Optimal stabilization of unstable periodic orbits embedded in chaotic systems

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A gradient-flow-based approach is proposed in this paper for stabilizing unstable periodic orbits (UPO) embedded in chaotic systems. In order to obtain an on-line stabilizing solution, the stabilization problem is considered to be an optimal control problem, and system state sensitivities with respect to the control input are introduced. The resulting feedback controller is able to stabilize UPO embedded in both kind of systems, with or without an odd Floquet number. Moreover, the proposed approach is easily extended to identifying the period of the UPO to be stabilized when it is unknown. Simulation experiments of the proposed controller are carried out on the Rössler and the Lorenz systems.

Keywords: Optimal stabilization; sensitivity theory; gradient flow; unstable periodic orbits.

En este artículo se propone un enfoque basado en el flujo gradiente con el propósito de estabilizar órbitas periódicas inestables inmersas en sistemas caóticos. Tal enfoque considera al problema de estabilización como un problema de control óptimo, lo cual permite obtener una solución en línea al problema de interés, para lo cual se introduce el cálculo de las sensibilidades de los estados con respecto a la entrada de control. El controlador por retroalimentación resultante permite estabilizar órbitas periódicas inestables inmersas en ambas clases de sistemas caóticos, con o sin número de Floquet impar. Además, el enfoque propuesto se extiende para identificar el periodo de la órbita periódica inestable a ser estabilizada, en el caso en que éste sea desconocido. Se realizan experimentos de simulación del controlador propuesto, para estabilizar órbitas periódicas inestables de los sistemas de Rossler y de Lorenz.

Descriptores: Estabilización óptima; teoría de sensitividad; flujo gradiente; órbitas periódicas inestables.

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1. Introduction

Chaos is a phenomenon that occurs in several physical systems. In the last few years, chaos control has been a topic of intensive research, and two main areas can be recognized: chaos generation, where the unforced system is not chaotic but the application requires it, and chaos suppression, where the chaotic behavior is dangerous or not required. In chaos suppression, the control target consists in stabilizing an unstable periodic orbit (UPO) embedded in the chaotic attractor, while the forcing is kept as small as possible.

For chaos suppression there are three major branches of controllers: the feed-forward control, the OGY method, and the delayed feedback control method [1]. Due to its simplicity of implementation, the delayed feedback control method has been successfully employed to stabilize unstable periodic orbits in a variety of experimental systems [2]. However, due to the odd number limitation [3], the delayed feedback controller is not able to stabilize UPO's embedded in any kind of chaotic systems. In order to remove the odd-number limitation, an unstable delayed controller has been proposed by Pyragas [4] to stabilize UPO embedded in the Lorenz system, but a general methodology for designing the controller is not given.

On the other hand, stabilization of UPO with the delayed feedback control method requires a time delay which is a multiple of the period. Several methods have been proposed to identify such a period. However, most of them are off-line methods [5], and that period should be known prior to the on line chaos suppression experiments. Online identification methods have also been proposed, but a close initial estimate is required [6].

In this work, an on-line optimal controller is proposed for stabilization of UPO's embedded in chaotic systems. The performance index considers the error between the actual and the delayed output of the system, as well as the energy consumption. As the solution to this optimal control problem is based on the gradient flow [7], the solution may be locally optimal if the objective function is not convex. Therefore, we consider the square of the error added to the square of the control effort as the objective function.

In order to compute the gradient of the objective function with respect to the control input (the independent variable), we propose to use the sensitivities of the states of the chaotic system with respect to the control input. The stability of periodic solutions is proved when convex functions are selected as performance indexes. Moreover, the controller design methodology is easily extended to consider the case when the period of the UPO to be stabilized is unknown. Numerical simulations show the effectiveness of the controller when it is applied to the stabilization of UPO's embedded in both the Rössler and the Lorenz systems.

2. Statement of the problem

Let us consider the nonlinear control system (1), where $x(t) \in R^n$ is the state vector, $f(\cdot) : R^{n+m} \rightarrow R^n$ is a continuously differentiable vector function, and $u(t) \in R^m$ is the control input.

$$\dot{x}(t) = f(x(t), u(t)); \quad x(0) = x_0 \quad (1)$$

Assumption 1. For the free system (1), *i.e.* $u = 0$, there exists an unstable periodic orbit $x_p^T(t) = x(t - T)$ with period T.

Under assumption 1, the control objective is to stabilize the unstable periodic orbit $x_p^T(t)$, which can be stated as an infinite time optimal control problem consisting of minimizing the nonnegative differentiable function (2) subject to the chaotic system dynamics (1), where $y = h(x)$ is the system output and $h(\cdot) : R^n \rightarrow R^1$ is a differentiable function

$$J = g(y(t), y(t - T), u) \quad (2)$$

Two cases are considered: the first one considers T as a known parameter and in the second one we assume that T is unknown.

3. Case 1. Period T is known

The gradient of the objective function (2) with respect to the control input u is given as

$$\nabla_u J = \frac{\partial g(y(t), y(t-T), u)}{\partial y(t)} \frac{\partial y(t)}{\partial u(t)}$$
$$+ \frac{\partial g(y(t), y(t-T), u)}{\partial y(t-T)} \frac{\partial y(t-T)}{\partial u(t)}$$
$$+ \frac{dg(y(t), y(t-T), u)}{du(t)},$$

where $\partial y(t)/\partial u(t)$ represents the output sensitivity with respect to the control input $u(t)$, given by Eq. (3)

$$\frac{\partial y(t)}{\partial u(t)} = \frac{\partial h(x(t))}{\partial x(t)} \frac{\partial x(t)}{\partial u(t)} \quad (3)$$

The state sensitivity vector with respect to the control input $\partial x(t)/\partial u(t)$ is the solution to the first order differential Eq. (4)

$$\frac{d}{dt}\left[\frac{\partial x(t)}{\partial u(t)}\right] = \frac{\partial f(x(t), u(t))}{\partial x(t)} \frac{\partial x(t)}{\partial u(t)}$$
$$+ \frac{df(x(t), u(t))}{du(t)} \quad (4)$$

Theorem 1. If $u^*(t)$ is the solution to the differential equation (5), with $\eta_u \in R^{m \times m}$ a definite positive matrix, then the objective function (2) is non-increasing along trajectories $x^*(t)$ and $u^*(t)$, where $x^*(t)$ is the solution to (1) with $u(t) = u^*(t)$

$$\frac{d}{dt}u(t) = -\eta_u \nabla_u J \quad (5)$$

Proof. The time evolution of $g(y(t), y(t-T), u)$ is given by (6), where the time derivative of $g(y(t), y(t-T), u)$ is given by (7), where for reading simplicity the time dependency is not considered and $y(t-T) = y_T$

$$g(y(t), y(t-T), u) = g(y(0), y(0-T), u(0))$$
$$+ \int_0^t \frac{dg(\cdot)}{d\tau} d\tau \quad (6)$$

$$\frac{dg(\cdot)}{dt} = \left[\frac{\partial g(y, y_T, u)}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial g(y, y_T, u)}{\partial y_T} \frac{\partial y_T}{\partial u}\right.$$
$$\left. + \frac{dg(y, y_T, u)}{du}\right]\left[\frac{du}{dt}\right]^T \quad (7)$$

Finally, considering (5), the time evolution of $g(\cdot)$ is given by (8), and the proof is finished

$$g(y(t), y(t-T), u) = g(y(0), y(0-T), u(0))$$
$$- \int_0^t \left(\nabla_u J \eta_u \nabla_u^T J\right) d\tau \quad (8)$$

Corollary 1. Under assumption 1, if an error function is defined as $e(t) = y(t) - y(t-T)$, and if $g(\cdot)$ is considered to be a convex function of the error with the minimum equal to zero, then the unstable periodic orbit embedded in the chaotic open loop system (1) will become stable for the closed loop system (1), (4) and (5).

4. Case 2. Period T is unknown

In this section, the proposed approach is extended to deal with systems where the period T of the unstable periodic orbit is unknown.

The derivative of the objective function (2) with respect to the time period T is given by (9). In this case, the sensitivity of the output with respect to T is required, and is given by (10)

$$\nabla_T J = \frac{\partial g(y(t), y(t-T), u)}{\partial y(t)} \frac{\partial y(t)}{\partial T}$$
$$+ \frac{\partial g(y(t), y(t-T), u)}{\partial y(t-T)} \frac{\partial y(t-T)}{\partial T}$$
$$+ \frac{dg(y(t), y(t-T), u)}{dT} \quad (9)$$

$$\frac{\partial y(t)}{\partial T} = \frac{\partial h(x(t))}{\partial x(t)} \frac{\partial x(t)}{\partial T} \quad (10)$$

The state sensitivity with respect to the time period T is given by (11)

$$\frac{\partial x(t)}{\partial T} = f(x(t-T), u(t-T)) \qquad (11)$$

Theorem 2. If $u^*(t)$ is the solution to the differential equation (5), with $\eta_u \in R^{m \times m}$ a definite positive matrix, and T^* is the solution to (12), with $\eta_T > 0$, then the objective function (2) is non-increasing along trajectories $x^*(t)$, $u^*(t)$ and T^* where $x^*(t)$ is the solution to (1) with $u(t) = u^*(t)$:

$$\frac{d}{dt}T = -\eta_T \nabla_T J. \qquad (12)$$

Proof. The proof is similar to the proof of Theorem 1, but in this case the time evolution of $g(\cdot)$ is given by (13).

$$g(y, y_T, u) = g(y(0), y(0-T), u(0))$$

$$- \int_0^t \left(\nabla_u J \eta_u \nabla_u^T J + \nabla_T J \eta_T \nabla_T J \right) d\tau \quad (13)$$

5. Applications

In this section, the feedback optimal control law is implemented on two representative chaotic systems, the Rössler system and the Lorenz system.

5.1. Rössler system assuming T known

The Rössler system is given as

$$\dot{x_1} = -x_2 - x_3$$

$$\dot{x_2} = x_1 + ax_2 - u$$

$$\dot{x_3} = b + x_1 x_3 - cx_3 \qquad (14)$$

where $a = b = 0.2$, $c = 5.7$ and $\dot{x}$ represents the time derivative of x. The state sensitivity with respect to the control input $u(t)$ is the solution to the differential equation (15), where $s_i = \partial x_i / \partial u$.

$$\dot{s_1} = -s_2 - s_3$$

$$\dot{s_2} = s_1 + as_2 - 1$$

$$\dot{s_3} = x_3 s_1 + s_3 (x_1 - c) \qquad (15)$$

For this example, x_2 is the output variable, *i.e.*, $y = x_2$, and the objective function is a sum of squares of the error $e(t) = y(t) - y(t-T)$ and the control action $u(t)$, as given in (16). The purpose of such an objective function is to stabilize the unstable periodic orbit while the control effort is minimized.

$$J = \frac{1}{2}\left[(x_2(t) - x_2(t-T))^2 + u^2(t) \right] \qquad (16)$$

According to Sec. 3, the optimal control law $u^*(t)$ is the solution to (17).

$$\dot{u} = -\eta_u \left[(x_2(t) - x_2(t-T))(s_2(t) - s_2(t-T)) + u \right] \quad (17)$$

The behavior of the controlled system when the target is the period-one UPO, *i.e.* $T = 5.861$, is shown in Figs. 1-5 for the initial conditions

$$x(0) = \begin{bmatrix} x_1(0) & x_2(0) & x_3(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

and

$$s(0) = \begin{bmatrix} s_1(0) & s_2(0) & s_3(0) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}.$$

The time evolution of the controlled states x_1, x_2 and x_3 are shown in Fig. 1. The time evolution of the sensitivities s_1, s_2 and s_3 are shown in Fig. 2. From the figures, it is clear that both the states and the sensitivities reach a stable period-one orbit. Figure 3 shows the control input, where it is noticed that after a transient of approximately seventy seconds, the control is also periodic with small magnitude. In Figs. 4 and 5 the stable periodic orbit for the states and the sensitivities, respectively are shown; for these figures only the stationary time series are considered.
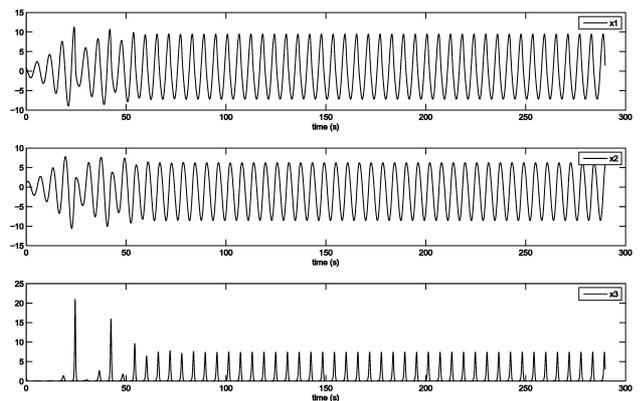
FIGURE 1. Time evolution of the controlled Rössler system states for T=5.861.
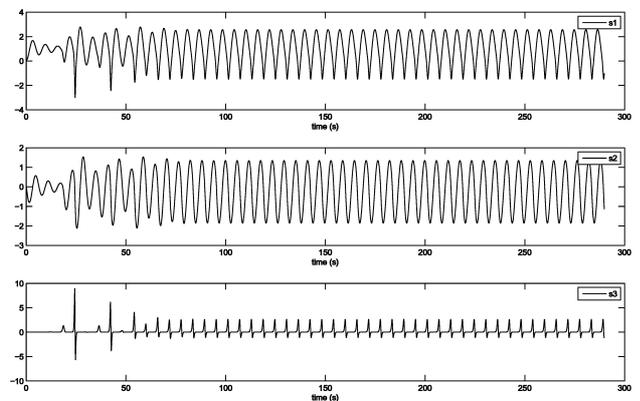
FIGURE 2. Time evolution of the controlled Rössler system sensitivities for T=5.861.
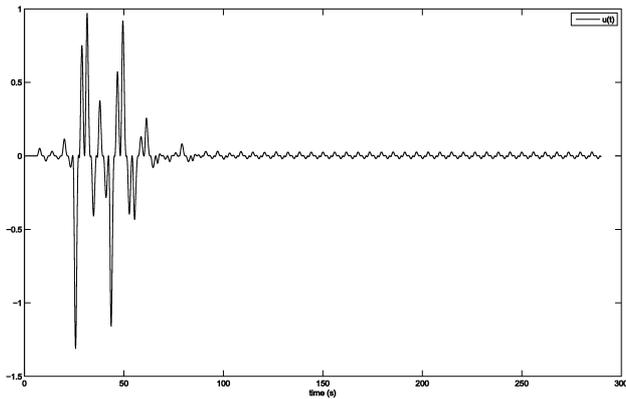
FIGURE 3. Control input to the Rössler system, for T=5.861.
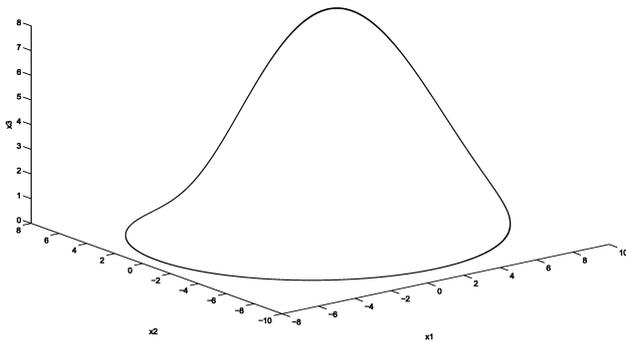
FIGURE 4. Stationary period-one orbit for the states of the controlled Rössler system.
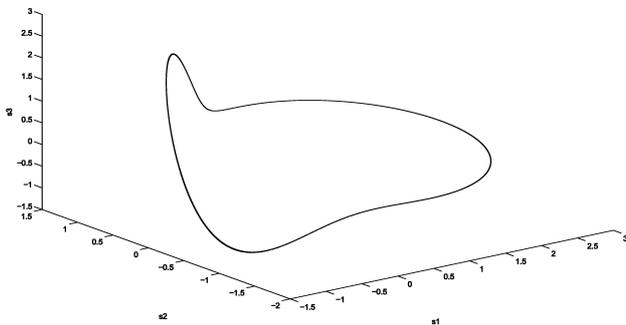
FIGURE 5. Stationary period-one orbit for the sensitivities of the controlled Rössler system.

5.2. Lorenz system assuming T known

The Lorenz system is given as

$$\dot{x}_1 = c(x_2 - x_1)$$
$$\dot{x}_2 = rx_1 - x_2 - x_1 x_3 + u$$
$$\dot{x}_3 = x_1 x_2 - bx_3 \tag{18}$$

where $c = 10$, $r = 28$ and $b = 8/3$. The state sensitivity with respect to the control input $u(t)$ is the solution to the

differential equation (19).

$$\dot{s}_1 = c(s_2 - s_1)$$
$$\dot{s}_2 = (r - x_3)s_1 - s_2 - x_1 s_3 + 1$$
$$\dot{s}_3 = s_1 x_2 + x_1 s_2 - bs_3 \tag{19}$$

In this example, x_2 is considered as the output variable, and for illustrative purposes, the objective function is a differentiable (but not continuously differentiable) convex function of the error $e(t) = x_2(t) - x_2(t - T)$ and the control action $u(t)$, as given in (20). In this case, the control objective is to stabilize the unstable period one orbit while the control effort is minimized.

$$J = |(x_2(t) - x_2(t - T))| + |u(t)| \tag{20}$$

The optimal control law $u^*(t)$ is the solution to (21), where $sign(\cdot)$ represents the sign function as defined in (22), [8].

$$\dot{u} = -\eta_u [(s_2(t) - s_2(t - T)) sign(x_2(t) \\ - x_2(t - T)) + sign(u)] \tag{21}$$

$$sign(x) = \begin{cases} 1 \text{ for } x > 0 \\ -1 \text{ for } x < 0 \\ \text{undefined for } x = 0 \end{cases} \tag{22}$$

For this experiment, the target is the period-one orbit, with $T = 1.5586$. Figures 6-8 show the results obtained when

$$x(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

and

$$s(0) = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}.$$

The time evolution of the control input is shown in Fig. 6, in this case the control input does not exhibit a periodic behavior; however, the control objective is achieved, which can be noticed in Figs. 7 and 8, where the stationary period-one orbit and its (x_1, x_2) projection are shown. In this case, the state sensitivities are not shown, as they are unstable.
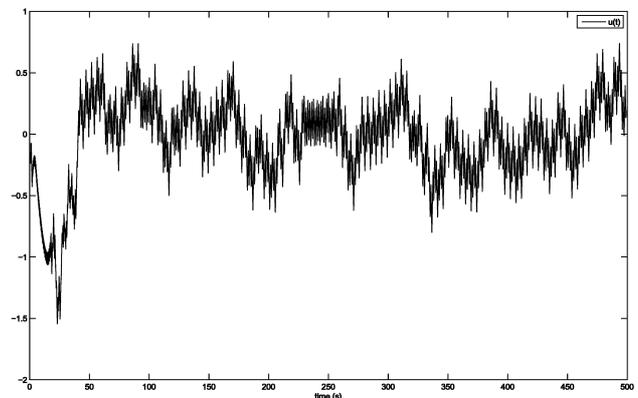
FIGURE 6.   Optimal control input for the Lorenz system, T=1.5586.
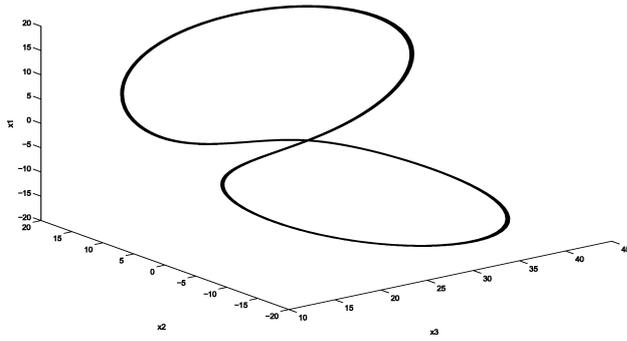
FIGURE 7. Stationary period-one orbit for the Lorenz system.
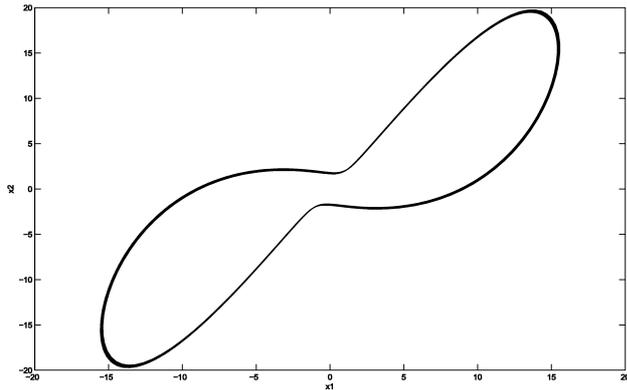
FIGURE 8. (x_1, x_2) projection of the period-one orbit of the Lorenz system.
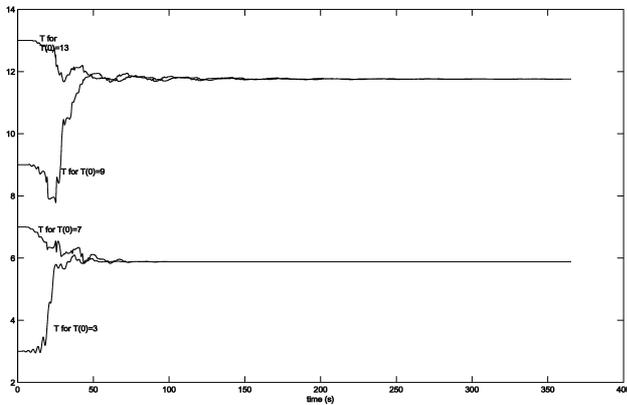
FIGURE 9. Time evolution of the Rössler system estimated delay for four different initial conditions.

5.3. Rössler system assuming T unknown

In this section, the results of the controlled Rössler system are presented when it is assumed that the period T of the unstable periodic orbit is unknown. The Rössler system (14), as well as the sensitivities (15) are considered. The control system is also the solution to Eq. (17), but in this case, T is the solution to the differential equation (23). It should be noticed that the initial condition $T(0)$ considerably affects the

stationary solution.

$$
\frac{dT}{dt} = -\eta_T \left(x_2(t) - x_2(t-T)\right)\left(x_1(t-T) \right.
$$
$$
\left. + ax_2(t-T) - u(t-T)\right) \tag{23}
$$
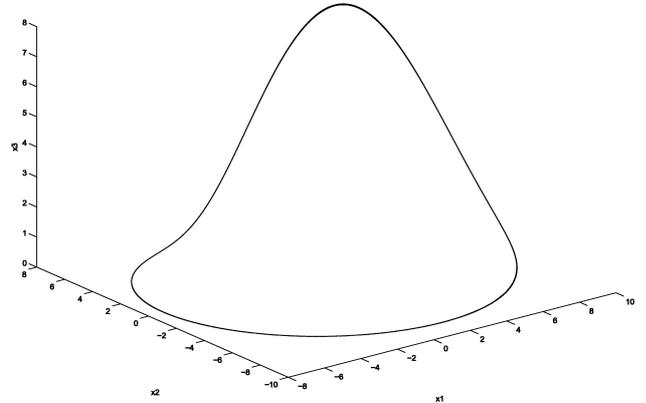
FIGURE 10. Stationary period orbit for the Rössler system, with estimated $T = 5.8811$ s.
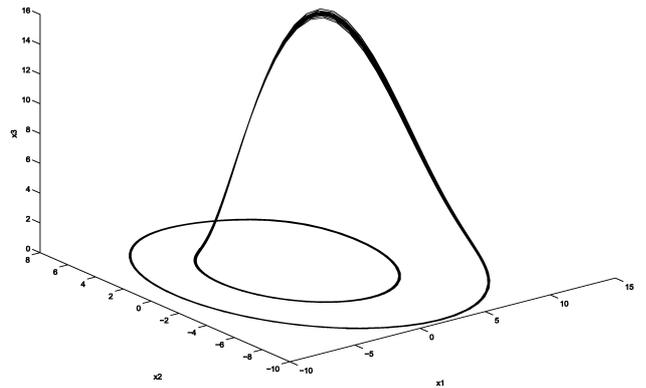
FIGURE 11. Stationary period two orbit for the Rössler system, with estimated period $T = 11.7586$ s.
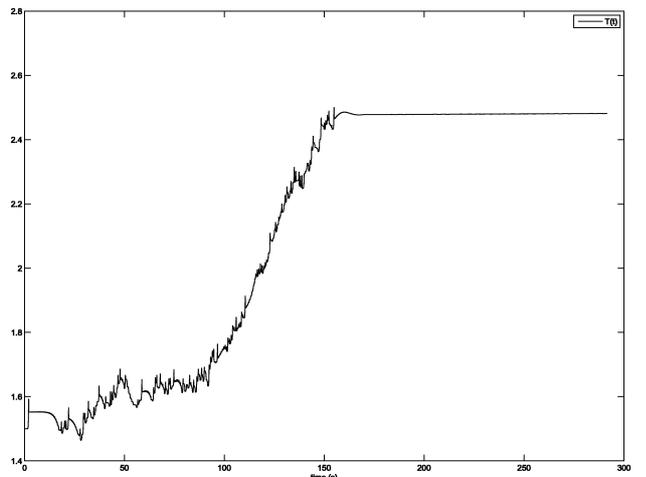
FIGURE 12. Time evolution of estimated period T for the controlled Lorenz system.
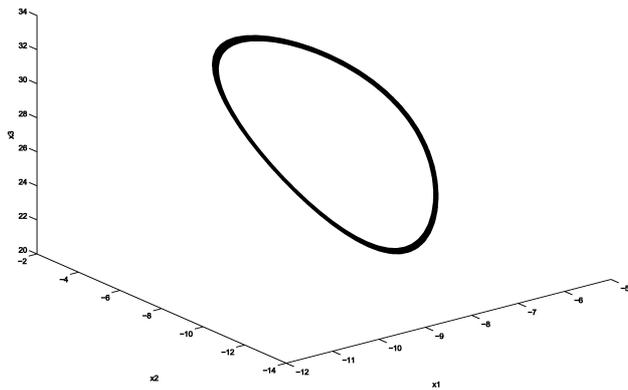
FIGURE 13. Stabilized period orbit of the controlled Lorenz system, for T=2.4986.
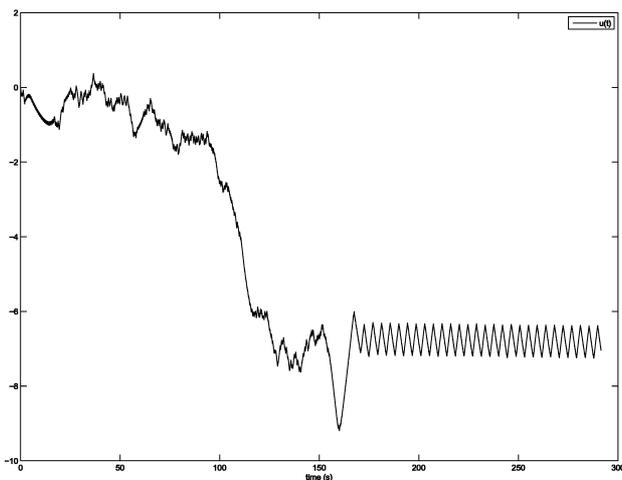
FIGURE 14. Control input for the Lorenz system assuming T unknown.

The time evolution of the period T is shown in Fig. 9 for four different initial conditions, we can observe that period T converges to two steady values, one at 5.8811 s, where the stationary period one orbit is shown in Fig. 10, and other at 11.7586 s, for which the stationary period two orbit is shown in Fig. 11.

5.4. Lorenz system assuming T unknown

In this section, the results of the controlled Lorenz system are presented when it is assumed that the period T of the unstable periodic orbit is unknown. The Lorenz system (18) as well as the sensitivities (19) are considered. The control system is also the solution to Eq. (21), but in this case, T is the solution to the differential equation (24).

$$
\begin{aligned}
\frac{dT}{dt} = & - \eta_T \left(x_2 \left(t \right) - x_2 \left(t - T \right) \right) \\
& \times \left(r x_1 \left(t - T \right) - x_2 \left(t - T \right) - x_1 \left(t - T \right) \right) \\
& \times x_3 \left(t - T \right) + u \left(t - T \right))
\end{aligned}
\tag{24}
$$

The time evolution of the period T is shown in Fig. 12; we can observe that period T converges to 2.4896 s, where the stationary period-one orbit is shown in Fig. 13. In Fig. 14, the time evolution of the control input is shown. It is clear from these figures that the stabilization of an unstable periodic orbit is fulfilled. However, for the case of the Lorenz system, with T unknown, the stabilization is not fulfilled in the optimal sense, since the energy consumption is not minimal.

6.   Conclusions

An on-line optimal control approach has been proposed to stabilize unstable periodic orbits embedded in chaotic systems. Such an approach strongly relies on the state sensitivities of the states with respect to both the control input and the delay period. It is shown via numerical experiments that the closed loop controller stabilizes the UPO embedded in systems with and without the odd number limitation. In the first case, the state sensitivities with respect to the control input are stable and periodic. In the second case, the state sensitivities are unstable; this fact agrees with theoretical studies previously performed by Pyragas [4].

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