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Calculation of temporal spreading of ultrashort pulses propagating through optical glasses

M. Rosete-Aguilar, F.C. Estrada-Silva, N.C. Bruce, C.J. Román-Moreno, and R. Ortega-Martínez
*Centro de Ciencias Aplicadas y Desarrollo Tecnológico, Universidad Nacional Autónoma de México,
Cd. Universitaria, Circuito Exterior S/N,
Apartado Postal 70-186, México, D.F., 04510, México.*

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The chromatic dispersion of optical materials causes an optical pulse to spread as it propagates through the material. The pulse spreading is produced by the dependence of the group velocity on the frequency. In this paper we evaluate the temporal spreading of a pulse as it propagates through optical glass. We evaluate the dependence of group velocity on frequency in terms of the dependence of the phase refractive index of the glass on the wavelength of light. The dependence of the refractive index on the wavelength in glass is well known through the Sellmeier formula. Results are presented for 50, 80 and 100 fs pulses propagating a distance L, in BK7, SF14 and Fused Silica Schott optical glasses and verified by a model of the sum of Gaussian modulated frequencies.

Keywords: Ultrashort pulses; group velocity dispersion; group velocity; temporal spreading.

La dispersión cromática de materiales ópticos ocasiona el ensanchamiento de un pulso óptico conforme se propaga a través del material. El ensanchamiento del pulso se produce por la dependencia de la velocidad de grupo con la frecuencia. En este artículo evaluamos el ensanchamiento temporal del pulso conforme se propaga en el vidrio óptico. Evaluamos la dependencia de la velocidad de grupo con la frecuencia en términos de la dependencia del índice de refracción de fase del vidrio con la longitud de onda de la luz. La dependencia del índice de refracción de fase con la longitud de onda en vidrios es bien conocida a través de la fórmula de Sellmeier. Se presentan resultados para pulsos de 50, 80 y 100 fs propagándose una distancia L en vidrios ópticos de Schott BK7, SF14 y Silica fundida los cuales son verificados usando un modelo de suma de frecuencias moduladas por una gaussiana.

Descriptores: Pulsos ultracortos; dispersión de la velocidad de grupo; velocidad de grupo; ensanchamiento temporal.

PACS: 42.65.Re

1. Introduction

Chromatic dispersion is an intrinsic property of practically all optical materials [1]. The phenomenon is manifested as a dependence of the phase velocity of a beam of light in a transparent material on the frequency or wavelength of that light. On the other hand, a wave packet or an optical pulse propagates at the group velocity, $v_g = c/n_g$, where n_g is called the group refractive index [2], which depends on the frequency. Due to the dependence of the group velocity on the frequency, the pulse spreads as it propagates through the material. There are two types of spreading: temporal and spatial. In this paper we shall only evaluate the temporal spreading of the pulse which is produced by the phenomenon termed group velocity dispersion (GVD) and which is given by

$$\mathrm{GDV} = \frac{d^2k}{dw^2}.$$

GVD is characterized by a parameter a defined as

$$a \equiv \frac{1}{2}\frac{d^2k}{dw^2}.$$

For pulses with a Gaussian profile, the duration of the pulse τ after propagating a distance L through the medium can be expressed in terms of parameter a as [3]

$$\tau(L) = \tau_0\sqrt{1+\left(\frac{8aL\ln 2}{\tau_0^2}\right)^2},$$

where τ_0 is the initial pulse duration at $L = 0$.

In the present paper we shall calculate parameter a, which is expressed in terms of the dispersion of the material described by the wavelength dependence $n(\lambda)$ of the index of refraction. This dependence is well known in optical glasses through the Sellmeier formula from the UV through the visible to the IR. Results are presented for 50, 80 and 100 fs pulses propagating a distance L, in BK7, SF14 and Fused Silica Schott optical glasses. These results are verified by a model of the sum of Gaussian modulated frequencies propagating through the material.

2. Theory

We assume a temporal Gaussian pulse given by [3]

$$E(z=0,t) = \exp\left(-\alpha t^2 + iw_0 t\right), \tag{1}$$

where α is a constant and w_0 is the optical carrier frequency.

We consider the case of a slowly varying envelope so that there are many optical oscillations within the envelope. We may express the input pulse $E(0,t)$ as a Fourier integral

$$E(z=0,t) = \int F(\Omega)\, e^{i(w_0+\Omega)t} d\Omega, \tag{2}$$

where $F(\Omega)$ is the Fourier transform of the Gaussian envelope $\exp\left(-\alpha t^2\right)$

$$F(\Omega) = \sqrt{\frac{1}{4\pi\alpha}}\exp\left(\frac{-\Omega^2}{4\alpha}\right). \tag{3}$$

We may view Eq. (2) as an assembly of harmonic fields, each with its unique frequency $(w_0+\Omega)$ and amplitude $F(\Omega)\,d\Omega$. To obtain the field at an output plane z, each frequency component $F(\Omega)\,d\Omega\exp[i(w_0+\Omega)t]$ in Eq. (2) is multiplied by its propagation phase delay factor,

$$\exp[-ikz], \tag{4}$$

where k is a function of $(w_0+\Omega)$, that is, $k(w_0+\Omega)$.

The result is

$$E(z,t)=\int F(\Omega)\exp[i(w_0+\Omega)t-k(w_0+\Omega)z]. \tag{5}$$

We can expand $k(w_0+\Omega)$ near the center of the optical frequency w_0 in a Taylor series

$$k(w_0{+}\Omega){=}k(w_0)+\left.\frac{dk}{dw}\right|_{w=w_0}\Omega+\frac{1}{2}\left.\frac{d^2k}{dw^2}\right|_{w=w_0}\Omega^2+\dots \tag{6}$$

where

$$k(w_0)\equiv k_0 \tag{7}$$

$$\left.\frac{dk}{dw}\right|_{w=w_0}=\frac{1}{v_g}=\frac{1}{Group\ velocity} \tag{8}$$

$$a\equiv\frac{1}{2}\left.\frac{d^2k}{dw^2}\right|_{w=w_0}=\frac{1}{2}\frac{d}{dw}\left(\frac{1}{v_g}\right)=-\frac{1}{2v_g^2}\frac{dv_g}{dw}. \tag{9}$$

After substituting Eqs. (3), (7), (8) and (9), Eq. (5) becomes

$$\begin{aligned}\mathcal{E}(z,t)&=\exp[i(\omega_0t-k_0z)]\int_{-\infty}^{\infty}d\Omega F(\Omega)\\&\times\exp\left\{i\left[\Omega t-\frac{\Omega z}{v_g}-\frac{1}{2}\frac{d}{d\omega}\left(\frac{1}{v_g}\right)\Omega^2z\right]\right\}\\&\equiv\exp[i(\omega_0t-k_0z)]\,E(z,t)\end{aligned}$$

where $\mathcal{E}(z,t)$ is the envelope of the field given by

$$\begin{aligned}\mathcal{E}(z,t)&=\sqrt{\frac{1}{4\pi\alpha}}\int_{-\infty}^{\infty}\\&\times\exp\left\{-\left[\Omega^2\left(\frac{1}{4\alpha}+iaz\right)-i\left(t-\frac{z}{v_g}\right)\Omega\right]\right\}d\Omega.\end{aligned} \tag{10}$$

The integration is carried out explicitly using "Siegman's lemma", namely,

$$\int_{-\infty}^{\infty}e^{-A\Omega^2-2B\Omega}d\Omega\equiv\sqrt{\frac{\pi}{A}}e^{B^2/A},\quad Re[A]>0, \tag{11}$$

where $A=(1/4\alpha)+iaz$ and $B=-(i/2)\left(t-\frac{z}{v_g}\right)$.

Carrying out the integration, the envelope of the electric field of a pulse after propagating a distance z through the medium is given by,

$$\begin{aligned}\mathcal{E}(z,t)&=\frac{1}{\sqrt{1+i4a\alpha z}}\exp\left(-\frac{(t-z/v_g)^2}{1/\alpha+16a^2z^2\alpha}\right)\\&\times\exp\left(i\frac{4az\,(t-z/v_g)^2}{1/\alpha^2+16a^2z^2}\right).\end{aligned} \tag{12}$$

The intensity of the pulse is given by the pulse envelope squared so that

$$I(z,t)=\frac{1}{\sqrt{1+16a^2\alpha^2z^2}}\exp\left(-\frac{2\left(t-\frac{z}{v_g}\right)^2}{\frac{1}{\alpha}+16a^2z^2\alpha}\right). \tag{13}$$

The pulse duration τ at z can be taken as the separation between the two times when the intensity is reduced by a factor of $1/2$ from its peak value (the so called FWHM), that is,

$$\tau(z)=\sqrt{2\ln 2}\sqrt{\frac{1}{\alpha}+16a^2z^2\alpha}\quad, \tag{14}$$

The initial pulse width is

$$\tau_0=\tau(0)=\sqrt{\frac{2\ln 2}{\alpha}} \tag{15}$$

so that the pulse width after propagating a distance L can be expressed as

$$\tau(L)=\tau_0\sqrt{1+\left(\frac{8aL\ln 2}{\tau_0^2}\right)^2}\quad. \tag{16}$$

3. Calculation of parameter a

The angular frequency of light is given by

$$w=\frac{2\pi c}{\lambda_0}, \tag{17}$$

where λ_0 is the wavelength of light in vacuum. Therefore,

$$\frac{d\lambda_0}{dw}=-\frac{2\pi c}{w^2}=-\frac{\lambda_0^2}{2\pi c}. \tag{18}$$

Using the chain rule, the first derivative with respect to frequency is given by

$$\frac{d}{dw}=\frac{d}{d\lambda_0}\left(\frac{d\lambda_0}{dw}\right)=\left(\frac{d\lambda_0}{dw}\right)\frac{d}{d\lambda_0}=-\frac{\lambda_0^2}{2\pi c}\frac{d}{d\lambda_0}. \tag{19}$$

From Eq. (19), the first derivative of the group velocity with respect to frequency is given by

$$\frac{dv_g}{dw}=-\frac{\lambda_0^2}{2\pi c}\frac{dv_g}{d\lambda_0} \tag{20}$$

where

$$v_g=\frac{c}{n_g}=\frac{c}{n-\lambda_0\frac{dn}{d\lambda_0}}, \tag{21}$$

and n is the phase refractive index, $n = c/v$, where v is the phase velocity.

Evaluating the first derivative of the group velocity with respect to the wavelength in a vacuum in Eq. (21), we have

$$\frac{dv_g}{d\lambda_0} = -\frac{v_g^2}{c}\lambda_0 \frac{d^2 n}{d\lambda_0^2}. \tag{22}$$

Substituting Eq. (22) into (20) and then into Eq. (9), parameter a is given by

$$a = -\frac{1}{2v_g^2}\frac{dv_g}{dw} = \frac{\lambda_0^3}{4\pi c^2}\frac{d^2 n}{d\lambda_0^2}. \tag{23}$$

The second derivative of the refractive index can be calculated from the Sellmeier formula, where the constants can be found in optical glass catalogs. If the Sellmeier equation is used, the units for the second derivative of n with wavelength, λ_0, are μm^{-2}. The wavelength λ_0 of the carrier frequency, and the velocity of light in vacuum, c, are most conveniently given in microns and microns/fs respectively, so parameter a is expressed in $fs^2 \mu m^{-1}$.

4. The Sellmeier equation for optical glasses

From classical dispersion theory, the phase refractive index as a function of wavelength in a vacuum can be calculated using the Sellmeier formula [5], which is suitable for describing the dispersion curve in the wavelength range from the UV through the visible to the IR, that is, from 0.36 to 2.3μm:

$$n^2(\lambda_0) - 1 = \frac{B_1\lambda_0^2}{\lambda_0^2 - C_1} + \frac{B_2\lambda_0^2}{\lambda_0^2 - C_2} + \frac{B_3\lambda_0^2}{\lambda_0^2 - C_3}, \tag{24}$$

where λ_0 is the wavelength in a vacuum in microns and $B_1, B_2, B_3, C_1, C_2, C_3$ are constants which are calculated for each optical glass and can be found in optical glass catalogs such as the Schott catalog.

The second derivative of the refractive index is given by:

$$\frac{d^2 n}{d\lambda_0^2} = \frac{\left(-\frac{2B_1\lambda_0^3}{\left(-C_1+\lambda_0^2\right)^2} + \frac{2B_1\lambda_0}{-C_1+\lambda_0^2} - \frac{2B_2\lambda_0^3}{\left(-C_2+\lambda_0^2\right)^2} + \frac{2B_2\lambda_0}{-C_2+\lambda_0^2} - \frac{2B_3\lambda_0^3}{\left(-C_3+\lambda_0^2\right)^2} + \frac{2B_3\lambda_0}{-C_3+\lambda_0^2}\right)^2}{4\left(1 + \frac{B_1\lambda_0^2}{-C_1+\lambda_0^2} + \frac{B_2\lambda_0^2}{-C_2+\lambda_0^2} + \frac{B_3\lambda_0^2}{-C_3+\lambda_0^2}\right)^{3/2}}$$
$$+\frac{\frac{8B_1\lambda_0^4}{\left(-C_1+\lambda_0^2\right)^3} - \frac{10B_1\lambda_0^2}{\left(-C_1+\lambda_0^2\right)^2} + \frac{2B_1}{-C_1+\lambda_0^2} + \frac{8B_2\lambda_0^4}{\left(-C_2+\lambda_0^2\right)^3} - \frac{10B_2\lambda_0^2}{\left(-C_2+\lambda_0^2\right)^2} + \frac{2B_2}{-C_2+\lambda_0^2} + \frac{8B_3\lambda_0^4}{\left(-C_3+\lambda_0^2\right)^3} - \frac{10B_3\lambda_0^2}{\left(-C_3+\lambda_0^2\right)^2} + \frac{2B_3}{-C_3+\lambda_0^2}}{2\sqrt{1+\frac{B_1\lambda_0^2}{-C_1+\lambda_0^2}+\frac{B_2\lambda_0^2}{-C_2+\lambda_0^2}+\frac{B_3\lambda_0^2}{-C_3+\lambda_0^2}}}. \tag{25}$$

TABLE I. Sellmeier coefficients for the dispersion formula.

Glass	B_1	B_2	B_3	C_1	C_2	C_3
BK7	1.03961212	2.31792344×10^{-1}	1.01046945	6.00069867×10^{-3}	2.00179144×10^{-2}	1.03560653×10^{2}
SF14	1.69182538	2.85919934×10^{-1}	1.12595145	1.33151542×10^{-2}	6.12647445×10^{-2}	1.18405242×10^{2}
Fused Silica	6.69422575×10^{-1}	4.34583937×10^{-1}	8.71694723×10^{-1}	4.48011239×10^{-3}	1.32847049×10^{-2}	9.5341482×10^{1}

TABLE II. Pulse width after propagating a distance L through different glasses. Initial pulse width, $\tau_0 = 50fs$ and $\lambda_0 = 800nm$.

L (mm)	Glass BK7 $\tau(L)$(fs)		Glass SF14 $\tau(L)$(fs)		Fused Silica $\tau(L)$(fs)	
	Theory	Model	Theory	Model	Theory	Model
0	50.00	50.50 ± 0.5	50.00	50.84 ± 0.5	50.00	50.37 ± 0.5
1	50.06	50.56 ± 0.5	50.97	50.76 ± 0.5	50.04	50.12 ± 0.5
2	50.24	50.13 ± 0.5	53.77	53.35 ± 0.5	50.16	50.76 ± 0.5
3	50.55	50.61 ± 0.5	58.18	54.85 ± 0.5	50.36	50.94 ± 0.5
4	50.97	50.76 ± 0.5	63.75	61.22 ± 0.6	50.64	50.76 ± 0.5
5	51.51	51.95 ± 0.5	70.31	70.36 ± 0.7	50.99	50.44 ± 0.5
10	55.78	54.85 ± 0.5	110.78	110.52 ± 1.1	53.87	52.42 ± 0.5
50	133.36	132.20 ± 1.3	496.80	498.80 ± 5.0	111.99	111.99 ± 1.1
100	252.26	253.32 ± 2.5	989.81	999.10 ± 9.9	206.56	202.75 ± 4.0

TABLE III. Pulse width after propagating a distance L through different glasses. Initial pulse width, τ_0 = 80f sand λ_0 = 800nm.

L (mm)	Glass BK7 $\tau\,(L)$(fs)		Glass SF14 $\tau\,(L)$(fs)		Fused Silica $\tau\,(L)$(fs)	
	Theory	Model	Theory	Model	Theory	Model
0	80.00	80.03 ± 0.8	80.00	80.03 ± 0.8	80.00	80.92 ± 0.8
1	80.01	80.64 ± 0.8	80.24	80.12 ± 0.8	80.01	80.32 ± 0.8
2	80.06	80.03 ± 0.8	80.95	80.94 ± 0.8	80.04	80.21 ± 0.8
3	80.13	80.07 ± 0.8	82.12	82.27 ± 0.8	80.09	80.48 ± 0.8
4	80.24	80.03 ± 0.8	83.73	82.60 ± 0.8	80.16	80.12 ± 0.8
5	80.37	80.81 ± 0.8	85.75	84.51 ± 0.8	80.24	80.32 ± 0.8
10	81.48	80.32 ± 0.8	101.08	98.87 ± 1.0	80.97	81.38 ± 0.8
50	111.22	110.60 ± 1.1	319.11	317.97 ± 3.2	101.6	98.83 ± 1.0
100	174.02	172.72 ± 1.7	623.00	619.82 ± 6.2	148.63	149.21 ± 1.5

TABLE IV. Pulse width after propagating a distance L through different glasses. Initial pulse width, τ_0 = 100fs and λ_0 = 800nm.

L (mm)	Glass BK7 $\tau\,(L)$(fs)		Glass SF14 $\tau\,(L)$(fs)		Fused Silica $\tau\,(L)$(fs)	
	Theory	Model	Theory	Model	Theory	Model
0	100.00	100.08 ± 1.0	100.00	100.08 ± 1.0	100.00	100.44 ± 1.0
1	100.00	100.89 ± 1.0	100.12	99.90 ± 1.0	100.00	99.91 ± 1.0
2	100.03	99.90 ± 1.0	100.49	99.55 ± 1.0	100.02	100.26 ± 1.0
3	100.07	100.23 ± 1.0	101.09	99.91 ± 1.0	100.05	99.74 ± 1.0
4	100.12	99.90 ± 1.0	101.94	101.38 ± 1.0	100.08	100.61 ± 1.0
5	100.19	100.22 ± 1.0	103.01	102.65 ± 1.0	100.13	100.07 ± 1.0
10	100.76	100.50 ± 1.0	111.55	111.71 ± 1.1	100.50	100.99 ± 1.0
50	117.56	117.53 ± 1.2	266.60	265.33 ± 2.6	111.85	111.71 ± 1.1
100	159.01	158.70 ± 1.6	504.29	504.02 ± 5.0	141.60	140.06 ± 1.4

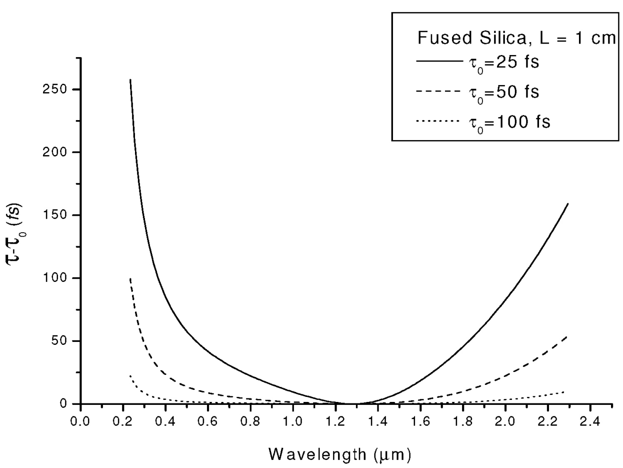

FIGURE 1. Temporal spreading for pulses with an initial width of 25fs, 50fs and 100 fs after propagating 1cm in Fused Silica.

The units for the second derivative of the refractive index with respect to wavelength are μm^{-2}.

In Table I the constants for the dispersion formula are given for Schott optical glasses BK7, SF14 and Fused Silica, which were used to evaluate the temporal spreading of the pulse.

5. Results

The temporal spreading of the pulse after propagating a distance L through a transparent and homogeneous material can be evaluated with Eqs. (16), (23) and (25).

The results are presented in Tables II, III and IV in the column labelled theory, for pulses of 50, 80 and 100 fs, respectively, and $\lambda_0 = 800$nm. BK7 is a low dispersive inexpensive glass widely used in lenses; Fused Silica is also a low dispersive glass widely used when working in the near IR or UV regions. SF14 is a high dispersive heavy flint glass which is sometimes used in the arrangement of pairs of prisms that can produce negative group velocity dispersion [4].

As we can see from the tables, the effect of the group dispersion velocity on the temporal spreading of the pulse becomes more important for shorter pulses.

In Fig. 1 the temporal width difference between the initial pulse, τ_0, and the pulse width τafter propagating a distance L through the material is plotted as a function of the wavelength between 0.246 and 2.3 microns for Fused Silica and after propagating 1cm in the material. The curves are presented for an initial pulse temporal width of 25, 50, and 100 fs. In Fig. 1 we can see that the group velocity dispersion is negligible, around 1.3 microns, and increases from this wavelength towards the IR and towards the UV regions.

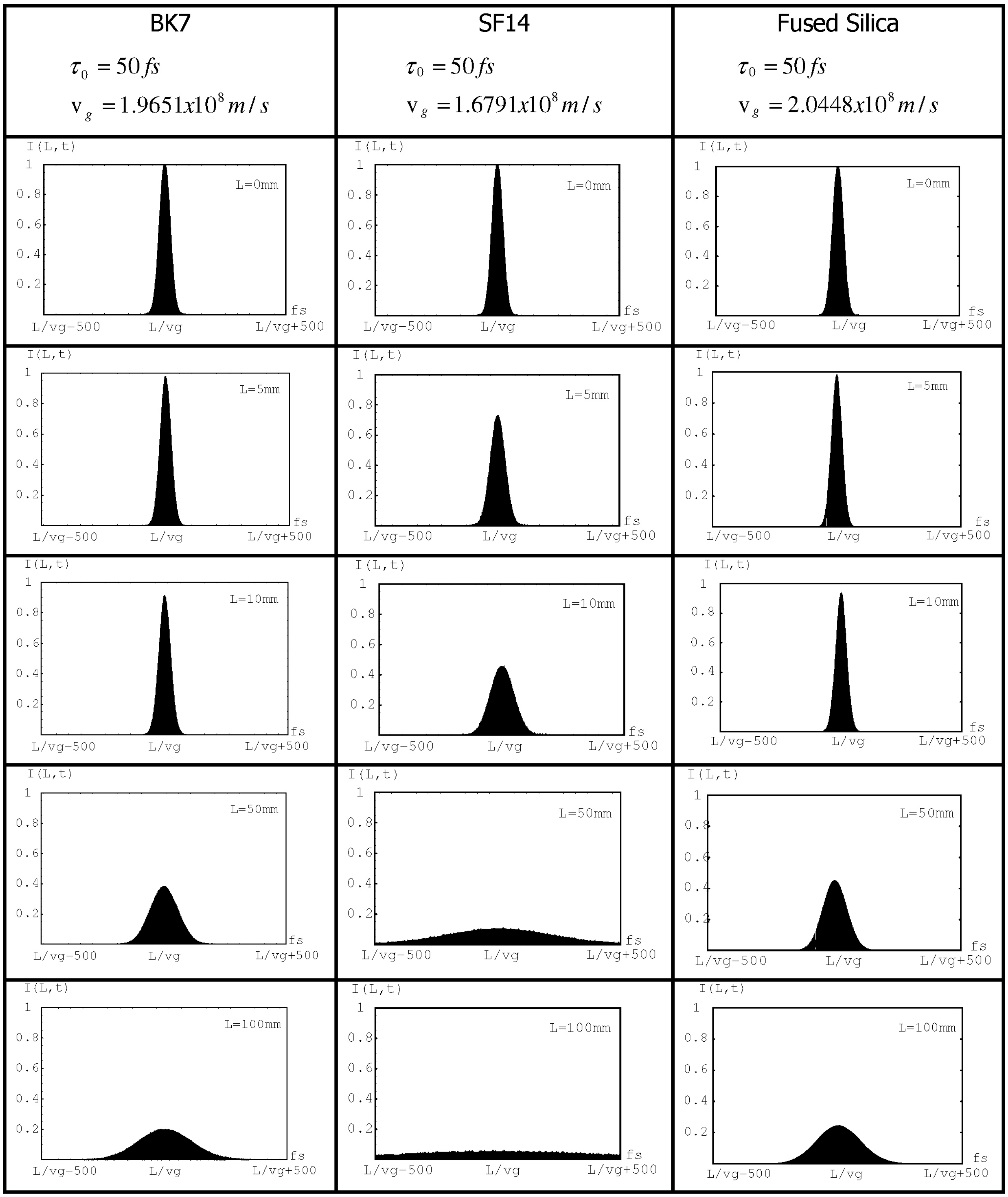

FIGURE 2. Temporal pulse spreading as it propagates through glass. The initial temporal pulse width is 50fs.

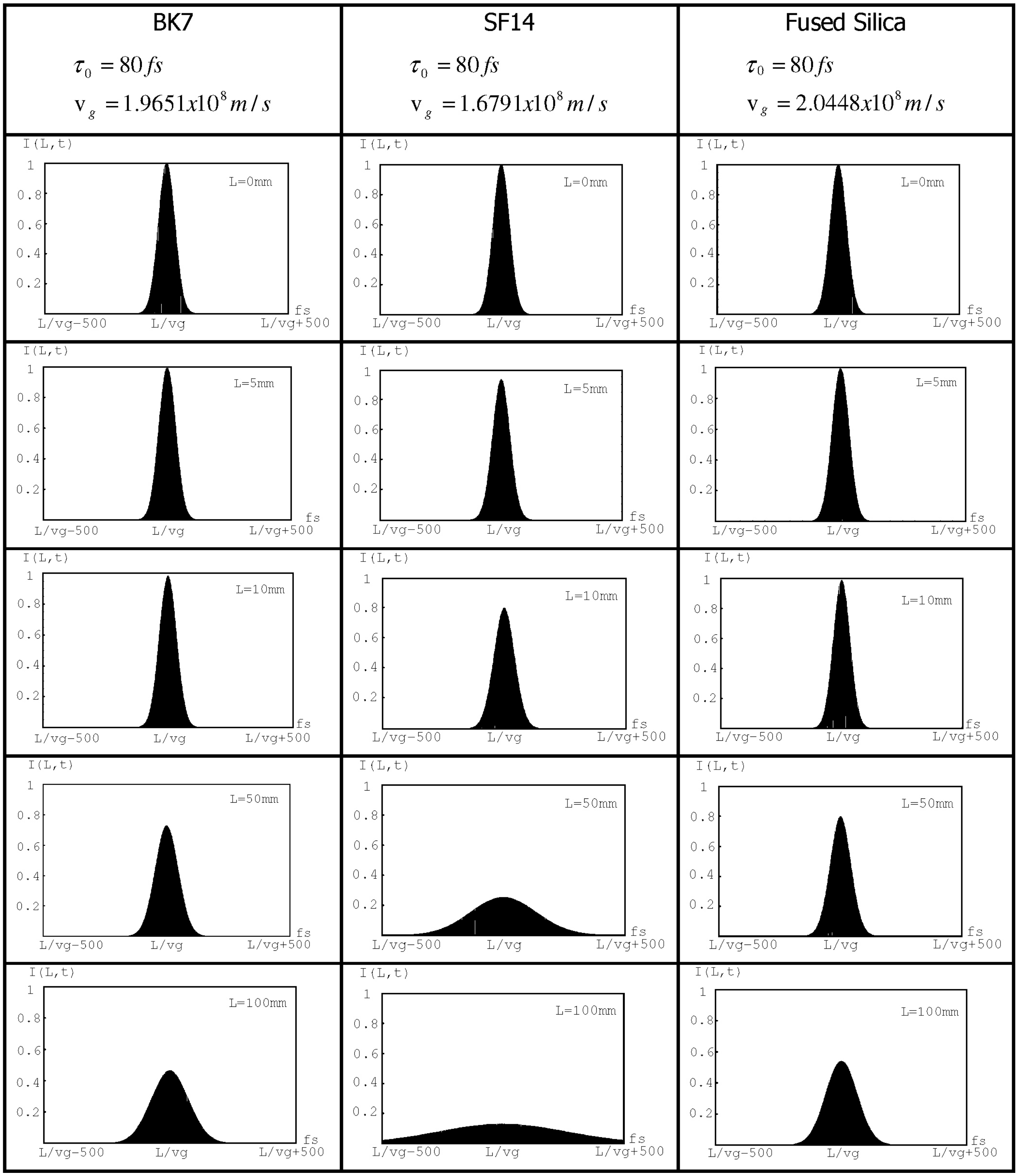

FIGURE 3. Temporal pulse spreading as it propagates through glass. The initial temporal pulse width is 80fs.

6. Generation of pulses with a Gaussian profile

The theoretical results were verified by directly calculating the electric field for a Gaussian pulse given by Eq. (5). The calculation of the electric field given by Eq. (5) was exact, $i.e.$, there is no approximation in the wave number. The normalized squared of the real electric field was plotted using the Mathematica software as a function of time, that is,

$$I\left(L,t\right) = \frac{\{Re\{E\left(L,t\right)\}\}^2}{\{Re\{E\left(0,0\right)\}\}^2}. \tag{26}$$

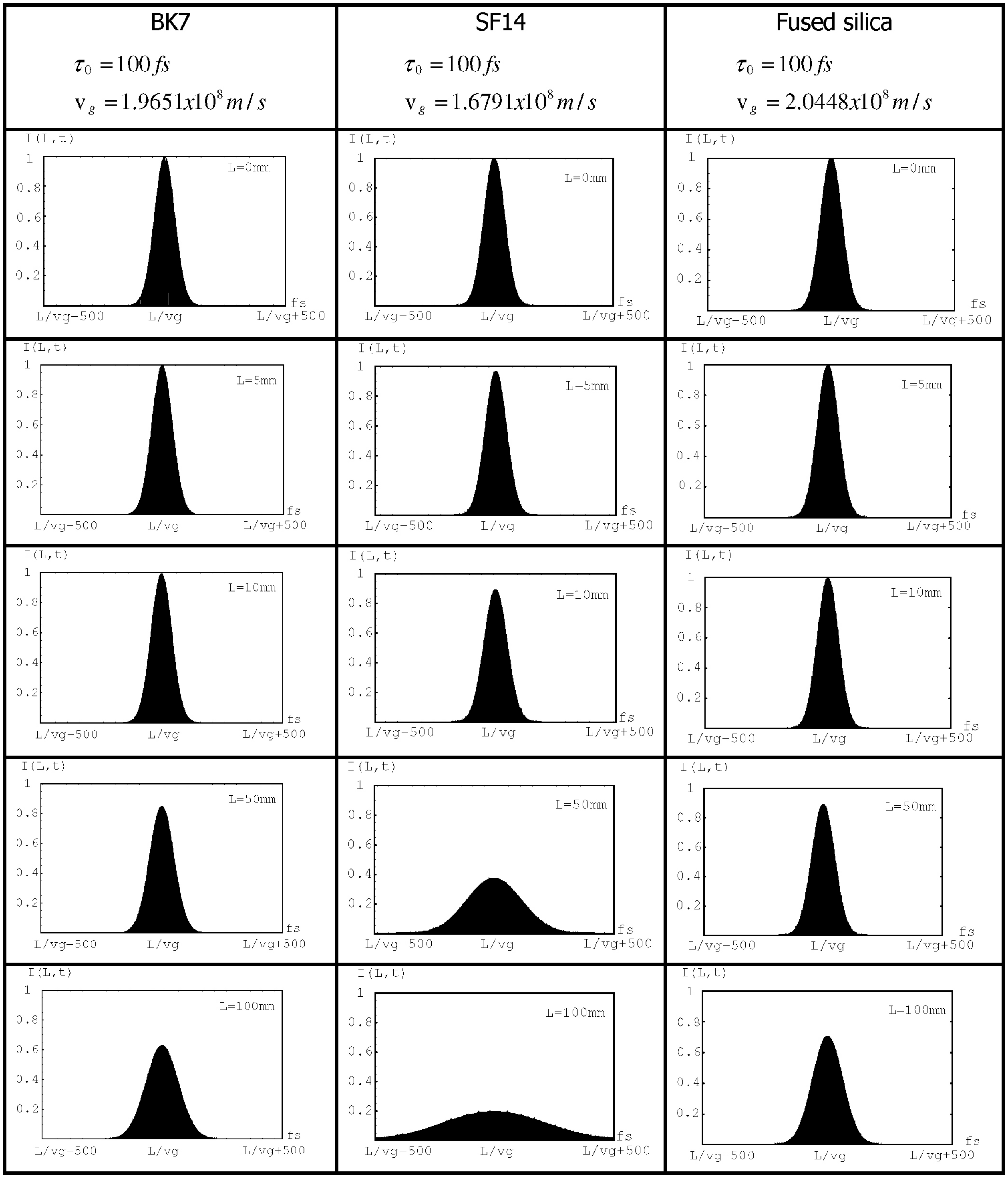

FIGURE 4. Temporal pulse spreading as it propagates through glass. The initial temporal pulse width is 100fs.

In Figs. 2 and 3 we present the pulses as they propagate through the glass after a distance L. The pulses are generated assuming that the wavelength of the carrier frequency is $\lambda_0 = 800nm$ and an initial pulse width of $\tau_0 = 50fs$ (Fig. 2) and $\tau_0 = 100fs$ (Fig. 3). The center of the abscissa is at the peak of the pulse located at L/v_g, where v_g is the group velocity given by Eq. (20). The abscissa covers a full range of 1000 fs. As we can see, the effect of the spreading after traveling 5 mm in the glass is almost negligible for pulses of 100 fs, but becomes more important for pulses of 50 fs in high dispersive materials such as SF14. In Tables II, III and IV, in the columns labelled Model, we give the temporal width of the 50, 80 and 100 fs pulses measured directly from Figs. 2, 3 and 4. The temporal width values of the pulses were estimated by measuring directly from the figures when the peak intensity falls to half. The values in the tables are a mean of several measurements of the pulse widths. The error was calculated as the RMS of the measured values and is below 2%.

7. Conclusions

In the present work we have derived the necessary equations for evaluating parameter$a$, in order to calculate the temporal spreading of a pulse as it propagates through optical glass. The theoretical model presented in Sec. 3 shows that the temporal spreading of the pulse depends on the initial duration of the pulse, the dispersion of the material and the distance that the pulse propagates through the material. The dispersion of the material is wavelength dependent and therefore the temporal spreading is also wavelength dependent. We presented a plot of temporal spreading as a function of wavelength to show this effect. In this plot we can see that at around 1.3 microns, the group velocity dispersion is negligible so there is no pulse temporal spreading at this wavelength, but increases towards the UV and IR regions. By using the equations derived in Secs. 3 and 4, the temporal spreading was evaluated for 50, 80 and 100 fs pulses propagating through BK7, SF14 and Fused Silica Schott glasses. The results show that the temporal spreading of the pulses increases for high dispersive materials, long propagating distances inside the material, and shorter pulses. For 100 fs pulses propagating 1 cm in BK7, the temporal spreading is less than 1fs, so pulses of 100 fs passing through single thin lenses made with glass of low dispersive materials will not modify the temporal width of these pulses. This is not the case for shorter pulses. The theoretical results were verified by a model of the sum of Gaussian modulated frequencies. The temporal width of the pulses was measured directly from the graphs and agrees with the theoretical results.

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