On recovering the parametric model of the Chua system via a gradient algorithm

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The Chua circuit parameter estimation problem is addressed in this paper. This circuit is algebraically observable and identifiable with respect to its two measurable voltages. This fact allows us to straightforwardly propose two linear estimators for recovering the unknown parameters, where the estimator gains are continuously adjusted by means of a gradient algorithm, until the estimated parameters converge to the actual values. The convergence of this method is demonstrated by using the Lyapunov method.

Keywords: Chua's circuit; chaos; reconstruction and observers; Lyapunov's approach.

En este trabajo se trata el problema de estimación de los parámetros del circuito de Chua. Este circuito es algebraicamente observable e identificable con respecto a sus dos voltajes disponibles. Este hecho nos permite proponer directamente dos estimadores lineales para la recuperación de los parámetros desconocidos, donde las ganancias de los estimadores son ajustadas continuamente mediante un algoritmo de gradiente, hasta que los parámetros estimados convergen con los valores reales. La convergencia de este método es demostrada empleando el método de Lyapunov.

Descriptores: Circuito de Chua; caos; reconstrucción y observadores; enfoque de Lyapunov.

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1. Introduction

The reconstruction of a chaotic system from one or more measurable variables has attracted the attention of many researchers, because these kinds of systems have an enormous potential for applications. For example, they can be used in communication engineering to encode and decode information [1, 2]. Roughly speaking, the reconstruction problem consists in recovering the underlying variables and the unknown parameters from a partial knowledge of a chaotic system that we desire to reconstruct [3]. That is, we want to extract some physical parameters and to estimate some non-available states from the available system outputs. In general, there are two ways of identifying and reconstructing a chaotic system. The first one relies on the embedding approach and the second one is based on control theory. The embedding approach, supported by Taken's theorem [4], allows us to estimate the attractor characteristics of a chaotic system by unfolding its time series into a higher dimensional phase space, which facilitates the reconstruction of the attractor [4–7]. Topologically, the embedding problem consists in finding a one-to-one map between points of both the original system and the attractor in the reconstructed phase space. Then, embedding consists in finding the optimal mapping which, when applied to the observed time series, will map it to a higher dimensional space, revealing information about the original attractor (see Prasad *et al*. [6–8]). The second approach exploits some control theoretical ideas, such as inverse system design [9] and system identification [10]. The system inversion design consists in seeing the vector of unknown parameters as an external input and the available measurable signal as the output of the system. Then, the objective is to find an asymptotic inverse of this mapping. In addition, the problem can be solved by means of standard identification methods, mainly supported by the traditional least square methods and gradient algorithms (for a detailed treatment of these topics see [2, 3, 11–13]).

In this work, we recover the unknown set of parameters of the Chua system (**CS**) using the adaptative control approach and assuming that the voltages of the capacitors are available. To do so, we show that the system is algebraically observable and identifiable with respect to the well chosen outputs. Afterwards, the selected parameter estimation method is carried out by proposing two linear estimators, where their gains are adjusted according to a gradient algorithm [10, 14]. It is worth mentioning that this problem has been solved by other authors. In [15], the authors present an estimation strategy based on the embedding approach using time delayed outputs. They firstly have to build a map and, secondly, the parameters are obtained by computing the inverse of the proposed map, which is nearly singular. A similar work with similar tools was presented in [16]. In that work, the authors experimentally compute some of the unknown parameters by monitoring two variables of the **CS**. In [17], the authors solve the problem by using the traditional least squares method, assuming that all the states of the **CS** are available. In [18], a non-asymptotic linear estimator is presented based on the construction of a parameter-linear system. To justify it, the authors had to assume that the **CS** displays a chaotic behavior in order to apply the Poincaré-Bendixon theorem. Our identification strategy has the advantage of being very easy to implement numerically. We do not need to find the inverse maps, nor do we need to compute the inverse of a matrix (thus avoiding dealing with numerical singularities), as must be done when using the works mentioned above.

This paper is organized as follows. Section 2 briefly introduces some algebraic properties that the **CS** satisfied. Section 3 is devoted to establishing the framework of the proposed identification schema, which is based on a gradient algorithm. In the same section, we show how the robustness of the estimates with respect to zero-mean high frequency measurements of noisy outputs was enhanced using an *invariant-filter*. Finally, the conclusions are given in Sec. 4.

2. The Chua system

The Chua system, shown in Fig. 1, consists of three energy-store elements (an inductor and two capacitors), a linear resistor and a single nonlinear resistor, called Chua's diode. A simplified nonlinear model of this system, which can be derived from Kirchoff's laws (see [8] and [19] for details), is given by:

$$C_1\frac{dv_{c_1}}{dt} = \frac{1}{R}(v_{c_2} - v_{c1}) - \phi(v_{c_1}),$$
$$C_2\frac{dv_{c_2}}{dt} = \frac{1}{R}(v_{c_1} - v_{c2}) + i_l, \tag{1}$$
$$L\frac{di_l}{dt} = -v_{c_2},$$

where R is a linear resistance, v_{c_1} and v_{c_2} are the voltages across capacitors C_1 and C_2, respectively, i_l is the current through the inductor L, and $\phi(v_{c_1})$ is the current through the non-linear resistor as a function of the voltage across capacitor C_1. This non-linear function is described by an odd-symmetric piecewise-linear function made of three straight-line segments and which has the following explicit representation:

$$\phi(x) = -\Bigg(\overline{m}_1 v_{c_1} v_{c_1} + \frac{\overline{m}_0 - \overline{m}_1}{2}\left(|v_{c_1} + B_p| - |v_{c_1} - B_p|\right)\Bigg), \tag{2}$$

where $\overline{m}_0$, $\overline{m}_1$ and B_p are three fixed constants of the diode.

The three equations in (1) can be rewritten in the dimensionless form (see [20]), as:

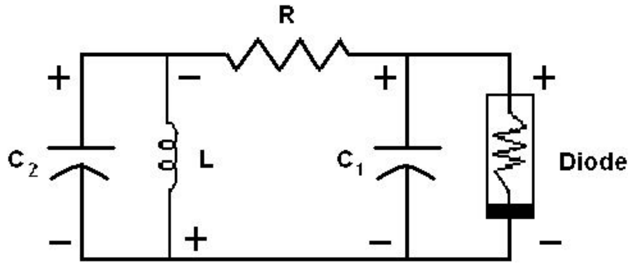

FIGURE 1. The Chua System.

$$\dot{x}_1 = \beta\left(-x_1 + x_2 - f(x_1)\right),$$
$$\dot{x}_2 = x_1 - x_2 + x_3, \tag{3}$$
$$\dot{x}_3 = -\gamma x_2,$$

with

$$f(x) = ax + b\left(|x+1| - |x-1|\right), \tag{4}$$

where

$$\beta = \frac{C_2}{C_1}, \quad \gamma = \frac{C_2 R^2}{L}, \quad a = \overline{m}_1 R, \quad b = \frac{\overline{m}_0 R - \overline{m}_1 R}{2},$$
$$x_1 = \frac{v_{c_1}}{B_p}, \quad x_2 = \frac{v_{c_2}}{B_p}, \quad x_3 = \frac{i_L R}{B_p}. \tag{5}$$

For the fixed values of parameters in a neighborhood of $\gamma = 27$, $\beta = 15.6$, $a = -5/7$, $b = -3/14$, we know that the **CS** has the so-called double scroll chaotic attractors.

Remark 1: Note that in other versions of the **CS** a resistor is added in the inductance, by adding to the right hand side of the third equation of (1) the voltage absorbed by the resistor $-R_0 i_L$, where R_0 is the resistant of the inductance element of the circuit. For simplicity, we assumed that $R_0 = 0$.

Comment 1: The **CS** is considered to be the standard paradigm of chaos and has been studied and applied by many researchers as a challenging benchmark to test advanced identification methods. On the other hand, the **CS** has the advantage of being one of the easiest chaotic systems to implement.

3. Problem statement

The main aim of this paper is to recover the set of unknown parameters β, γ, a and b, under the assumption that the two variables $x_1(t)$ and $x_2(t)$are available for all $t > 0$. That is, the two voltages of the circuit are monitored continuously.

Algebraic properties of the CS *Consider a smooth nonlinear system, described by a state vector $\mathbf{x} = \{x_i\}_1^{i=n} \in R^n$ and by the vector output $\mathbf{y} = \{y_i\}_1^{i=m} \in R^m$, of the form,*

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{p}), \quad \mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{6}$$

where $\mathbf{h}(\cdot)$ is a smooth vector and $\mathbf{p}^T \in R^l$ is a constant parameter vector, with $l < n$. We say that vector state x is algebraically observable if it can be uniquely expressed as

$$\mathbf{x} = \mathbf{s}(\mathbf{y}, .., \mathbf{y}^{(m)}, \mathbf{p}), \tag{7}$$

for some smooth function $\mathbf{s}$. *Moreover, if the vector of parameters* $\mathbf{p}$ *satisfies the following linealr relation*

$$\mathbf{s}_1(\mathbf{y},..,\mathbf{y}^{(m)}) = \mathbf{s}_2(\mathbf{y},..,\mathbf{y}^{(m)})\mathbf{p}, \tag{8}$$

where $\mathbf{s}_1(\cdot)$ *and* $\mathbf{s}_2(\cdot)$ *are respectively* $n\times 1$ *and* $n\times n$ *smooth matrices, then* $\mathbf{p}$ *is said to be algebraically linearly identifiable with respect to the output* $\mathbf{y}$ (see [21] for details).

Evidently, system (3) is algebraically observable with respect to the outputs $y_1 = x_1$ and $y_2 = x_2$, since all the system variables can be rewritten as

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \dot{y}_2 + y_2 - y_1. \tag{9}$$

Besides, from the first equation of (3), we easily have[i]

$$\dot{y}_1 = \Lambda^T(\overline{\mathbf{y}})\mathbf{q} \tag{10}$$

where

$$\begin{aligned} \mathbf{q}^T &= \left[\begin{array}{ccc} -\beta(1+a) & \beta & \beta b \end{array}\right], \\ \Lambda^T(\overline{\mathbf{y}}) &= \left[\begin{array}{ccc} y_1 & y_2 & |1-y_1| - |y_1+1| \end{array}\right]. \end{aligned} \tag{11}$$

In similarly form, the third equation of (3), leads to

$$\ddot{y}_2 = -\dot{y}_2 - \gamma y_2 + \dot{y}_1. \tag{12}$$

From the two differential relations given in (11) and (12), we claim that the nonlinear system (3) is linearly identifiable with respect to the outputs y_1 and y_2.

4. Model parameter estimation

We establish the framework for recovering the unknown parameters of the **CS.** Firstly, we analyze the hypothetical case where the signals $\dot{y}_1$, $\dot{y}_2$ and $\ddot{y}_2$ are available and noise free.

First of all, we need to introduce the following two assumptions:

A1 *The set of parameters* $\mathbf{q}$ *and* γ *belongs in some neighborhood, in such a way that all the states of* ***CS*** *remain oscillating around the origin, for all* $t > 0$.

A2 *The states* y_1, $\dot{y}_2$ and $\ddot{y}_2$ *are available or can be estimated with great accuracy.*

Remark 1: *Evidently signals* $\dot{y}_1$, $\dot{y}_2$*and* $\ddot{y}_2$ *must be estimated, to a high degree of accuracy. To compute these derivatives, we use the spline interpolant method proposed in [13]. This method consists in approximating a window of data (set of recorded data) by means of an interpolating polynomial, where the coefficients of the desired polynomial are computed according to the least square method. For instance, using a window of data set* $\{y_t, y_{t-\tau}, y_{t-2\tau}\}$, *where* τ *is the sampling time, it is easy to show that the first and second time derivative of* y *can be estimated by*

$$\begin{aligned} \dot{\widehat{y}} &= \frac{0.5y_{t-2\tau} - 2y_{t-\tau} + 1.5y_t}{\tau}, \\ \ddot{\widehat{y}} &= \frac{y_{t-2\tau} - 2y_{t-\tau} + y_t}{\tau^2}. \end{aligned} \tag{13}$$

4.1. Parameter estimations

Based on the two differential parametrization of the outputs (see (12 and (11)), we propose the following estimators

$$\begin{aligned} \dot{\widehat{y}}_1 &= \Lambda^T(\overline{\mathbf{y}})\widehat{\mathbf{q}}, \\ \ddot{\widehat{y}}_2 &= \widehat{\gamma} y_2 - \dot{y}_2 + \dot{y}_1, \end{aligned} \tag{14}$$

where $\widehat{\mathbf{q}}$ and $\widehat{\gamma}$ are the estimates of $\mathbf{q}$ and γ, respectively, and these are computed continuously according to

$$\begin{aligned} \frac{d}{dt}\widehat{\mathbf{q}} &= \frac{k_{p_1}\Lambda(\overline{\mathbf{y}})e_1}{k_{k_1} + \Lambda^T(\overline{\mathbf{y}})\Lambda(\overline{\mathbf{y}})}, \\ \frac{d}{dt}\widehat{\gamma} &= \frac{k_{p_2} y_2 e_2}{k_{d_2} + y_2^2}, \end{aligned} \tag{15}$$

e_1 and e_2 being the measurable errors given by

$$\begin{aligned} e_1 &= \dot{y}_1 - \Lambda^T(\overline{\mathbf{y}})\widehat{\mathbf{q}} = \Lambda^T(Y)(\mathbf{q} - \widehat{\mathbf{q}}) = \Lambda^T(\overline{\mathbf{y}})\widetilde{\mathbf{q}}; \\ e_2 &= \ddot{y}_2 - \ddot{\widehat{y}}_2 = (\gamma - \widehat{\gamma})y_2 = \widetilde{\gamma} y_2. \end{aligned} \tag{16}$$

Here k_{p_1}, k_{p_2}, k_{d_1} and k_{d_2} are strictly positive gains. Now, to show that the previous estimators converge to zero, we propose the following candidate Lyapunov function:

$$V(\widetilde{\mathbf{q}}, \widetilde{\gamma}) = \frac{1}{2}\widetilde{\mathbf{q}}^T\widetilde{\mathbf{q}} + \frac{1}{2}\widetilde{\gamma}^2. \tag{17}$$

Differentiating the proposed V now with respect to time along the trajectories of (15), this yields

$$\dot{V}(\widetilde{\mathbf{q}}, \widetilde{\gamma}) = -k_{p_1}\frac{\widetilde{\mathbf{q}}^T\Lambda(\overline{\mathbf{y}})e_1}{k_{d_1} + \Lambda^T(\overline{\mathbf{y}})\Lambda(\overline{\mathbf{y}})} - k_{p_2}\frac{\widetilde{\gamma} y_2 e_2}{k_{d_2} + y_2^2}. \tag{18}$$

Applying the definitions of e_1 and e_2, given in (16), it is easy to show that

$$\dot{V}(\widetilde{\mathbf{q}}, \widetilde{\gamma}) = \frac{-k_{p_1}e_1^2}{k_{d_1} + \Lambda^T(\overline{\mathbf{y}})\Lambda(\overline{\mathbf{y}})} - \frac{k_{p_2}e_2^2}{k_{d_2} + y_2^2}. \tag{19}$$

As $\dot{V}$ is semi-definite negative we guarantee that e_1 and e_2 are bounded. Now, to show that e_1 and e_2 converge to zero, as long as $t \to \infty$, we apply Barbalat's Lemma[ii]. Integrating both sides of (19) we obtain

$$\begin{aligned} &\bar{k}_1 \int e_1^2 ds + \bar{k}_2 \int e_2^2 ds \\ &\leq \int_0^t \left(\frac{k_{p_1}e_1^2}{k_{d_1} + \Lambda^T(\overline{\mathbf{y}})\Lambda(\overline{\mathbf{y}})} + \frac{k_{p_2}e_2^2}{k_{d_2} + y_2^2}\right) \leq V(0), \end{aligned} \tag{20}$$

where $\bar{k}_1$ and $\bar{k}_2$ are defined as

$$\begin{aligned} \bar{k}_1 &= \frac{k_{p_1}}{k_{d_1} + \max\limits_{0<s\leq t}\Lambda^T\bar{y}(s)\Lambda(\bar{y}(s))}, \\ \bar{k}_2 &= \frac{k_{p_2}}{k_{d_2} + \max\limits_{0<s\leq t} y_2^2(s)} \end{aligned} \tag{21}$$

Notice that $\bar{k}_1$ and $\bar{k}_2$ are well defined because states y_1 and y_2 are bounded. Therefore, e_1 and e_2 belong to L_2 space. Now, from Eq. (19) we conclude that $\dot{\widehat{\mathbf{q}}}$ and $\dot{\widehat{\gamma}}$ are bounded. Thus, from the Barbalat Lemma it follows that e_1 and e_2 converge to zero as long, as $t \to \infty$. We should recall that as y_2 is almost always different to zero, then clearly $\tilde{\gamma}$ converge to zero. Following similar arguments it is possible to show that $\lim_{t\to\infty} \Lambda^T(\bar{y})\widehat{\mathbf{q}} = 0$.

However, we cannot guarantee that $\widehat{\mathbf{q}}$ converges to zero. If we want to guarantee it, we need to impose a persistency of excitation condition on signals y_1 and y_2. And we did not do so because it is beyond the scope of this work.

To finish this section we establish the following proposition:

Proposition 1: *Consider the CS, given in (3), under assumptions A1 and A2. Then, the two proposed estimators (14) assure that*

$$\|\widehat{\mathbf{q}} - \mathbf{q}\| < \epsilon \quad \text{and} \quad \lim_{t\to\infty} \widehat{\gamma}(t) = \gamma \tag{22}$$

for the entire set of strictly positive constants k_{p_1}, k_{p_2}, k_{d_1} and k_{d_2}, where ϵ is a very small positive estimation constant (it depends on how persistent the available signals are). ■

Note: for a profound treatment on the topics of Persistency of Excitation and Barbalat's Lemma, we recommend books [12] and [14].

5. Numerical Simulations

To test the performance of the proposed method a digital simulation was carried out. In this simulation the step size integration and the sampling time were chosen equal to 0.001 and 0.005, respectively. The initial conditions were fixed as $x_{10} = -0.9, x_{20} = -0.15$, $x_{30} = 1.47$, $\widehat{q}_{10} = -4$, $\widehat{q}_{20}(0) = -4, \widehat{q}_{30} = -2$ and $\widehat{\gamma}_0 = 25^{iii}$. Finally, the design gains of the two estimators were given by

$$k_{p_1} = 10; \quad k_{d_1} = 1; \quad k_{p_2} = 2.5; \quad k_{d_1} = 0.5. \tag{23}$$

Figures 2 and 3 show comparison between the estimated and the actual values of the parameters. From these simulations, it is concluded that the proposed method reconstructs all the parameters after $t > 50$ [seconds] with the errors $|\widetilde{\mathbf{q}}|$ and $|\widetilde{\gamma}|$ close to 10^{-2} and 10^{-3}, respectively. We would expect it to be possible to obtain better parameter estimation for longer times. Besides, the estimation is improved by selecting a smaller step size integration and a smaller sampling time than the ones used in the previous simulation. Recall that from (11) we have $q_1 = -\beta(1+a)$, $q_2 = \beta$ and $q_3 = \beta b$.

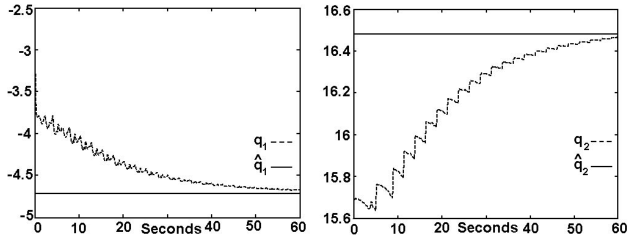

FIGURE 2. Estimates for parameters q_1 and q_2.

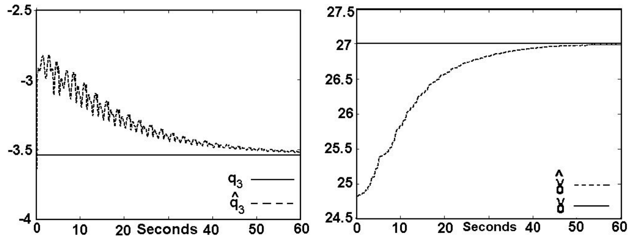

FIGURE 3. Estimates for parameters q_3 and γ.

6. Conclusions

We have proposed an estimation scheme for revealing the parameters of the **CS** on the basis of our knowledge of variables x_1 and x_2, which are the available voltages of the Chua circuit. The fact that the original system is algebraically observable and identifiable with respect to the defined outputs allows us to obtain two differential parameterizations of the outputs. Based on these parameterizations, two linear parametric estimators can be introduced to recover the desired parameters, where the gains of the estimators are continually adjusted by means of a gradient algorithm. The convergence analysis of the proposed identification method is tested by the Lyapunov method in conjunction with the Barbalat Lemma. The performance of the identification process has been illustrated with numerical simulations, where the unknown parameters were obtained with very low error.

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i Here $\overline{\mathbf{y}} = [y_1, y_2, y_1', y_2']$.

ii Lemma (Barbalat): If the differential function $f(t)$ has a finite limit as $t \to \infty$, and if $df/dt \to 0$ as $t \to \infty$. A consequence of this Lemma is that if $f \in L_2$ and df/dt is bounded then $f \to 0$ as $t \to \infty$.

iii The symbol x_0 denotes $x(0)$.

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