Dynamics of solitons of the nonlinear dispersion Drinfel'd-Sokolov system by ansatz method and He's varitional principle

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In this article, the exact-special solutions of the nonlinear dispersion Drinfel'd-Sokolov (shortly $D(m,n)$) system are analyzed. We use the ansatz approach and the He's variational principle for the mentioned equation. The general formulae for the compactons, solitary patterns, solitons and periodic solutions are acquired. These types of solutions are useful and attractive for clarifying some types of nonlinear physical phenomena. These two methods will be used to carry out the integration.

Keywords: Nonlinear dispersion $D(m;n)$ system; ansatz method; he's variational principle; compacton.

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1. Introduction

Nonlinear partial differential equations (NPDE) have been analyzed by different type of mathematical approach, among which include the Darboux transformation, the inverse scattering method, the Hirota method, the Backlund transformation, the tanh method, the sine-cosine method, the exp-function method, the variational iteration method, the homogenous balance method and among others [1-41].

In this article, the $D(m,n)$ system [1] are considered:

$$\begin{aligned} &u_t + (\nu^m)_x = 0, \\ &\nu_t + a(\nu^n)_{xxx} + bu_x\nu + cu\nu_x = 0. \end{aligned} \tag{1}$$

For $m = 2$ and $n = 1$, the system (1) is called "The normal Drinfel'd-Sokolov system"

$$\begin{aligned} &u_t + (\nu^2)_x = 0, \\ &\nu_t + a\nu_{xxx} + bu_x\nu + cu\nu_x = 0, \end{aligned} \tag{2}$$

where a, b, c are unchanged. The system (2) is considered as an example of a system of nonlinear equations possessing Lax pairs of a special form [13]. Wang obtained its Hamiltonian, recursion, symplectric and cosymplectric operators and roots of its symmetries and scaling symmetry of the system (2) [14].

For $n = 1$ the system (1) changes to "The generalized Drinfel'd-Sokolov system" as:

$$\begin{aligned} &u_t + (\nu^m)_x = 0, \\ &\nu_t + a\nu_{xxx} + bu_x\nu + cu\nu_x = 0. \end{aligned} \tag{3}$$

Wazwaz obtained some exact traveling wave solutions of the $D(m,n)$ system with compact and noncompact structures by applying the tanh method and the sine-cosine method [15].

We firstly apply the ansatz method [16-20] to obtain the exact special solutions to the $D(m,n)$ system. Then, we use the He's variational approach [21] to obtain unknown traveling wave solution to the subsidiaries of the $D(m,n)$ system. We give a comparison between the obtained solutions and those exist in the literature.

2. Ansatz method

We start by considering the solution of the equation

$$\left(\frac{dw}{dz}\right)^2 = a_0 - b_0 w^2, \tag{4}$$

where $a_0 \neq 0$ and $b_0 \neq 0$ are constants. When $b_0 > 0$, Eq. (4) admits two solutions as:

$$\begin{aligned} w_1 &= \pm\sqrt{\frac{a_0}{b_0}}\sin[\sqrt{b_0}(z+A)], \\ w_2 &= \pm\sqrt{\frac{a_0}{b_0}}\cos[\sqrt{b_0}(z+A)], \end{aligned} \tag{5}$$

where A is an arbitrary unchanged. If $b_0 < 0$, recognizing that $\cosh^2 z + \sinh^2 z = 1$ we know that Eq. (4) has two solutions of the form as:

$$\begin{aligned} w_1 &= \pm\sqrt{\frac{a_0}{b_0}}\sinh[\sqrt{b_0}(z+A)], \\ w_2 &= \pm i\sqrt{\frac{a_0}{b_0}}\cosh[\sqrt{b_0}(z+A)], \quad i^2 = -1. \end{aligned} \tag{6}$$

Secondly, we make a consideration of the solutions of the equation of the form

$$\left(\frac{dw}{dz}\right)^2 = w^2(c_0 + d_0 w^2) \tag{7}$$

where $c_0 \neq 0$ and $d_0 \neq 0$ are constants. If $c_0 < 0$, Eq. (7) confesses two solutions

$$w_5 = \pm\sqrt{\frac{c_0}{d_0}}\sec[\sqrt{-c_0}(z)],$$

$$w_6 = \pm\sqrt{\frac{a_0}{b_0}}\csc[\sqrt{-c_0}(z)]. \tag{8}$$

If $c_0 > 0$ in Eq. (7), then the equation offers two solutions of the form

$$w_7 = \pm\sqrt{\frac{c_0}{d_0}}\,\mathrm{sech}[\sqrt{-c_0}(z)],$$

$$w_6 = \pm i\sqrt{\frac{a_0}{b_0}}\,\mathrm{csch}[\sqrt{-c_0}(z)]. \tag{9}$$

2.1. Nonlinear Dispersion $D(m,n)$ System

We assume that the traveling wave solution has the form $u(x,t) = u(\xi)$ with wave variable $\xi = k(x-\lambda t)$, $(k, \lambda \neq 0)$. Then, we get the following ordinary differential equation:

$$-k\lambda u' + k(\nu^m)' = 0, \tag{10}$$

$$-k\lambda\nu' + ak^3(\nu^n)''' + bku'\nu + cku\nu' = 0. \tag{11}$$

We get

$$u = \frac{1}{\lambda}\nu^m + c_1, \tag{12}$$

by (11), where c_1 is arbitrary constant. Substituting (12) into (11) we obtain

$$-\lambda\nu' + ak^2(\nu^n)^m + b\left(\frac{m}{\lambda}\nu^{m-1}\nu'\right)\nu$$

$$+ c\left(\frac{1}{\lambda}\nu^m + c_1\right)\nu' = 0. \tag{13}$$

By integrating Eq. (13), we get

$$(cc_1 - \lambda)\nu + ak^2(\nu^n)'' + \frac{bm+c}{\lambda(m+1)}\nu^{m+1} = c_2. \tag{14}$$

where c_2 is integration constant.

Case 1. When $c = -bm = \lambda/c_1$, the nonlinear ODE (14) becomes

$$ak^2(\nu^n)'' - c_2 = 0. \tag{15}$$

Therefore, we get the rational solution of Eq. (15),

$$\nu(x,t) = \left\{\frac{c_2}{2ak^2}(x-\lambda t)^2 + c_3(x-\lambda t) + c_4\right\}^{\frac{1}{n}}, \tag{16}$$

and

$$u(x,t) = \frac{1}{\lambda}\left\{\frac{c_2}{2ak^2}(x-\lambda t)^2 + c_3(x-\lambda t) + c_4\right\}^{\frac{m}{n}} + c_1, \tag{17}$$

where c_3 and c_4 are arbitrary constants.

Case 2. If we take $m = n-1$ and $c = \lambda/c_1$ in Eq. (14), the following traveling wave solutions are obtained

$$(\nu^n)'' = -\frac{bm+c}{ak^2\lambda(m+1)}\nu^n + \frac{c_2}{ak^2}.$$

$$\nu(x,t) = \left\{\frac{c_2\lambda(m+1)}{bm+c} + c_5\sin\left(\frac{1}{k}\sqrt{\frac{bm+c}{a\lambda(m+1)}}(x-\lambda t)\right) + c_6\cos\left(\frac{1}{k}\sqrt{\frac{bm+c}{a\lambda(m+1)}}(x-\lambda t)\right)\right\}^{\frac{1}{n}} \tag{18}$$

and

$$u(x,t) = \frac{1}{\lambda}\left\{\frac{c_2\lambda(m+1)}{bm+c} + c_5\sin\left(\frac{1}{k}\sqrt{\frac{bm+c}{a\lambda(m+1)}}(x-\lambda t)\right) + c_6\cos\left(\frac{1}{k}\sqrt{\frac{bm+c}{a\lambda(m+1)}}(x-\lambda t)\right)\right\}^{\frac{m}{n}} + c_1 \tag{19}$$

where c_5 and c_6 arbitrary constants. In view of (18) and (19), we clearly see that these solutions exist provided that $(bm+c)/(a\lambda(m+1)) > 0$.

$$\nu(x,t) = \left\{-\frac{c_2\lambda(m+1)}{bm+c} + c_7\sinh\left(\frac{1}{k}\sqrt{-\frac{bm+c}{a\lambda(m+1)}}(x-\lambda t)\right) + c_8\cosh\left(\frac{1}{k}\sqrt{-\frac{bm+c}{a\lambda(m+1)}}(x-\lambda t)\right)\right\}^{\frac{1}{n}} \tag{20}$$

and

$$u(x,t) = \frac{1}{\lambda}\left\{-\frac{c_2\lambda(m+1)}{bm+c} + c_7\sin\left(\frac{1}{k}\sqrt{\frac{bm+c}{a\lambda(m+1)}}(x-\lambda t)\right) + c_8\cosh\left(\frac{1}{k}\sqrt{-\frac{bm+c}{a\lambda(m+1)}}(x-\lambda t)\right)\right\}^{\frac{m}{n}} + c_1 \tag{21}$$

where c_7 and c_8 arbitrary constants. In view of (20) and (21), we clearly see that these solutions exist provided that $(bm+c)/(a\lambda(m+1)) < 0$.

Case 3. $c + bm \neq 0$, $cc_1 - \lambda \neq 0$ and specially $c_2 = 0$.

Let

$$\frac{d\nu^n}{d\xi} = z, \quad \frac{d^2\nu^n}{d\xi^2} = z\frac{dz}{d\nu^n}. \tag{22}$$

Substituting (22) into (14) leads to the following equation

$$ak^2\left(n\nu^{\frac{n-3}{2}}\frac{d\nu}{d\xi}\right)^2 = \frac{-2n(cc_1-\lambda)}{n+1} - \frac{2n(bm+c)}{\lambda(m+1)(m+n+1)}\nu^m. \tag{23}$$

Letting $\nu = w^{2/m}$, we have

$$\nu = w^{\frac{2}{m}} \Rightarrow dv = \frac{2}{m}w^{\frac{2}{m}-1}dw, \tag{24}$$

which changes Eq. (23) to

$$\frac{4ak^2n^2}{m^2}\left(w^{\frac{(n-m-1)}{m}}\frac{dw}{d\xi}\right)^2 = \frac{-2n(cc_1-\lambda)}{n+1} - \frac{2n(bm+c)}{\lambda(m+1)(m+n+1)}w^2. \tag{25}$$

If we take $n = 2m+1$ in Eq. (25), we get the algebraic traveling wave solution of the form:

$$\nu(\xi) = \left[\frac{1}{B}(A - B^2\xi^2 - 2B^2\xi C - B^2C^2)\right]^{\frac{1}{m}}, \tag{26}$$

and

$$u(\xi) = \frac{1}{\lambda B}(A - B^2\xi^2 - 2B^2\xi C - B^2C^2) + c_1, \tag{27}$$

where

$$A = \frac{-(cc_1-\lambda)m^2}{4ak^2(m+1)(2m+1)}$$

and

$$B = \frac{m^2(bm+c)}{2ak^2\lambda(m+1)(3m+2)(2m+1)}.$$

Case 4. If $m = n-1$, we know that Eq. (25) becomes

$$\left(\frac{dw}{d\xi}\right)^2 = \frac{(n-1)^2}{4ak^2n^2} \times \left[\frac{-2n(cc_1-\lambda)}{n+1} - \frac{2(b(n-1)+c)}{\lambda n^2}w^2\right], \tag{28}$$

where $n \neq 1$ and $a, k, n, \lambda \neq 0$.

If we take $an\lambda(c + bn - b) > 0$, then we acquire from Eqs. (5) and (28)

$$\nu(x,t) = \left\{\frac{2n^2\lambda(\lambda - cc_1)}{(n+1)(bn-b+c)}\sin^2 \times \left[\frac{n-1}{2\mid kn\mid}\sqrt{\frac{bn-b+c}{\lambda an}}(\xi + A)\right]\right\}^{\frac{1}{n-1}},$$

$$u(x,t) = \frac{2n^2(\lambda - cc_1)}{(n+1)(bn-b+c)}\sin^2 \times \left[\frac{n-1}{2\mid kn\mid}\sqrt{\frac{bn-b+c}{\lambda an}}(\xi + A)\right] + c_1, \tag{29}$$

and

$$\nu(x,t) = \left\{\frac{2n^2\lambda(\lambda - cc_1)}{(n+1)(bn-b+c)}\cos^2 \times \left[\frac{n-1}{2\mid kn\mid}\sqrt{\frac{bn-b+c}{\lambda an}}(\xi + A)\right]\right\}^{\frac{1}{n-1}},$$

$$u(x,t) = \frac{2n^2(\lambda - cc_1)}{(n+1)(bn-b+c)}\cos^2 \times \left[\frac{n-1}{2\mid kn\mid}\sqrt{\frac{bn-b+c}{\lambda an}}(\xi + A)\right] + c_1. \tag{30}$$

Theorem 1. The $D(m,n)$ system has solutions in Eq. (28) described as follows:

1. When $an\lambda(c + bn - b) > 0$,

$$\nu = \begin{cases} \left\{\frac{2n^2\lambda(\lambda - cc_1)}{(n+1)(bn-n+c)}\cos^2\left[\frac{n-1}{2\mid kn\mid}\sqrt{\frac{bn-b+c}{\lambda an}}(\xi + A)\right]\right\}^{\frac{1}{n-1}}, & \mid\sqrt{b_0}(\xi)\mid \leq \frac{\pi}{2}, \\ 0, & \text{Otherwise} \end{cases} \tag{31}$$

is a solitary wave solution with compact support.

2. When $an\lambda(c + bn - b) > 0$,

$$\nu = \begin{cases} \left\{\frac{2n^2\lambda(\lambda - cc_1)}{(n+1)(bn-n+c)}\cos^2\left[\frac{n-1}{2\mid kn\mid}\sqrt{\frac{bn-b+c}{\lambda an}}(\xi + A)\right]\right\}^{\frac{1}{n-1}}, & 0 \leq\mid\sqrt{b_0}(\xi)\mid \leq \pi, \\ 0, & \text{Otherwise} \end{cases} \tag{32}$$

is a compacton solution for Eq. (1) and

$$\sqrt{b_0} = \frac{n-1}{2\mid kn\mid}\sqrt{\frac{bn-b+c}{\lambda an}}.$$

3. Equation (25) can be written as following

$$\left(\frac{dw}{d\xi}\right)^2 = (w^2)^{\frac{m-n+1}{m}}\left\{\frac{-2m^2(cc_1-\lambda)}{4ak^2n(n+1)} - \frac{2m^2(bm+c)}{4ak^2n\lambda(m+1)(m+n+1)}w^2\right\}. \quad (33)$$

If $n=1$ in Eq. (33) that yields as

$$\left(\frac{dw}{d\xi}\right)^2 = (w^2)\left\{\frac{-m^2(cc_1-\lambda)}{4ak^2} + \frac{-m^2(bm+c)}{2ak^2\lambda(m+1)(m+2)}w^2\right\}. \quad (34)$$

When $a(cc_1-\lambda)<0$,

$$\nu(x,t) = \left(-\frac{2(bm+c)}{\lambda(cc_1-\lambda)(m+1)(m+2)}\operatorname{csch}^2\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right]\right)^{-\frac{1}{m}},$$

$$u(x,t) = -\frac{2(bm+c)}{\lambda^2(cc_1-\lambda)(m+1)(m+2)}\operatorname{csch}^{-2}\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right] + c_1, \quad (35)$$

which is a singular soliton solution for the $D(m,n)$ equation for

$$0<\xi<\frac{\pi m}{2k}\sqrt{\frac{(cc_1-\lambda)}{a}}$$

4. When $a(cc_1-\lambda)>0$ and $m<0$,

$$\nu(x,t) = \left\{-\frac{2(bm+c)}{\lambda(cc_1-\lambda)(m+1)(m+2)}\sec^2\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right]\right\}^{-\frac{1}{m}},$$

$$u(x,t) = -\frac{2(bm+c)}{\lambda^2(cc_1-\lambda)(m+1)(m+2)}\sec^{-2}\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right] + c_1, \quad (36)$$

is a traveling wave solution for the $D(m,n)$ equation for

$$\xi<\frac{\pi m}{4k}\sqrt{\frac{(cc_1-\lambda)}{a}}.$$

Remark 1. If $(bn-b-c)/(\lambda an) < 0$, it follows from (6) and (28) that

$$\nu(x,t) = \left\{-\frac{2n^2\lambda(\lambda-cc_1)}{(n+1)(bn-b+c)}\sinh^2\left[\frac{n-1}{2\mid nk\mid}\sqrt{-\frac{bn-b+c}{\lambda an}}(\xi+A)\right]\right\}^{\frac{1}{n-1}},$$

$$u(x,t) = -\frac{2n^2(\lambda-cc_1)}{(n+1)(bn-b+c)}\sinh^2\left[\frac{n-1}{2\mid nk\mid}\sqrt{-\frac{bn-b+c}{\lambda an}}(\xi+A)\right] + c_1, \quad (37)$$

and

$$\nu(x,t) = \left\{\frac{2n^2\lambda(\lambda-cc_1)}{(n+1)(bn-b+c)}\cosh^2\left[\frac{n-1}{2\mid nk\mid}\sqrt{-\frac{bn-b+c}{\lambda an}}(\xi+A)\right]\right\}^{\frac{1}{n-1}},$$

$$u(x,t) = \frac{2n^2(\lambda-cc_1)}{(n+1)(bn-b+c)}\cosh^2\left[\frac{n-1}{2\mid nk\mid}\sqrt{-\frac{bn-b+c}{\lambda an}}(\xi+A)\right] + c_1, \quad (38)$$

Theorem 2. The $D(m,n)$ equation with when $m=n$ equation has the following solutions:

1. When $(c+bn-b)<0$, $an\lambda>0$, $(\lambda-cc_1)(n+1)<0$ and $n\neq 1$

$$\nu(x,t) = \left\{-\frac{2n^2\lambda(\lambda-cc_1)}{(n+1)(bn-b+c)}\sinh^2\left[\frac{n-1}{2\mid nk\mid}\sqrt{-\frac{bn-b+c}{\lambda an}}(\xi+A)\right]\right\}^{\frac{1}{n-1}},$$

$$u(x,t) = -\frac{2n^2(\lambda-cc_1)}{(n+1)(bn-b+c)}\sinh^2\left[\frac{n-1}{2\mid nk\mid}\sqrt{-\frac{bn-b+c}{\lambda an}}(\xi+A)\right] + c_1, \quad (39)$$

is a solitary solution of Eq. (1).

2. When $(c+bn-b)<0$, $an\lambda<0$, $(\lambda-cc_1)(n+1)>0$

and $n \neq 1$

$$\nu(x,t) = \left\{ \frac{2n^2\lambda(\lambda - cc_1)}{(n+1)(bn-b+c)} \cosh^2 \left[\frac{n-1}{2 \mid nk \mid} \sqrt{-\frac{bn-b+c}{\lambda an}} (\xi + A) \right] \right\}^{\frac{1}{n-1}},$$
$$u(x,t) = \frac{2n^2(\lambda - cc_1)}{(n+1)(bn-b+c)} \cosh^2 \left[\frac{n-1}{2 \mid nk \mid} \sqrt{-\frac{bn-b+c}{\lambda an}} (\xi + A) \right] + c_1, \quad (40)$$

is a solitary solution of Eq. (1). If we take $n < 1$, then Eq. (40) is a bounded solution.

3. $m = n < 1$ solutions (39) and (40) turns to solitary wave solutions

$$\nu(x,t) = \left\{ \frac{(n+1)(bn-b+c)}{2n^2\lambda(\lambda - cc_1)} \csc^2 \left[\frac{n-1}{2 \mid nk \mid} \sqrt{-\frac{bn-b+c}{\lambda an}} (\xi) \right] \right\}^{\frac{1}{n-1}},$$
$$u(x,t) = \frac{(n+1)(bn-b+c)}{2n^2(\lambda - cc_1)} \csc^2 \left[\frac{n-1}{2 \mid nk \mid} \sqrt{-\frac{bn-b+c}{\lambda an}} (\xi) \right] + c_1, \quad (41)$$

and

$$\nu(x,t) = \left\{ \frac{(n+1)(bn-b+c)}{2n^2\lambda(\lambda - cc_1)} \sec^2 \left[\frac{n-1}{2 \mid nk \mid} \sqrt{-\frac{bn-b+c}{\lambda an}} (\xi) \right] \right\}^{\frac{1}{n-1}},$$
$$u(x,t) = \frac{(n+1)(bn-b+c)}{2n^2(\lambda - cc_1)} \sec^2 \left[\frac{n-1}{2 \mid nk \mid} \sqrt{-\frac{bn-b+c}{\lambda an}} (\xi) \right] + c_1. \quad (42)$$

Case 5. $a(cc_1 - \lambda) > 0$ and $m > 0$

Thus, by using (8) and (34), the periodic solutions of Eq. (1) are obtained as:

$$\nu(x,t) = \left\{ -\frac{\lambda(m+1)(m+2)(cc_1 - \lambda)}{bm+c} \csc^2 \left[\frac{m}{2 \mid k \mid} \sqrt{\frac{cc_1 - \lambda}{a}} (\xi) \right] \right\}^{\frac{1}{m}},$$
$$u(x,t) = -\frac{(m+1)(m+2)(cc_1 - \lambda)}{bm+c} \csc^2 \left[\frac{m}{2 \mid k \mid} \sqrt{\frac{cc_1 - \lambda}{a}} (\xi) \right] + c_1. \quad (43)$$

and

$$\nu(x,t) = \left\{ -\frac{\lambda(m+1)(m+2)(cc_1 - \lambda)}{bm+c} \sec^2 \left[\frac{m}{2 \mid k \mid} \sqrt{\frac{cc_1 - \lambda}{a}} (\xi) \right] \right\}^{\frac{1}{m}},$$
$$u(x,t) = -\frac{(m+1)(m+2)(cc_1 - \lambda)}{bm+c} \sec^2 \left[\frac{m}{2 \mid k \mid} \sqrt{\frac{cc_1 - \lambda}{a}} (\xi) \right] + c_1. \quad (44)$$

Case 6. $a(cc_1 - \lambda) < 0$ and $m > 0$

Therefore, by considering (9) and (34), solitary pattern and bell-shaped solitary wave solutions of (1) are obtained as:

$$\nu(x,t) = \left\{ -\frac{\lambda(m+1)(m+2)(cc_1 - \lambda)}{bm+c} \cosh^2 \left[\frac{m}{2 \mid k \mid} \sqrt{\frac{cc_1 - \lambda}{a}} (\xi) \right] \right\}^{\frac{1}{m}},$$
$$u(x,t) = -\frac{(m+1)(m+2)(cc_1 - \lambda)}{bm+c} \cosh^2 \left[\frac{m}{2 \mid k \mid} \sqrt{\frac{cc_1 - \lambda}{a}} (\xi) \right] + c_1. \quad (45)$$

and

$$\nu(x,t) = \left\{ -\frac{\lambda(m+1)(m+2)(cc_1 - \lambda)}{bm+c} \operatorname{sech}^2 \left[\frac{m}{2 \mid k \mid} \sqrt{\frac{cc_1 - \lambda}{a}} (\xi) \right] \right\}^{\frac{1}{m}},$$
$$u(x,t) = -\frac{(m+1)(m+2)(cc_1 - \lambda)}{bm+c} \operatorname{sech}^2 \left[\frac{m}{2 \mid k \mid} \sqrt{\frac{cc_1 - \lambda}{a}} (\xi) \right] + c_1, \quad (46)$$

Case 7. $a(cc_1-\lambda)<0$, $m<0$ and $m\neq 0$.

Using the solutions of (43) and (44), gives the following compacton solutions as:

$$\nu(x,t)=\left\{-\frac{bm+c}{\lambda(m+1)(m+2)(cc_1-\lambda)}\cos^2\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right]\right\}^{-\frac{1}{m}},$$
$$u(x,t)=\frac{bm+c}{\lambda^2(m+1)(m+2)(cc_1-\lambda)}\cos^{-2}\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right]+c_1, \tag{47}$$

for

$$\left|\frac{\pi m}{2k}\sqrt{\frac{(cc_1-\lambda)}{a}}\right|\leq\frac{\pi}{2}$$

and $u=0$, otherwise.

$$\nu(x,t)=\left\{-\frac{\lambda(m+1)(m+2)(cc_1-\lambda)}{bm+c}\sin^2\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right]\right\}^{\frac{1}{m}},$$
$$u(x,t)=\frac{(m+1)(m+2)(cc_1-\lambda)}{bm+c}\sin^{-2}\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right]+c_1, \tag{48}$$

for

$$0\leq\left|\frac{\pi m}{2k}\sqrt{\frac{(cc_1-\lambda)}{a}}\right|\leq\pi$$

and $u=0$, otherwise.

Case 8. $a(cc_1-\lambda)>0$, $m<0$ and $m\neq 1$.

Using $\cosh(x)=\cos(ix)$ and $\sinh(x)=-\sin(ix)$ we have the following solitary pattern solutions of Eq. (1):

$$\nu(x,t)=\left\{-\frac{bm+c}{\lambda(m+1)(m+2)(cc_1-\lambda)}\cosh^2\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda)}{a}}(\xi)\right]\right\}^{-\frac{1}{m}},$$
$$u(x,t)=\frac{bm+c}{\lambda^2(m+1)(m+2)(cc_1-\lambda)}\cosh^{-2}\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right]+c_1, \tag{49}$$

and

$$\nu(x,t)=\left\{-\frac{\lambda(m+1)(m+2)(cc_1-\lambda)}{bm+c}\sinh^2\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right]\right\}^{\frac{1}{m}},$$
$$u(x,t)=\frac{(m+1)(m+2)(cc_1-\lambda)}{bm+c}\sinh^2\left[\frac{m}{2\mid k\mid}\sqrt{\frac{cc_1-\lambda}{a}}(\xi)\right]+c_1, \tag{50}$$

Remark 2. When $m=n<1$, the obtained solution (42) agrees with the outcomes (2.13a), (2.13b) in [36] and (25) in [37]. The solution (42) changes to the compacton solution (2.12a) and the periodic solution (2.12b) in [36,37].

If $m>0$, the obtained solution (46) is consented with the outcomes (3.18a) and (3.18b) described in [36] and (26) in [37]. The solution (46) changes to the solitary pattern solution (3.17a) and solitary wave solution (3.17b) in [36].

Remark 3. If $a(cc_1-\lambda)<0$, the obtained solution (35) agrees with the outcomes (3.9a) and (3.9b) in [36] and (32) in [37]. The solution (35) changes to the singular solitary wave solution.

Remark 4. If we take $(c+bn-b)<0$, $an\lambda>0$, $(\lambda-cc_1)(n+1)<0$ then the obtained solution (39) similar to the solitary pattern solutions (3.7a) and (3.7b) in [36] and (46) in [37].

3. Variational principle

In this Section, He's variational principle will be applied to the system (1). This technique was first proposed by He [21] and it is popularly known as He's semi-inverse variational principle. Some years back, it was applied mainly to extract soliton solutions of nonlinear PDEs and systems by many authors [16-24]. Biswas and co-workers [17-20] obtained optical solitons and soliton solutions with higher order dispersion by applying the He's variational principle. Xu and Zhang's [25] used a variational principle to construct catalytic reactions in short monoliths by He's semi-inverse approach. He's variational method was used to the effective nonlinear oscillators with high nonlinearity by Liu [26]. Zheng *et al*. [27] established a class of generalized variational principles for the initial-boundary value problem of micromorphic magneto electrodynamics by He's semi-inverse technique. In order to seek traveling wave solutions of the system (1). We consider

$$(cc_1-\lambda)\nu+ak^2(\nu^n)''+\frac{bm+c}{\lambda(m+1)}\nu^{m+1}=c_2. \tag{51}$$

Let $\nu^n = V$

$$J = \int_{\infty}^{\infty} \left[\frac{n(cc_1 - \lambda)}{n+1} V^{\frac{n+1}{n}} - \frac{ak^2}{2}(V')^2 + \frac{n(bm+c)}{\lambda(m+1)(m+n+1)} V^{\frac{m+n+1}{n}} \right] d\xi, \tag{52}$$

the 1-soliton solution ansatz, given by

$$V(\xi) = \{p \operatorname{sech}^2(q\xi)\}^{\frac{1}{n}}, \tag{53}$$

is substituted into (51). Here, in (53), the parameters p and q represent the amplitude and inverse width of the soliton, respectively.

$$J = \int_{-\infty}^{\infty} \{p_1(p \operatorname{sech}^2[q\xi] + 2p_2 pq^2(-2 + \cosh[2q\xi]) \times \operatorname{sech}^4[q\xi] + p_3(p \operatorname{sech}^2[q\xi])^{(m+1)}\} d\xi, \tag{54}$$

where $p_1 = cc_1 - \lambda$, $p_2 = ak^2$ and

$$p_3 = \frac{bm+c}{\lambda(m+1)}.$$

From the above equation it is obtained as

$$J = \frac{8p}{15q} \Bigg\{ 2pq^2(-cc_1 + \lambda) - \frac{154^{1+m}(c+bm)p^{1+m}}{(1+m)(2+m)^2\lambda} \times F[2+m, 2(2+m), 3+m, -1] + \frac{154^n ak^2 p^n}{(1+n)^2} F[1+n, 2(1+n), 2+n, -1] \Bigg\}, \tag{55}$$

where F is Gauss' hypergeometric function defined as

$$F[\alpha, \beta, \gamma, z] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{z^n}{n!} \tag{56}$$

and Re$[(2+m)q] > 0$, Re$[(1+n)q] > 0$, Re$[q] > 0$.

Making J stationary with respect to p and q results in

$$\frac{dJ}{dp} = -\frac{8}{15q\lambda(1+m)(2+m)(1+n)} 154^{(l+m)} \times (c+bm)(1+n)p^{(l+m)} \times F[2+m, 2(2+m), 3+m, -1] + \lambda(2+3m+m^2)(4(1+n)pq^2(cc_1 - \lambda) - 154^n ak^2 p^n F[1+n, 2(1+n), 2+n, -1]) = 0, \tag{57}$$

$$\frac{dJ}{dp} = -\frac{8}{15} p \Bigg\{ 2p(-cc_1 + \lambda) + \frac{154^{l+m}(c+bm)p^{l+m}}{(1+m)(2+m)^2 q^2 \lambda} \times F[2+m, 2(2+m), 3+m, -1] - \frac{154^n ak^2 p^n}{(1+n)^2 q^2} F[1+n, 2(1+n), 2+n, -1] \Bigg\} = 0. \tag{58}$$

Solving Eqs. (57) and (58) for $m = n = 2$ simultaneously, we get

$$p = \frac{35ak^2\lambda}{9(2b+c)}, \quad q = \frac{k}{3}\sqrt{\frac{ap}{\lambda - cc_1}}, \tag{59}$$

Therefore, by substituting p and q in (52) we have the following a new solitary wave solution for the system (1) as:

$$\nu(x,t) = \left\{ \frac{35ak^2\lambda}{9(2b+c)} \operatorname{sech}^2 \times \left(\frac{k}{3}\sqrt{\frac{ap}{\lambda - cc_1}} [k(x - \lambda t)] \right) \right\}^{\frac{1}{n}},$$

$$u(x,t) = \frac{35ak^2\lambda}{9(2b+c)} \operatorname{sech}^{-2} \times \left(\frac{k}{3}\sqrt{\frac{ap}{\lambda - cc_1}} [k(x - \lambda t)] \right) + c_1. \tag{60}$$

So, the solitary wave solution (60) will exist for $ap(\lambda - cc_1) > 0$.

4. Results and Discussions

In this article, we investigated the nonlinear dispersion $D(m,n)$ system and obtained some traveling wave solutions by applying the ansatz technique and the He's variational principle. Several forms of solutions including topological, non-topological, compacton, solitary pattern, singular soliton, algebraic and periodic wave solutions were acquired. The approaches can be used to a lot of other nonlinear differential equations and coupled systems. Some new obtained exact solutions were previously unknown by other methods. We proved the existence of these solutions for a generalized form of the $D(m,n)$ system under specific conditions. In general, the outcome expose that the ansatz approach and the He's variational principle are important mathematical techniques for solving nonlinear partial differential equations in terms of correctness and ability to avoid errors.

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Conflict of Interests

The authors declare that they have no conflict of interests.

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