The Maxwell equations in a uniformly accelerated frame

G.F. Torres del Castillo
Departamento de Física Matemática, Instituto de Ciencias,
Universidad Autónoma de Puebla, 72570 Puebla, Pue., México.

C.I. Pérez Sánchez
Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla,
Apartado postal 1152, 72001 Puebla, Pue., México.

Recibido el 11 de julio de 2006; aceptado el 16 de enero de 2007

The solution of the source-free Maxwell equations in a uniformly accelerated frame of reference is expressed in terms of a single complex scalar potential that obeys a second-order equation. The field of a static electric charge is obtained as an example of a stationary axisymmetric field.

Keywords: Special relativity; Maxwell's equations; accelerated charge.

La solución de las ecuaciones de Maxwell sin fuentes en un sistema de referencia uniformemente acelerado se expresa en términos de un solo potencial escalar complejo que obedece una ecuación de segundo orden. El campo de una carga eléctrica estática se obtiene como ejemplo de un campo estacionario axialmente simétrico.

Descriptores: Relatividad especial; ecuaciones de Maxwell; carga acelerada.

PACS: 03.30.+p; 03.50.De; 02.30.Gp

1. Introduction

In a recent paper [1], some kinematic effects produced by the uniform acceleration of a reference frame on the propagation of light were rigorously studied employing special relativity (see also Ref. 2). These kinematic effects include the redshift and the bending of light rays, which are analogous to the effects that, according to the general theory of relativity, must be present in a real gravitational field. In such derivations, it is assumed that light propagates along light rays, which are null geodesics in space-time.

In this paper we continue investigating the effects produced by a uniform acceleration, considering the source-free Maxwell equations. We want to be able to find, exactly, what influence that acceleration can have on the electromagnetic field. According to the principle of equivalence, an analogous behavior can be expected, at least locally, in a real gravitational field. This idea has been applied in determining whether a static electric charge in a gravitational field should radiate (see Refs. 3-9 and the references cited therein). In this paper the source-free Maxwell equations in a uniformly accelerated reference frame are solved assuming that the electromagnetic field is stationary and axially symmetric. In the general case, without symmetries, the complete solution of the Maxwell equations is expressed in terms of a single scalar potential.

In Sec. 2, we briefly summarize the relevant results of Ref. 1 (see also Refs. 2,4,5,10-13) and the source-free Maxwell equations are written down in useful explicit forms. Considering an exact solution that represents monochromatic plane waves traveling in a direction parallel to the acceleration of the frame, the redshift formula is derived. In Sec. 3, the stationary, axially symmetric solutions of the Maxwell equations are obtained and, as an example, the field of a static electric point charge is explicitly calculated; our result coincides with the field of a uniformly accelerated charge given in Ref. 4. In Sec. 4 we give the general solution to the source-free Maxwell equations, in a form adapted to the accelerated frame, in terms of a single complex potential that must obey a linear second-order partial differential equation.

2. Electromagnetic fields in a uniformly accelerated frame

As shown in Ref. 1, if the origin of a reference frame S′ has a constant proper acceleration g (which means that the acceleration of the origin of S′, with respect to an inertial frame that instantaneously accompanies S′, is equal to g), the relation between the Cartesian coordinates of a suitably chosen inertial frame S and those of S′ are given by

$$\begin{aligned} ct &= (z' + c^2/g)\sinh(gt'/c), \\ x &= x', \\ y &= y', \\ z &= (z' + c^2/g)\cosh(gt'/c), \end{aligned} \tag{1}$$

if the coordinate axes are parallel and S′ is accelerated along the z-axis (see also Refs. 2,4,5,10-13).

The coordinates (ct, x, y, z), measured in the inertial frame S, have the usual well-known meaning encountered in the elementary treatment of special relativity, where it is possible to synchronize the clocks stationary in S. The coordinates (x', y', z') of an event P are determined by the distances from the origin, O, of S′ in the usual manner, and t' is

the value that the proper time of an observer at O has simultaneously (with respect to this observer) with the occurrence of P. These facts can also be derived from the expression

$$\begin{aligned} \mathrm{d}s^2 &= -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= -(1+gz'/c^2)^2 c^2 \mathrm{d}t'^2 + \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2 \end{aligned} \quad (2)$$

that follows from Eqs. (1). (The metric (2) defines what is sometimes called Rindler space-time; though, of course, it actually corresponds to Minkowski space-time.)

In order to simplify the notation, in what follows the coordinates associated with the uniformly accelerated frame will be denoted by symbols without primes; thus, the (flat) space-time metric (2) will read

$$\mathrm{d}s^2 = -\left(1+\frac{gz}{c^2}\right)^2 c^2\mathrm{d}t^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2 + \mathrm{d}z^2, \quad (3)$$

in terms of circular cylindrical coordinates $(x^0, x^1, x^2, x^3) = (ct, \rho, \phi, z)$.

The Maxwell equations for the source-free electromagnetic field on a possibly curved space-time can be written as

$$\begin{aligned} \partial_\alpha f_{\beta\gamma} + \partial_\gamma f_{\alpha\beta} + \partial_\beta f_{\gamma\alpha} &= 0, \\ \frac{1}{\sqrt{|\tilde{g}|}}\partial_\alpha(\sqrt{|\tilde{g}|}\, f^{\alpha\beta}) &= 0, \end{aligned} \quad (4)$$

where $f^{\alpha\beta}$ denotes the contravariant components of the electromagnetic field tensor, $\partial_\alpha \equiv \partial/\partial x^\alpha$, the x^α are space-time coordinates, $\tilde{g} \equiv \det(g_{\alpha\beta})$, with $g_{\alpha\beta}$ being the components of the metric tensor in the coordinate system x^α, $f_{\alpha\beta} = g_{\alpha\gamma}g_{\beta\delta}f^{\gamma\delta}$ and the Greek lower case indices run from 0 to 3 (see, for example, Ref. 14). With the metric given by Eq. (3), we have

$$(g_{\alpha\beta}) = \mathrm{diag}\big(-h^2(z), 1, \rho^2, 1\big), \quad (5)$$

with

$$h(z) \equiv 1 + \frac{gz}{c^2}, \quad (6)$$

and $\sqrt{|\tilde{g}|} = \rho h(z)$.

Making use of Eq. (5) and the definitions

$$\begin{aligned} F_0 &= hf^{03} + \mathrm{i}\rho f^{12}, \\ F_{\pm1} &= hf^{01} + \mathrm{i}\rho f^{23} \pm \mathrm{i}(\rho h f^{02} + \mathrm{i}f^{31}), \end{aligned} \quad (7)$$

a straightforward computation shows that the Maxwell equations (4) are equivalent to

$$\begin{aligned} &\left(\frac{1}{h}\partial_0 + \partial_z\right)F_0 + \frac{1}{\rho}\left(\partial_\rho - \frac{\mathrm{i}}{\rho}\partial_\phi\right)(\rho F_{+1}) = 0, \\ &\left(\frac{1}{h}\partial_0 - \partial_z\right)F_0 - \frac{1}{\rho}\left(\partial_\rho + \frac{\mathrm{i}}{\rho}\partial_\phi\right)(\rho F_{-1}) = 0, \\ &\left(\frac{1}{h}\partial_0 + \partial_z\right)(hF_{-1}) - h\left(\partial_\rho - \frac{\mathrm{i}}{\rho}\partial_\phi\right)F_0 = 0, \\ &\left(\frac{1}{h}\partial_0 - \partial_z\right)(hF_{+1}) + h\left(\partial_\rho + \frac{\mathrm{i}}{\rho}\partial_\phi\right)F_0 = 0. \end{aligned} \quad (8)$$

The combinations F_0 and $F_{\pm1}$, defined by Eqs. (7), also amount to

$$\begin{aligned} F_0 &= E_z + \mathrm{i}B_z, \\ F_{\pm1} &= E_\rho + \mathrm{i}B_\rho \pm \mathrm{i}(E_\phi + \mathrm{i}B_\phi) \\ &= (E_\rho \pm \mathrm{i}E_\phi) + \mathrm{i}(B_\rho \pm \mathrm{i}B_\phi), \end{aligned} \quad (9)$$

where E_ρ, E_ϕ, and E_z are the components of the electric field with respect to the *orthonormal* basis $\{\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z\}$ induced by the circular cylindrical coordinates ρ, ϕ, z, with an analogous meaning for B_ρ, B_ϕ, and B_z (the factors h and ρ contained in Eqs. (7) are just scale factors).

From Eqs. (8) and (9) one finds that, in vector form, the source-free Maxwell equations are

$$\begin{aligned} \nabla\cdot\mathbf{B} &= 0, \qquad \nabla\times(h\mathbf{E}) = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \\ \nabla\cdot\mathbf{E} &= 0, \qquad \nabla\times(h\mathbf{B}) = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \end{aligned}$$

where the divergence and curl are the usual operators associated with a flat Euclidean three-dimensional space.

2.1. Redshift

Some particular solutions of the Maxwell equations can be readily obtained from the various expressions given above. For instance, an electromagnetic field with $F_{+1} = 0 = F_0$ corresponds to electromagnetic radiation, and from Eqs. (8) one finds that any of these solutions is of the form

$$\rho h F_{-1} = G(\rho\mathrm{e}^{\mathrm{i}\phi}, (z + c^2/g)\mathrm{e}^{-gt/c}), \quad (10)$$

where G is an *arbitrary* (complex-valued) function of two variables. (A field with $F_{+1} = 0 = F_0$ is *algebraically special*, which means that both Lorentz invariants of the electromagnetic field, $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$, are equal to zero.)

Thus, the field given by

$$\rho h F_{-1} = \rho\mathrm{e}^{\mathrm{i}\phi}\cos\left[\omega\left(t - \frac{c}{g}\ln(z + c^2/g)\right)\right], \quad (11)$$

where ω is a constant (with $F_{+1} = 0 = F_0$), is a solution of the Maxwell equations, which corresponds to a monochromatic plane wave propagating along the z-axis. For a fixed value of z, Eq. (11) represents an oscillating field; however, for $z \neq 0$, the angular frequency of this field differs from ω, since t is only the time measured by a clock at $z = 0$. According to Eq. (3), the time, τ, measured by a stationary clock at $z = z_0$, is related to t by $\tau = h(z_0)t$; this means that the angular frequency of the wave at $z = z_0$ is

$$\omega' = \frac{\omega}{h(z_0)} = \frac{\omega}{1 + gz_0/c^2},$$

which agrees with the redshift formula obtained in Ref. 1 (see also the references cited therein). Equation (11) yields two additional things. The presence of the factor h on the left-hand side of Eq. (11) means that the amplitude of this plane

wave diminishes as z increases (though the fractional change of the amplitude per unit of length is g/c^2) and, owing to the presence of the logarithm, the field given by Eq. (11) *does not* have a well-defined wavelength.

3. Static axisymmetric fields

In this section we show that the solutions to the source-free Maxwell equations that, in the uniformly accelerated frame, are static and invariant under rotations about the direction of the acceleration possess certain additional symmetries. Assuming that the cylindrical components of the electromagnetic field depend on ρ and z only, Eqs. (8) reduce to

$$\begin{aligned} \partial_z F_0 + \frac{1}{\rho}\partial_\rho(\rho F_{\pm 1}) &= 0, \\ \frac{1}{h}\partial_z(hF_{\pm 1}) - \partial_\rho F_0 &= 0, \end{aligned} \tag{12}$$

and by combining these equations we obtain

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho F_0) = -\frac{1}{h}\partial_z(h\partial_z F_0). \tag{13}$$

(Similarly, one can derive the decoupled equation for $F_{\pm 1}$

$$\partial_\rho\left[\frac{1}{\rho}\partial_\rho(\rho F_{\pm 1})\right] = -\partial_z\left[\frac{1}{h}\partial_z(hF_{\pm 1})\right].)$$

Looking for separable solutions to Eq. (13) of the form

$$F_0(\rho, z) = A(\rho)B(z),$$

we obtain the ordinary differential equation

$$\frac{\mathrm{d}^2A}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}A}{\mathrm{d}\rho} + \alpha^2 A = 0$$

(where α is a constant), whose solutions are linear combinations of order zero Bessel functions, $J_0(\alpha\rho)$ and $N_0(\alpha\rho)$, and, using $\tilde{z} \equiv z + c^2/g$ as the independent variable in place of z,

$$\frac{\mathrm{d}^2B}{\mathrm{d}\tilde{z}^2} + \frac{1}{\tilde{z}}\frac{\mathrm{d}B}{\mathrm{d}\tilde{z}} - \alpha^2 B = 0,$$

whose solutions are linear combinations of order zero modified Bessel functions, $I_0(\alpha\tilde{z})$ and $K_0(\alpha\tilde{z})$. When the separation constant α is equal to zero, A is a linear combination of 1 and $\ln\rho$ and, similarly, B is a linear combination of 1 and $\ln\tilde{z}$. Thus, the static axisymmetric solutions of the source-free Maxwell equations that are regular for $\tilde{z} > 0$ are given by

$$F_0 = \int_0^\infty f(\alpha)J_0(\alpha\rho)K_0(\alpha\tilde{z})\,\mathrm{d}\alpha, \tag{14}$$

where f is an arbitrary function. Making use of Eqs. (12) and the recurrence relations for the Bessel functions one finds that the remaining field components are given by

$$F_{\pm 1} = \int_0^\infty f(\alpha)J_1(\alpha\rho)K_1(\alpha\tilde{z})\,\mathrm{d}\alpha. \tag{15}$$

Despite the symmetry of Eq. (13), a much more convenient and useful set of expressions is obtained employing in place of ρ and $\tilde{z}$, the variables u, v defined by

$$u \equiv \tfrac{1}{2}(\rho^2 - \tilde{z}^2), \qquad v \equiv \rho\tilde{z} \tag{16}$$

(these definitions are identical to those of the parabolic coordinates, considering ρ and $\tilde{z}$ as Cartesian coordinates on the plane). Indeed, making use of the chain rule, a straightforward computation shows that Eqs. (12) are equivalent to

$$\partial_v F_0 + \partial_u F_{\pm 1} = 0, \qquad \partial_u F_0 - \frac{1}{v}\partial_v(vF_{\pm 1}) = 0 \tag{17}$$

and by combining these equations, one obtains the second-order decoupled equations

$$\partial_u^2 F_0 + \frac{1}{v}\partial_v(v\partial_v F_0) = 0, \tag{18}$$

and

$$\partial_u^2 F_{\pm 1} + \partial_v\left[\frac{1}{v}\partial_v(vF_{\pm 1})\right] = 0. \tag{19}$$

It is readily seen that Eqs. (18) and (19) can be solved by separation of variables. Equation (18) admits solutions of the form

$$F_0 = \int_0^\infty f(\alpha)\mathrm{e}^{\pm\alpha u}J_0(\alpha v)\,\mathrm{d}\alpha, \tag{20}$$

where $f(\alpha)$ is an arbitrary function. Then, Eqs. (17) show that the components $F_{\pm 1}$ accompanying (20) are

$$F_1 = F_{-1} = \pm\int_0^\infty f(\alpha)\mathrm{e}^{\pm\alpha u}J_1(\alpha v)\,\mathrm{d}\alpha. \tag{21}$$

3.1. The field of a point charge

As an application of the preceding results, we shall obtain the field of an electric point charge fixed at the origin of the accelerated frame. This is the electromagnetic field produced by a point charge with a constant proper acceleration which, in its comoving frame, must be static and axially symmetric.

Thus, the coordinates of the point charge are $\rho = 0$, $\tilde{z} = c^2/g$ or, equivalently, $v = 0$, $u = -c^4/(2g^2)$. With respect to the accelerated frame, the electromagnetic field must be given by expressions of the form

$$\begin{aligned} F_0 &= \int_0^\infty f_+(\alpha)\mathrm{e}^{-\alpha u}J_0(\alpha v)\,\mathrm{d}\alpha, \\ F_{\pm 1} &= -\int_0^\infty f_+(\alpha)\mathrm{e}^{-\alpha u}J_1(\alpha v)\,\mathrm{d}\alpha, \end{aligned} \tag{22}$$

for $u > -c^4/(2g^2)$, where $f_+(\alpha)$ is some function to be determined, while for $u < -c^2/(2g^2)$,

$$F_0 = \int_0^\infty f_-(\alpha)\mathrm{e}^{\alpha u}J_0(\alpha v)\mathrm{d}\alpha,$$

$$F_{\pm 1} = \int_0^\infty f_-(\alpha)\mathrm{e}^{\alpha u}J_1(\alpha v)\mathrm{d}\alpha, \tag{23}$$

where f_- is a second function to be determined.

In order to find the explicit expressions of the functions $f_\pm(\alpha)$, we apply the boundary conditions for the electromagnetic field on the surface $u = -c^4/(2g^2)$ (corresponding to the hyperboloid $\rho^2 - \tilde{z}^2 = -c^4/g^2$, which passes through the origin). The components $F_{\pm 1}$ must be continuous at the boundary $u = -c^4/(2g^2)$, and F_0, which involves the normal component of the electric field to the boundary at the origin, must be continuous on the surface $u = -c^4/(2g^2)$, except at the origin, owing to the presence of a point charge q. These conditions give

$$f_+(\alpha)\mathrm{e}^{\alpha c^4/(2g^2)} = -f_-(\alpha)\mathrm{e}^{-\alpha c^4/(2g^2)} \tag{24}$$

and

$$2\pi\rho\int_0^\infty \left[f_-(\alpha)\mathrm{e}^{-\alpha c^4/(2g^2)} - f_+(\alpha)\mathrm{e}^{\alpha c^4/(2g^2)}\right] \times J_0(\alpha v)\,\mathrm{d}\alpha = 4\pi q\delta(\rho) \tag{25}$$

(Gauss' law).

On the other hand, Hankel's integral theorem (see, for example, Ref. 15, Sec. 5.14) implies that the delta function can be expressed in the form

$$\delta(r-\rho) = \int_0^\infty \alpha J_0(\alpha r)\rho J_0(\alpha\rho)\,\mathrm{d}\alpha,$$

hence,

$$\delta(\rho) = \int_0^\infty \alpha\rho J_0(\alpha\rho)\,\mathrm{d}\alpha.$$

Combining this last equation with Eqs. (24) and (25), one finds that

$$-\rho\int_0^\infty f_+(\alpha)\mathrm{e}^{\alpha c^4/(2g^2)}J_0(\alpha v)\,\mathrm{d}\alpha = q\int_0^\infty \alpha\rho J_0(\alpha\rho)\,\mathrm{d}\alpha.$$

With the change of variable $\alpha = \beta c^2/g$ and the second equation in (16), the integral on the right-hand side of the last equation is equivalent to

$$q\frac{c^4}{g^2}\int_0^\infty \beta\rho J_0(\beta\rho c^2/g)\,\mathrm{d}\beta = q\frac{c^4}{g^2}\int_0^\infty \alpha\rho J_0(\alpha v)\,\mathrm{d}\alpha$$

and hence,

$$f_+(\alpha) = -\frac{qc^4}{g^2}\alpha\,\mathrm{e}^{-\alpha c^4/(2g^2)}. \tag{26}$$

Substituting Eq. (26) into Eq. (22), making use of the so-called Lipschitz's integral (see, *e.g.*, Ref. 15, Sec. 5.15)

$$\int_0^\infty \mathrm{e}^{-ax}J_0(bx)\,\mathrm{d}x = \frac{1}{\sqrt{a^2+b^2}},$$

one obtains

$$F_0 = -\frac{qc^4}{g^2}\frac{u + c^4/(2g^2)}{\{[u+c^4/(2g^2)]^2 + v^2\}^{3/2}} = -\frac{4qc^4}{g^2}\frac{\rho^2 - \tilde{z}^2 + c^4/g^2}{[(\rho^2-\tilde{z}^2+c^4/g^2)^2 + 4\rho^2\tilde{z}^2]^{3/2}}. \tag{27}$$

Then, Eqs. (17) or (22) yield

$$F_{\pm 1} = \frac{qc^4}{g^2}\frac{v}{\{[u+c^4/(2g^2)]^2 + v^2\}^{3/2}} = \frac{8qc^4}{g^2}\frac{\rho\tilde{z}}{[(\rho^2-\tilde{z}^2+c^4/g^2)^2 + 4\rho^2\tilde{z}^2]^{3/2}}. \tag{28}$$

According to Eqs. (9), the components F_0 and $F_{\pm 1}$, which in the present case are real, are equal to E_z and E_ρ, respectively. Thus, the solution given by Eqs. (27) and (28) coincides with the field of an accelerated charge given in Ref. 4. In the accelerated frame, there is no magnetic field and, therefore, there is no radiation (however, this issue has generated long discussions in the literature; see, for example, Refs. 3 to 9).

4. General solution in terms of a single scalar potential

Going back to the source-free Maxwell equations without symmetry restrictions, we shall show that, as in the case of an inertial frame [16,17], the Maxwell equations can be solved by separation of variables and the general solution can be expressed in terms of a single complex potential.

The combinations (9) show a simple behavior under the spatial rotations about the z-axis. In fact, under the rotation about $\mathbf{e}_z$ through an angle θ given by

$$\mathbf{e}_\rho + \mathrm{i}\mathbf{e}_\phi \mapsto \mathrm{e}^{\mathrm{i}\theta}(\mathbf{e}_\rho + \mathrm{i}\mathbf{e}_\phi),$$

we have $F_s \mapsto \mathrm{e}^{\mathrm{i}s\theta}F_s$, for $s = 0, \pm 1$.

By definition [16,17], η has spin weight s if $\eta \mapsto \mathrm{e}^{\mathrm{i}s\theta}\eta$ when $\mathbf{e}_\rho + \mathrm{i}\mathbf{e}_\phi \mapsto \mathrm{e}^{\mathrm{i}\theta}(\mathbf{e}_\rho + \mathrm{i}\mathbf{e}_\phi)$; hence, F_{-1}, F_0, and F_{+1} have spin weight -1, 0, and 1, respectively. If η has spin weight s, $\eth\eta$ and $\bar{\eth}\eta$, defined by

$$\eth\eta \equiv -\rho^s\left(\partial_\rho + \frac{\mathrm{i}}{\rho}\partial_\phi\right)(\rho^{-s}\eta),$$

$$\bar{\eth}\eta \equiv -\rho^{-s}\left(\partial_\rho - \frac{\mathrm{i}}{\rho}\partial_\phi\right)(\rho^{s}\eta), \tag{29}$$

have spin weight $s+1$ and $s-1$, respectively [16,17]. Thus, the source-free Maxwell equations can also be written as

$$\begin{aligned}\left(\frac{1}{h}\partial_0+\partial_z\right)F_0-\overline{\eth}F_{+1}&=0,\\ \left(\frac{1}{h}\partial_0-\partial_z\right)F_0+\eth F_{-1}&=0,\\ \left(\frac{1}{h}\partial_0+\partial_z\right)(hF_{-1})+h\,\overline{\eth}F_0&=0,\\ \left(\frac{1}{h}\partial_0-\partial_z\right)(hF_{+1})-h\,\eth F_0&=0.\end{aligned}\tag{30}$$

We look for separable solutions of Eqs. (30) of the form

$$F_s=g_s(z,t)\,{}_sZ_{\alpha m}(\rho,\phi),\qquad s=0,\pm1,\tag{31}$$

where the ${}_sZ_{\alpha m}$ are spin-weighted cylindrical harmonics [16,17], α is a real number and m is an integer. The spin-weighted cylindrical harmonics satisfy the relations

$$\eth({}_sZ_{\alpha m})=\alpha_{s+1}Z_{\alpha m},\qquad \overline{\eth}({}_sZ_{\alpha m})=-\alpha_{s-1}Z_{\alpha m}\tag{32}$$

and, therefore, substituting Eqs. (31) into Eqs. (30), we obtain

$$\begin{aligned}\left(\frac{1}{h}\partial_0+\partial_z\right)g_0+\alpha g_{+1}&=0,\\ \left(\frac{1}{h}\partial_0-\partial_z\right)g_0+\alpha g_{-1}&=0,\\ \left(\frac{1}{h}\partial_0+\partial_z\right)(hg_{-1})-\alpha h g_0&=0,\\ \left(\frac{1}{h}\partial_0-\partial_z\right)(hg_{+1})-\alpha h g_0&=0.\end{aligned}\tag{33}$$

These equations can be combined to obtain a *second-order* partial differential equation for g_0, g_1, or g_{-1}. For instance, from the first and the fourth equation in (33), one finds that

$$\left(\frac{1}{h}\partial_0-\partial_z\right)h\left(\frac{1}{h}\partial_0+\partial_z\right)g_0=-\alpha^2hg_0$$

or, equivalently,

$$\frac{1}{h^2}\partial_0^2g_0-\frac{1}{h}\partial_z(h\partial_zg_0)+\alpha^2g_0=0.\tag{34}$$

Letting

$$\chi\equiv\frac{g_0}{\alpha^2}\,{}_0Z_{\alpha m},$$

for $\alpha\neq0$, one finds that Eq. (34) is equivalent to

$$\frac{1}{h^2}\partial_0^2\chi-\frac{1}{h}\partial_z(h\partial_z\chi)-\overline{\eth}\eth\chi=0,\tag{35}$$

which reduces to the scalar wave equation when $h=1$. According to Eqs. (33), the functions $g_{\pm1}$ are given in terms of g_0 by

$$g_{\pm1}=-\frac{1}{\alpha}\left(\frac{1}{h}\partial_0\pm\partial_z\right)g_0,$$

and hence, using Eqs. (31) and (32) one finds that all components of the electromagnetic field can be expressed in terms of the complex scalar potential χ:

$$\begin{aligned}F_{+1}&=-\left(\frac{1}{h}\partial_0+\partial_z\right)\eth\chi,\\ F_0&=-\overline{\eth}\eth\chi,\\ F_{-1}&=\left(\frac{1}{h}\partial_0-\partial_z\right)\overline{\eth}\chi.\end{aligned}\tag{36}$$

The linearity of Eqs. (30) and (35) and the completeness of the spin-weighted cylindrical harmonics implies that the general solution to the Maxwell equations is given by Eqs. (36), with χ being a solution of Eq. (35). [In fact, one can directly verify that Eqs. (36) satisfy Eqs. (30), provided that χ is a solution of Eq. (35).]

5. Concluding remarks

As we have shown, the seemingly complex problem of solving the Maxwell equations in a uniformly accelerated frame becomes tractable by means of appropriate definitions of the independent and dependent variables.

The procedure followed in this paper is applicable to the equations governing other fields and, according to the principle of equivalence, the local behavior of the fields should resemble that occurring in the presence of a gravitational field.

From the mathematical point of view, the equivalence of expressions (14) and (20) leads to some relations between Bessel functions that have not been explored here.

Acknowledgments

The authors wish to thank the referee for helpful comments. One of the authors (C.I.P.S.) thanks the Vicerrectoría de Investigación y Estudios de Posgrado of the Universidad Autónoma de Puebla for financial support through the programme “Jóvenes investigadores”.

1. G.F. Torres del Castillo and C.I. Pérez Sánchez, *Rev. Mex. Fís.* **52** (2006) 70.

2. J.B. Formiga and C. Romero, gr-qc/0607003.

3. W. Pauli, *Theory of Relativity*, (Pergamon Press, Oxford, 1958) reprinted by Dover, New York, 1981.

4. F. Rohrlich, *Ann. Phys. (NY)* **22** (1963) 169.

5. R. Peierls, *Surprises in Theoretical Physics* (Princeton University Press, Princeton, N.J., 1979).
6. F.J. Alexander and U.H. Gerlach, *Phys. Rev. D* **44** (1991) 3887.
7. A.K. Singal, *Gen. Rel. Grav.* **27** (1995) 953; **29** (1997) 1371.
8. S. Parrott, *Gen. Rel. Grav.* **29** (1997) 1463.
9. A. Harpaz, *Eur. J. Phys.* **26** (2005) 219.
10. C. Møller, *Kgl. Danske Videnskab. Selskab* **20** (1943) No. 19.
11. C. Møller, *The Theory of Relativity*, (Clarendon Press, Oxford, 1952).
12. W. Rindler, *Am. J. Phys.* **34** (1966) 1174.
13. E.A. Desloge, *Am. J. Phys.* **57** (1989) 1121.
14. L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, Oxford, 1975), §90.
15. N.N. Lebedev, *Special Functions and their Applications*, (Prentice-Hall, Englewood Cliffs, N.J., 1965) reprinted by Dover, New York, 1972.
16. G.F. Torres del Castillo, *Rev. Mex. Fís.* **38** (1992) 19.
17. G.F. Torres del Castillo, *3-D Spinors, Spin-Weighted Functions and their Applications* (Birkhäuser, Boston, 2003).