Sarlet-Bahar Method

In order to verify that the method proposed in this paper works, it will be applied to a well-known problem, namely the forced harmonic oscillator whose equation of motion can be written as

$$\ddot{x} + \Omega^2(t)x = \frac{1}{m} F(t), \quad (21)$$

where  $m$  is the mass attached to the spring,  $\Omega(t)$  the oscillation frequency and  $F(t)$  the external force.

By comparing Eq. (21) with Eq. (1), one notes that the external force  $F(t)$  depends explicitly on time and that  $\alpha(t) = 1/m$  is a constant.

Following the method proposed in Sec. 2, one can find the auxiliary equation

$$\ddot{\rho} + \Omega^2(t)\rho = \frac{1}{\rho^2} \frac{d\phi}{d\xi} \quad (22)$$

and the invariant

$$\begin{aligned} \mathcal{I} = \rho^2 \left[ \frac{\dot{x}^2}{2} + \frac{1}{2} \left( \frac{\dot{\rho}}{\rho} \right)^2 x^2 - \left( \frac{\dot{\rho}}{\rho} \right) x\dot{x} \right] \\ + \int^{\frac{x}{\rho}} F(\eta) d\eta + \phi\left(\frac{x}{\rho}\right). \quad (23) \end{aligned}$$

The invariant contains the kinetic energy, the potential energy and the momentum, all in mass units. In this expression, the potential energy and the momentum appear modified by

an expansion rate. By setting  $\phi(x/\rho) = x^2/\rho^2$ , one reproduces the results obtained by Ray *et al.* and Lutzky, and this term appears as an additional potential energy. The term

$$\int^{\frac{x}{\rho}} F(\eta) d\eta$$

provides the contribution of the external force  $F(t)$  to the invariant.

#### 4. Conclusions

In this paper, we have started from the non-linear equation of motion

$$\ddot{\rho} + \frac{1}{2}\omega^2(t)\rho = \alpha(t)F(\beta(t)\rho) \quad (24)$$

and, applying the Sarlet-Bahar formalism to this equation of motion, one has determined the form of the functions  $\alpha(t)$  and  $\beta(t)$  in such a way that Eq. (24) admits a time-dependent invariant. This method leads to an auxiliary Eq. (15) and the

invariant given by Eq. (20). These results generalize those obtained by Ray *et al.* [7] for conservative systems. Besides, we reproduce their results when the arbitrary function  $\varphi(\rho/x)$  equals  $k/2(\rho/x)^2$ . Applications of the formalism presented in this paper to other equations of motion and detailed comparison with other methods, in order to generate time-dependent invariants, are topics of further study.

To elucidate the physical meaning of the invariant, the method was applied to the equation of motion of a forced harmonic oscillator. It is found that the invariant contains terms that have to do with the kinetic energy, the potential energy, the momentum, and a contribution due to the external force.

#### Acknowledgments

We wish to thank Dr. A.L. Gelover-Santiago for valuable discussions and for reading and suggesting improvements to the manuscript. M.G. Corona-Galindo wishes to thank the DAAD and CONACyT for financial support.

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