Making hidden symmetries obvious

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It is shown that the Hamiltonian of a particle in a uniform gravitational field which possesses a constant of motion not related to transformations in the configuration space, can be expressed in a system of canonical coordinates such that a maximal set of independent constants of motion follows from the existence of ignorable coordinates.

Keywords: Hidden symmetries; Hamiltonian formalism.

Se muestra que la hamiltoniana de una partícula en un campo gravitacional uniforme, la cual posee una constante de movimiento no relacionada con transformaciones en el espacio de configuración, puede expresarse en un sistema de coordenadas canónicas tal que un conjunto máximo de constantes de movimiento sigue de la existencia de coordenadas ignorables.

Descriptores: Simetrías ocultas; formalismo hamiltoniano.

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1. Introduction

The invariance of the Lagrangian of a mechanical system under continuous transformations of the configuration space allows one to readily find constants of motion. However, in some cases, there exist constants of motion not related to symmetries in the configuration space. Two well-known examples are the isotropic harmonic oscillator and the Kepler problem; in both cases, in addition to the angular momentum, whose conservation follows from the invariance of the standard Lagrangian under rotations about the center of force, there exist constants of motion that are not associated with symmetries of the Lagrangian.

It may be remarked that, for a given set of equations of motion, the Lagrangian is not unique (see, *e.g.*, Ref. 1) and the symmetries of a Lagrangian may not be shared by the alternative Lagrangians. For example, the equations of motion $\ddot{x} = 0$, $\ddot{y} = -g$, considered in this paper, can be obtained from the Euler–Lagrange equations making use of the usual Lagrangian, given by Eq. (1) below, or the function $L' = m\dot{x}\dot{y} - mgx$; in this case one Lagrangian does not depend on x and the other does not depend on y.

On the other hand, in the Hamiltonian formalism, every constant of motion is associated with a symmetry of the Hamiltonian function; in fact, each constant of motion *is* the infinitesimal generator of a one-parameter group of canonical transformations that leave the Hamiltonian invariant. The corresponding symmetry of the Hamiltonian may not be obvious since these transformations may mix coordinates and momenta.

In this paper we consider a particle in a uniform gravitational field, which is an example of a system with a hidden symmetry. We find a set of canonical coordinates for which the existence of two constants of motion, in addition to the Hamiltonian, follows from the obvious symmetries of the Hamiltonian. This coordinate system is obtained by looking for the orbits of the transformations generated by the constants of motion in the phase space. A similar procedure can be applied in other examples to make the hidden symmetries obvious. The main results of this paper are presented in Sec. 2, and in Sec. 3 we show how the new coordinates are obtained.

2. The canonical transformation

The usual Lagrangian for a particle in a uniform gravitational field, written in Cartesian coordinates, is given by

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy, \tag{1}$$

where m is the mass of the particle and g is the acceleration of gravity. Since the coordinate x is ignorable, $p_x = m\dot{x}$ is conserved. Making use of the equation of motion $\ddot{y} = -g$, a straightforward computation shows that $\dot{x}y + gx$ is also conserved, but the fact that this constant of motion is not a homogeneous function of degree 1 in the components of the linear momentum implies that its existence is not associated with the invariance of the Lagrangian (1) under a continuous set of transformations on the configuration space (see, *e.g.*, Refs. 2 and 3). In other words, besides the obvious invariance of L under translations along the x-axis, L possesses a hidden symmetry.

Turning to the Hamiltonian formalism, the standard procedure applied to the Lagrangian (1) leads to the Hamiltonian function

$$H = \frac{p_x^2 + p_y^2}{2m} + mgy. \tag{2}$$

It can be readily verified that the coordinate transformation

$$z = x + \frac{p_x p_y}{m^2 g}, \qquad p_z = p_x,$$

$$w = y + \frac{p_x^2}{2m^2 g}, \qquad p_w = p_y \qquad (3)$$

is canonical; in fact, making use of Eqs. (3) one finds that

$$p_z \mathrm{d}z + p_w \mathrm{d}w = p_x \left(\mathrm{d}x + \frac{p_y}{m^2 g} \mathrm{d}p_x + \frac{p_x}{m^2 g} \mathrm{d}p_y \right)$$

$$+ p_y \left(\mathrm{d}y + \frac{p_x}{m^2 g} \mathrm{d}p_x \right)$$

$$= p_x \mathrm{d}x + p_y \mathrm{d}y + \mathrm{d} \left(\frac{p_x^2 p_y}{m^2 g} \right).$$

In terms of the new coordinates, the Hamiltonian (2) takes the form

$$H = \frac{p_w^2}{2m} + mgw, \qquad (4)$$

which is a function of w and p_w only; hence, z and p_z are constants of motion. Thus, the conservation of p_x and $\dot{x}\dot{y}+gx$ becomes an immediate consequence of the fact that z and p_z are ignorable coordinates in expression (4). It may be noticed that Eq. (4) corresponds to a particle in a uniform gravitational field in *one* dimension.

Making use of the conservation H, z, and p_z, one readily shows that the orbits in the xy-plane are parabolas.

Quantum version

Following the standard rules, from the Hamiltonian function (4) one obtains the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial w^2} + mgw\phi = E\phi, \qquad (5)$$

whose solution can be expressed in terms of Airy functions or, equivalently, of Bessel functions of order $\pm 1/3$ (see, *e.g.*, Ref. 4). The question now is, given a solution $\phi(w)$ of Eq. (5), how can we obtain a wave function depending on the original coordinates x, y?

In terms of the original coordinates one has the time-independent Schrödinger equation [see Eq. (2)]:

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + mgy\psi = E\psi. \qquad (6)$$

Taking into account that the wave function ϕ depends on the variable w, which is a mixture of y and p_x, one may guess that a solution $\psi(x,y)$ to Eq. (6) can be obtained from $\phi(w)$ by means of a partial Fourier transform

$$\psi(x,y) = \frac{1}{\sqrt{2\pi\hbar}} \int\limits_{-\infty}^{\infty} \phi(y + p_x^2/2m^2 g) \, \mathrm{e}^{\mathrm{i}xp_x/\hbar} \mathrm{d}p_x, \qquad (7)$$

where we have replaced the variable w by its equivalent expression in terms of y and p_x given by Eqs. (3). In fact, we

can show that, for each value of the parameter p_x, the expression $\phi(y + p_x^2/2m^2 g) \, \mathrm{e}^{\mathrm{i}xp_x/\hbar}$, appearing in the integral in Eq. (7), is a solution to the Schrödinger equation (6). In effect, making use of the chain rule, one finds that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi(y + p_x^2/2m^2 g) \, \mathrm{e}^{\mathrm{i}xp_x/\hbar}$$

$$= \left(-\frac{p_x^2}{\hbar^2}\phi + \phi'' \right) \mathrm{e}^{\mathrm{i}xp_x/\hbar}. \qquad (8)$$

On the other hand, Eq. (5) yields

$$\phi''|_{w=y+p_x^2/2m^2 g} = -\frac{2m}{\hbar^2} \left[E - mg(y + p_x^2/2m^2 g) \right] \phi,$$

which, substituted into Eq. (8), shows that the function $\phi(y + p_x^2/2m^2 g) \, \mathrm{e}^{\mathrm{i}xp_x/\hbar}$ satisfies the Schrödinger equation (6). This function contains the parameter p_x in addition to the parameter E contained in ϕ.

3. Derivation

Each (differentiable) function, f, defined on the phase space of a mechanical system with n degrees of freedom gives rise to a linear partial differential operator (or vector field), X_f, given by

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} \right) \qquad (9)$$

in an arbitrary system of canonical coordinates q_i, p_i. Then, the Poisson bracket can be expressed, or defined, by

$$\{f, g\} = X_f g. \qquad (10)$$

Furthermore, for any pair of differentiable functions, f, g,

$$[X_f, X_g] = X_{\{f,g\}}. \qquad (11)$$

Since $X_f f = 0$, the vector field X_f is tangent to the hypersurfaces $f = \mathrm{const}$.

If $f = f(q_i, p_i)$ is a constant of motion (that does not depend explicitly on time), the Poisson bracket $\{f, H\}$ vanishes or, equivalently, $X_f H = 0$, which means that X_f is also tangent to the hypersurfaces $H = \mathrm{const}$.

As pointed out above, in the specific case of the Hamiltonian (2), the functions p_x and $A \equiv x + p_x p_y/m^2 g$ are constants of motion; the vector fields associated with them are [see Eq. (9)]

$$X_{p_x} = -\frac{\partial}{\partial x}, \quad X_A = \frac{\partial}{\partial p_x} - \frac{p_y}{m^2 g}\frac{\partial}{\partial x} - \frac{p_x}{m^2 g}\frac{\partial}{\partial y} \qquad (12)$$

and a straightforward computation shows that

$$[X_{p_x}, X_A] = 0. \qquad (13)$$

(which also follows from Eq. (11), noting that $\{p_x, A\} = X_{p_x}A = -1$). According to Frobenius' theorem, Eq. (13) implies that the phase space is foliated

by two-dimensional surfaces characterized by the fact that X_{p_x} and X_A are tangent to them (see, *e.g.*, Ref. 5). Since $X_{p_x} H = 0 = X_A H$, the Hamiltonian is constant on these two-dimensional surfaces.

Looking for the simultaneous solutions to the linear partial differential equations $X_{p_x} f = 0$ and $X_A f = 0$, one readily finds that the above-mentioned surfaces are also given by

$$p_y = \text{const.}, \qquad \tfrac{1}{2} p_x^2 + m^2 gy = \text{const.} \qquad (14)$$

Hence, the Hamiltonian must be a function of p_y and

$$\frac{1}{2} p_x^2 + m^2 gy$$

only; in fact, one finds that

$$H = p_y^2/2m + \left(\frac{1}{2} p_x^2 + m^2 gy \right) / m.$$

Since

$$\left\{ \tfrac{1}{2} p_x^2 + m^2 gy, p_y \right\} = m^2 g,$$

the functions

$$w \equiv y + \frac{p_x^2}{2m^2 g}, \qquad p_w \equiv p_y, \qquad (15)$$

are conjugate variables and can be part of a set of canonical coordinates, and

$$H = \frac{p_w^2}{2m} + mgw. \qquad (16)$$

The remaining two canonical coordinates, z and p_z, say, will not appear in the expression for H and must be constants of motion, which implies that z and p_z must be functions of

p_x and A, such that $\{z, p_z\} = 1$. Since $\{A, p_x\} = 1$, we can choose

$$z \equiv x + \frac{p_x p_y}{m^2 g}, \qquad p_z \equiv p_x. \qquad (17)$$

As shown in the preceding section, Eqs. (15) and (17) define a canonical transformation.

4. Concluding remarks

As pointed out above, the constants of motion A and p_x satisfy the relation $\{A, p_x\} = 1$, and the corresponding vector fields X_A and X_{p_x} commute [see Eq. (13)]; thus, the symmetry group generated by X_A and X_{p_x} in the phase space is Abelian and its action consists of translations along z and p_z.

In the quantized system, A and p_x are operators that do not commute with each other, and the symmetry group generated by A, p_x, and 1 is the Heisenberg group; the states with a given energy form a representation space for this group. Specifically, the time-independent Schrödinger equation (6) admits separable solutions of the form

$$\mathrm{e}^{\mathrm{i}kx} f_{E,k}(y), \qquad (18)$$

where $f_{E,k}(y)$ depends parametrically on E and k. As is well known (and can be readily verified), the effect of the operator $\mathrm{e}^{\mathrm{i}ap_x/\hbar}$, with $a \in \mathbb{R}$, on function (18) amounts to replacing x by $x + a$ and, as one can show, the operator $\mathrm{e}^{\mathrm{i}bA/\hbar}$, with $b \in \mathbb{R}$, applied to function (18) yields another function of the same form with k replaced by $k + b/\hbar$.

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