Time dependent quantum harmonic oscillator subject to a sudden change of mass: continuous solution

H. Moya-Cessa
INAOE, Coordinación de Óptica,
Apartado Postal 51 y 216, 72000 Puebla, Pue., México.

M. Fernández Guasti
Depto. de Física, CBI, Universidad Autónoma Metropolitana - Iztapalapa,
09340, México, D.F., Apartado Postal 55-534, México.

Recibido el 24 de octubre de 2006; aceptado el 18 de diciembre de 2006

We show that a harmonic oscillator subject to a sudden change of mass produces squeezed states. Our study is based on an approximate analytic solution to the time-dependent harmonic oscillator equation with a subperiod function parameter. This continuous treatment differs from former studies that involve the matching of two time-independent solutions at the discontinuity. This formalism requires an ad hoc transformation of the original differential equation and is also applicable for rapid, although not necessarily instantaneous, mass variations.

Keywords: Solutions in closed form; quantum mechanics; exact invariants.

Mostramos que un oscilador armónico sujeto a un cambio repentino de masa produce estados comprimidos. Nuestro estudio está basado en una solución analítica aproximada para el oscilador armónico dependiente del tiempo. El tratamiento continuo que estudiamos difiere de estudios anteriores en los cuales se igualan las soluciones en la discontinuidad. Nuestro formalismo requiere una transformación *ad hoc* de la ecuación diferencial original y es aplicable también para variaciones de masa rápidas, no solo instantáneas.

Descriptores: Soluciones analíticas; mecánica cuántica; invariantes exactos.

PACS: 03.65.La; 03.65.Ge; 02.30.Ik

1. Introduction

The oscillator differential equation with a sudden change of mass has been tackled by considering two time regions with constant parameters and matching the solutions at the time where the abrupt change takes place [1]. This procedure is analogous to the problem of an oscillator with constant mass but a sudden change in frequency due to an abrupt change in the potential. In quantum optics, the time dependent mass is particularly relevant because it describes an external influence on the quantized electromagnetic field, *e.g.*, a decaying or driven Fabry-Pérot cavity [2].

The harmonic oscillator equation with time-dependent parameters [3–7] has been solved for a sudden frequency change using a continuous treatment based on an invariant formalism [8]. This analytic treatment requires that the time-dependent parameter be a monotonic function whose variation is short compared with the typical period of the system. This procedure allowed us to obtain analytic solutions that may or may not exhibit squeezing depending on time when departing from an initial coherent state.

In this communication, a continuous treatment of a harmonic oscillator whose mass suddenly changes is considered. This continuous approach is not as straightforward as in the case where two solutions are matched at the abrupt interface, the reason being that the transformation leading to a differential equation without involving first derivatives yields a time-dependent parameter that is no longer a monotonic finite function. This hurdle must be overcomed by invoking another transformation that produces a monotonic time-dependent parameter. The transformed equation can then be solved with the continuum approach, leading to squeezing of the momentum variables at certain times. An interesting advantage of this method is that the change of mass need not be a step function in the strict mathematical sense. The requirement, which is physically more plausible, is that the change in mass should take place in a time span much shorter than the characteristic period of the oscillator.

The plan of the manuscript is the following: in Sec. 2, we write the quadratic and linear invariants for a time-dependent mass and exhibit their relation with the Hamiltonian. In Sec. 3, we discuss two transformations that translate the classical time-dependent mass equation into the time dependent frequency problem. The analytic solution for a rapid variation of the time dependent mass using the appropriate transformation is then presented. In Sec. 4, we exhibit how a sudden change of mass in the quantum oscillator produces squeezing. Section 5 is left for conclusions.

2. Time dependent mass

The harmonic Hamiltonian with time-dependent mass $M(t)$ reads

$$\hat{H} = \frac{1}{2}\left[\frac{\hat{p}^2}{M(t)} + M(t)\Omega^2\left(t\right)\hat{q}^2\right], \tag{1}$$

where $\Omega^2\left(t\right)$ is a time-dependent parameter. It is well-known that a quantum invariant for this type of interaction has the form [9]

$$\hat{I} = \frac{1}{2}\left[\left(\frac{G\hat{q}}{\rho_v}\right)^2 + (\rho_v \hat{p} - M(t)\dot{\rho}_v \hat{q})^2\right], \tag{2}$$

where the overdot represents derivative with respect to time. The amplitude ρ_v obeys the Ermakov-type equation

$$\frac{d^2\rho_v}{dt^2} + \frac{\dot{M}}{M}\frac{d\rho_v}{dt} + \Omega^2\rho_v = \frac{G}{M^2\rho_v^3} \tag{3}$$

and G is a constant often set equal to one in the literature. This equation forms an Ermakov pair with the classic harmonic oscillator equation for the coordinate variable v with time dependent mass

$$\ddot{v} + \frac{\dot{M}}{M}\dot{v} + \Omega^2 v = 0. \tag{4}$$

The orthogonal function procedure leads to the classical invariant

$$G = M\left(v_1\dot{v}_2 - v_2\dot{v}_1\right), \tag{5}$$

where the functions v_1 and v_2 are linearly independent solutions to the time-dependent mass differential equation (4). The quantum orthogonal functions' linear invariants may then be obtained using an analogous procedure to a previous derivation [15] with the identifications $v_j \to \hat{q}$, $M\dot{v}_j \to \hat{p}$, for $j = 1$ or 2 yielding

$$\hat{G}_1 = v_1\hat{p} - M\dot{v}_1\hat{q}, \qquad \hat{G}_2 = -v_2\hat{p} + M\dot{v}_2\hat{q}. \tag{6}$$

The quadratic Ermakov Lewis invariant is related to the these linear invariants by

$$\hat{I} = 1/2\left(\hat{G}_1^2 + \hat{G}_2^2\right).$$

The constant mass results $\left(\dot{M} = 0\right)$, albeit with a time-dependent parameter, are then modified by the change of variables

$$G_{\left(\dot{M}=0\right)} \to \frac{G_{\left(\dot{M}\neq 0\right)}}{M}, \qquad \hat{q}_{\left(\dot{M}=0\right)} \to M\hat{q}_{\left(\dot{M}\neq 0\right)}. \tag{7}$$

3. Differential equation transformation

The transformation

$$v = \varkappa \exp[-1/2\int h dt]$$

is commonly invoked in order to eliminate terms of the form $h(t)\,\dot{v}$ that involve a first derivative in second-order differential equations [10]. Equation (4) with the function $h(t) = \dot{M}/M$ then transforms to

$$\frac{d^2\varkappa}{dt^2} + \Omega_\varkappa^2\varkappa = 0, \tag{8}$$

with an effective time-dependent parameter $\Omega_\varkappa^2$ given by

$$\Omega_\varkappa^2 = \Omega^2 - \frac{1}{2}\frac{\ddot{M}}{M} + \frac{1}{4}\frac{\dot{M}^2}{M^2}. \tag{9}$$

For a step function time-dependent mass, this effective parameter Ω_κ^2 acquires divergent values, as may be seen from the mass derivatives involved in the above expression. This issue is not a problem if piecewise integration is used to obtain the solution. However, since the parameter Ω_κ^2 obtained under the transformation is no longer a monotonic function or a step function in the appropriate limit, this procedure is not suitable if the continuous analytical approach is to be invoked.

There is however an alternative approach that involves the derivative of the time-dependent harmonic oscillator equation with constant mass $\ddot{\psi} + \Omega^2\psi = 0$ together with the substitution $v = d\psi/dt$:

$$\ddot{v} - 2\frac{\dot{\Omega}}{\Omega}\dot{v} + \Omega^2 v = 0. \tag{10}$$

Let the time-dependent parameter be written in terms of stiffness and time-dependent mass in the usual way

$$\Omega^2(t) = \frac{k(t)}{M(t)}, \tag{11}$$

where stiffness $k(t)$ may also be a time-dependent function. The above equation then reads

$$\ddot{v} + \left(\frac{\dot{M}}{M} - \frac{\dot{k}}{k}\right)\dot{v} + \Omega^2 v = 0. \tag{12}$$

But this is the time-dependent mass equation (4) that needs to be solved, provided that stiffness is constant. Therefore the transformation $v = d\psi/dt$ and first integration of the resulting equation also eliminates the first order derivative term. However, it maintains the same functional dependence on the parameter Ω^2 rather than introducing an effective parameter Ω_κ^2. The inverse of a monotonic time-dependent mass function is then also a monotonic function without infinite values, provided that the mass is not zero. A decreasing (increasing) mass as a function of time produces an increasing (decreasing) time-dependent parameter Ω^2. If a step function $\Omega^2 = k/M$ is considered, then it remains a step throughout the transformation without nasty divergences involved.

Therefore, the time-dependent mass problem that involves first-order derivatives may be translated into a time-dependent frequency case that does not involve such terms. However, the above derivations show that there are distinct transformations leading to the desired equation form. The latter is in fact a particular form of a Darboux transformation [11]. These results have been abridged in the table below:

	conventional transformation	derivative approach
original equation	$\ddot{u}+\frac{\dot{M}}{M}\dot{u}+\frac{k}{M}u=0$	$\ddot{v}+\frac{\dot{M}}{M}\dot{v}+\frac{k}{M}v=0$
transformation	$u=\sqrt{\frac{k}{M}}\varkappa$	$v=\frac{d\psi}{dt},\quad \frac{dv}{dt}=-\frac{k}{M}\psi$
transformed equation	$\ddot{\varkappa}+\Omega_{\varkappa}^2\varkappa=0$	$\ddot{\psi}+\Omega^2\psi=0$
time dependent parameter	$\Omega_{\varkappa}^2=\frac{1}{M}\left(k-\frac{1}{2}\ddot{M}+\frac{1}{4}\frac{\dot{M}^2}{M}\right)$	$\Omega^2=\frac{k}{M}$

From a physical point of view, it is interesting to consider the relationship between these results. Let a perturbation (*i.e.* displacement) ψ obey a TDHO equation with a time-dependent parameter but constant mass. The equation governing the time derivative of such a perturbation (*i.e.* velocity) is given by the time derivative of the perturbation equation. If the system has a time-dependent mass, the perturbation (*i.e.* displacement) now obeys an equation that is identical to that fulfilled by the velocity in the case of constant mass with time-dependent stiffness.

A. Analytic solution

An analytic approximate solution to the TDHO has been obtained for a time-dependent parameter $\Omega^2(t)$ that varies monotonically in a time span much shorter than the characteristic period of the system [12]. The solution in amplitude and phase variables $\psi=\rho\exp(i\varphi)$ is given by

$$\rho(t)=\frac{a_1}{\sqrt{2}}\sqrt{1+\frac{\Omega_1^2}{\Omega^2(t)}+\left(1-\frac{\Omega_1^2}{\Omega^2(t)}\right)\cos\left(2\int_{t_s}^{t}\Omega(t')\,dt'\right)}, \tag{13}$$

and

$$\varphi(t)=\arctan\left[\frac{\Omega_1}{\Omega(t)}\tan\left(\int_{t_s}^{t}\Omega(t')\,dt'\right)+(t_s-t_0)\,\Omega_1\right]. \tag{14}$$

a_1 and Ω_1 are the initial amplitude and frequency at a time well before the transient behaviour takes place. t_s is the time where the variation is maximum and t_0 is an arbitrary initial time. The frequency, defined as the derivative of the phase is given by

$$\dot{\varphi}=\frac{\Omega_1}{\left(\frac{\Omega_1}{\Omega(t)}\right)^2\sin^2\left(\int_{t_s}^{t}\Omega(t')\,dt'\right)+\cos^2\left(\int_{t_s}^{t}\Omega(t')\,dt'\right).} \tag{15}$$

The solution for the displacement v in the classical time-dependent mass problem is then

$$v=\dot{\psi}=\dot{\rho}e^{i\varphi}+i\dot{\varphi}\rho e^{i\varphi}=\left(\frac{\dot{\rho}}{\rho}+i\dot{\varphi}\right)\psi$$

but

$$\frac{\dot{\rho}}{\rho}+i\dot{\varphi}=\sqrt{\left(\frac{\dot{\rho}}{\rho}\right)^2+\dot{\varphi}^2}\exp\left[i\arctan\left(\frac{\rho\dot{\varphi}}{\dot{\rho}}\right)\right].$$

Therefore, in polar variables $v=\rho_v\exp(i\varphi_v)$, the amplitude and phase are given by

$$\rho_v=\sqrt{\rho^2\dot{\varphi}^2+\dot{\rho}^2},\qquad \varphi_v=\varphi+\arctan\left(\frac{\rho\dot{\varphi}}{\dot{\rho}}\right). \tag{16}$$

Allow for the time-dependent parameter to be given by (11) with constant k and let the function

$$M(t)=M_1\left[1+\frac{M_2-M_1}{2M_1}\left(1+\tanh\left[\alpha_s\left(t-t_s\right)\right]\right)\right] \tag{17}$$

model a step function in the limit when the slope $\alpha_s\to\infty$ as shown in Fig. 1. It is interesting to note that the approximate analytic solution that is being used is adequate even if the time-dependent parameter does not vary in a strictly abrupt fashion. The solution is appropriate in the so-called subperiod regime that requires variations of the time-dependent parameters in an interval much shorter than the period of the system although not necessarily infinitesimal. The amplitude ρ_v for this function is plotted in Fig. 2 using Eqs. (16) together with (13) and (14).

4. Quantum oscillator

The Ermakov-Lewis invariant $\hat{I}$ may be related to the Hamiltonian with time-independent mass by a unitary transformation of the form (for simplicity we set $\hbar=1$)

$$\hat{T}=\exp\left(i\frac{\ln(\rho_v)}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})\right)\exp\left(-i\frac{M(t)\dot{\rho}_v}{2\rho_v}\hat{q}^2\right), \tag{18}$$

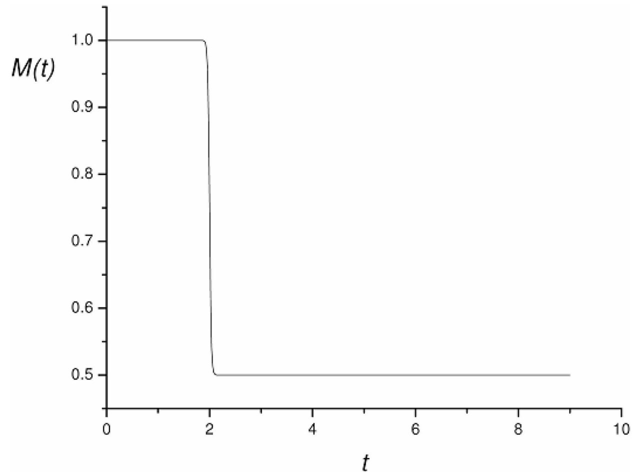

FIGURE 1. The mass $M(t)$ evolution as a function of time for $M_1 = 1$ and $M_2 = 0.5$; $\alpha_s = 20$ and $t_s = 2$.

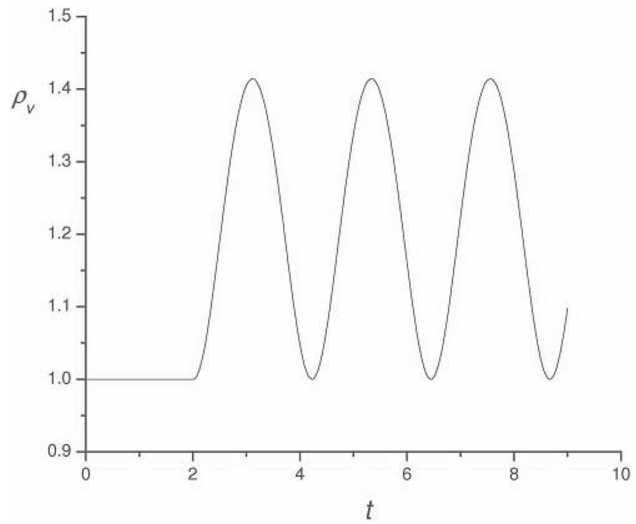

FIGURE 2. The amplitude $\rho_u(t)$ as a function of time for $M_1 = 1$, $M_2 = 0.5$, $\alpha_s = 20$ and $t_s = 2$.

with

$$\hat{\mathcal{H}} = \hat{T}\hat{I}\hat{T}^\dagger = \frac{1}{2}(\hat{p}^2 + G^2\hat{q}^2) \equiv G\left(\hat{n} + \frac{1}{2}\right), \qquad (19)$$

with $\hat{n}$ the so-called number operator with eigenstates $|n\rangle$. States of the form $|n\rangle_t = \hat{T}^\dagger|n\rangle$ are eigenstates of the Ermakov Lewis invariant. This invariant plays in the time-dependent case, the role that the quantized Hamiltonian does in the time-independent case [13, 14]. The Ermakov Lewis invariant can be written in terms of annihilation and creation operators as $\hat{I} = \hat{a}^\dagger\hat{a} + G/2$ with

$$\hat{a} = \frac{1}{\sqrt{2}}\left[\frac{G\hat{q}}{\rho_v} + i(\rho_v\hat{p} - M(t)\dot{\rho}_v\hat{q})\right],$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left[\frac{G\hat{q}}{\rho_v} - i(\rho_v\hat{p} - M(t)\dot{\rho}_v\hat{q})\right]. \qquad (20)$$

We can obtain coherent states of the TDHO with time-dependent mass as

$$|\alpha\rangle_t = \hat{D}_t(\alpha)|0\rangle_t,$$

with

$$\hat{D}_t(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$$

and

$$\hat{a}|\alpha\rangle_t = \alpha|\alpha\rangle_t.$$

Recently we have shown that the Schrödinger equation for the one-dimensional time-dependent harmonic Hamiltonian has a solution of the form [15]

$$|\psi(t)\rangle = e^{-i\hat{I}\int_0^t \omega(t')dt'}\hat{T}^\dagger\hat{T}(0)|\psi(0)\rangle, \qquad (21)$$

with $\omega(t) = 1/\rho_v^2$.

A. Squeezed states

Consider that, at time $t = 0$, the system is in the initial coherent state $|\alpha\rangle$. The initial state $|\psi(0)\rangle{=}\hat{T}^\dagger(0)|\alpha\rangle{=}|\alpha\rangle_0{=}|\alpha\rangle$ then evolves according to (21) as

$$|\psi(t)\rangle = \exp\left(-i\hat{I}\int_0^t \omega(t)dt\right)\hat{T}^\dagger|\alpha\rangle$$

$$= \hat{T}^\dagger|\alpha e^{-i\int_0^t \omega(t)dt}\rangle = |\alpha e^{-i\int_0^t \omega(t')dt'}\rangle_t. \qquad (22)$$

Therefore, coherent states remain coherent throughout the system's time evolution. This statement has been made before regarding an oscillator with constant mass but time-dependent frequency [8]. This result is now being extended to an oscillator with time dependent mass. From Fig. 2 we can see that $\hat{T}(0) = 1$, since $\dot{\rho}_v = 0$ and $\ln\rho_v = 0$. It may thus be seen how ideal squeezed states may be generated: the maxima of the function tell us when squeezing occurs, since as for such times $\dot{\rho}_v(t_{max}) = 0$ and $\ln\rho_v(t_{max}) \neq 0$, so that we obtain

$$|\psi(t_{max})\rangle$$

$$= \exp\left(\frac{i\ln\rho_v(t_{max})}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})\right)|\alpha e^{-i\int_0^{t_{max}}\omega(t)dt}\rangle, \quad (23)$$

where the operator

$$\exp[\frac{i}{2}\ln\rho_v(t_{max})(\hat{q}\hat{p} + \hat{p}\hat{q})]$$

is the well-known *squeeze operator* [16, 17]. The squeezed state

$$|\psi(t_{max})\rangle = |\alpha\exp\left(-i\int_0^{t_{max}}\omega(t)dt\right); \ln\rho_v(t_{max})\rangle$$

is then generated. Squeezed states, like as coherent states, are also minimum uncertainty states. However the uncertainties for $\hat{q}$ and $\hat{p}$ are not equal; in particular, we have

$$\Delta\hat{q}=\frac{\rho_v(t_{max})}{\sqrt{2}}, \quad \Delta\hat{p}=\frac{1}{\sqrt{2}\rho_v(t_{max})}, \quad \Delta\hat{q}\Delta\hat{p}=\frac{1}{2} \quad (24)$$

i.e. the momentum uncertainty is squeezed (as $\rho_v(t_{max}) > 1$, see Fig. 2). This result should be compared with the squeezing found for the TDHO when the frequency is suddenly doubled, thus yielding squeezing in the coordinate variable [8].

5. Conclusions

The real linear quantum invariants or orthogonal function invariants have been generalized for the one dimensional harmonic oscillator with time-dependent mass [Eq. (6)].

The TDHO with time-dependent mass $M(t)$ has been translated into a problem with constant mass but time-dependent parameter $\Omega^2(t)$. The transformation has been shown not to be unique. A Darboux type transformation yields an equation for the perturbation with time-dependent mass that is formally identical to that fulfilled by the velocity in the constant mass case. The transformed time dependent parameter $\Omega^2(t)$ then remains monotonic and finite, provided that the time-dependent mass is monotonic and finite, even if it varies in an abrupt fashion.

The problem has been solved using an approximate analytical solution whose validity holds when the time-dependent parameters vary monotonically in a time span that is much shorter than the period, although it need not be instantaneous. This feature, which describes a more realistic scenario of parameter variations with finite duration, is clearly unattainable when the problem is solved using the two steady-state solution approach.

A sudden change of mass beginning with a mass M_1 produces squeezing in the momentum variable provided that there is a loss of mass $M_2 < M_1$. In contrast, a variation of the potential from Ω_1 to Ω_2 with $\Omega_2 > \Omega_1$ produces squeezing in the coordinate variable [8]. These results are consistent with the view described above that the coordinate transformation is formally equivalent to the role played by the velocity variable when the mass is constant. Coherent states have been shown to keep their form throughout the system's evolution, whether or not the mass and/or the potential are time dependent.

1. T. Kiss, J. Janszky, and P. Adam, *Phys. Rev. A* **49** (1994) 4935.
2. R.K. Colegrave and M.S. Abdulah, *Optica Acta* **28** (1981) 495.
3. H.R. Lewis, *Phys. Rev. Lett.* **18** (1967) 510.
4. V.V. Dodonov and V.I. Man'ko, *Phys. Rev. A* **20** (1979) 550.
5. J. Janszky and Y.Y. Yushin, *Opt. Commun.* **59** (1986) 151.
6. F. Haas and J. Goedert, *Phys. Lett. A* **279** (2001) 181.
7. S. Bouquet and H.R. Lewis, *J. Math. Phys.* **37** (1996) 5509.
8. H. Moya Cessa and M. Fernández Guasti, *Phys. Lett. A* **311** (2003) 1.
9. I.A. Pedrosa and I Guedes, *Int. J. of Mod. Phys. B* **18** (2004) 1379.
10. J.R. Ray and J.L. Reid, *Phys. Rev. A* **26** (1982) 1042.
11. V.B. Matveev and M.A. Salle, *Darboux transformations and solitons* (Springer-Verlag, 1991).
12. M. Fernández Guasti, *Physica D: Nonlinear phenomena* **189** (2004) 188.
13. F. Haas, *J. Phys. A: Math. Gen.* **35** (2002) 2925.
14. M. Fernández Guasti and H. Moya Cessa, *Phys. Rev. A* **67** (2003) 063803-1-5.
15. M. Fernández Guasti and H. Moya Cessa, *J. Phys. A: Math. Gen.* **36** (2003) 2069.
16. S.M. Barnett and P.L. Knight, *J. of Mod. Optics* **34** (1987) 2069.
17. K. Zaheer and M.S. Zubairy, *Squeezed states of the radiation-field* (Academic Press, New York, 1990).