Noise-assisted synchronization of the transition times of a set of uncoupled bistable elements

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We study the noise-induced synchronization of the transition times of a set of non-interacting bistable systems undergoing an activation process in the presence of an external periodic field. Starting from a collection of time series corresponding to these transition times, we first explore graphically the extent of the synchronization for different values of the noise intensity and the frequency of the periodic signal. Then we quantify this phenomenon by means of the fractional fluctuation of the transition times. We show that this quantity has an optimal behavior when we plot it as a function of the noise intensity and the frequency of the periodic external force. Three-dimensional plots of the fractional fluctuation versus these two parameters clearly exhibit the parameter region of optimal synchronization.

Keywords: Synchronization; noise; activation processes.

En este trabajo se estudia la sincronización inducida por ruido, de los tiempos de transición de un conjunto de sistemas biestables. Estos sistemas llevan a cabo un proceso de activación en presencia de un campo periódico externo. Se parte de un conjunto de series de tiempo que corresponden a los tiempos de transición de los distintos sistemas y se explora en forma gráfica el grado de sincronización para diferentes valores de la intensidad del ruido, así como de la frecuencia de la señal periódica. Una vez hecho esto, se cuantifica este fenómeno por medio de la fluctuación fraccional de los tiempos de transición. Se muestra que esta cantidad tiene un comportamiento óptimo al graficarla contra la intensidad del ruido y la frecuencia de la fuerza periódica externa. En gráficas tridimensionales de la fluctuación fraccional contra estos dos parámetros, se observa con claridad la región de comportamiento óptimo en estos parámetros.

Descriptores: Sincronización; ruido; procesos de activación.

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A great deal of interest has been generated by the fact that noise and nonlinearity can produce unexpected organized behaviors, as occurs in noise-induced phase transitions [1], noise-induced pattern formation [2], noise-induced transport [3], stochastic resonance [4,5] and in the synchronization of sets of dynamical systems [6, 7, 9]. One topic of particular interest in this last field is the analysis of the synchronization of sets of bistable elements which are coupled through some interaction mechanism, and are also subjected to the influence of an external oscillatory field. In the linear response regime the synchronization phenomenon is expected to behave in a similar way to stochastic resonance; however, they are different phenomena that do not always have a cause-effect connection. As a matter of fact, in a bistable system the synchronization is related to transitions, whereas stochastic resonance is defined as a ratio of output-input amplitudes.

Short-range interactions have been thoroughly studied, and more recently long-range interactions have also been considered [6, 7]. The simultaneous consideration of a particular interaction mechanism, an external field, and the effect of noise, leads to very interesting, but also very complex behaviors.

In the present communication we analyze the limit case when the influences of the noise and the external field are dominant, and the interaction among the bistable subsystems may be neglected. Here the synchronization represents a coupling of the mean-first-passage times for each subsystem, with the periodic external force. We establish a measure of the synchronization as the fractional fluctuation of the transition times. It is shown that the dependence of this quantity as a function of noise amplitude and field frequency has a well-defined structure which, to our knowledge, has not been considered before [8]. It is expected that this structure will manifest itself in any collection of bistable systems where the interaction is sufficiently weak.

We shall assume that the behavior of each of the bistable subsystems can be described by the Langevin equation of a single particle moving in a one-dimensional double-well potential in the presence of an additive, delta-correlated noise, and driven by a periodic force. In the present communication

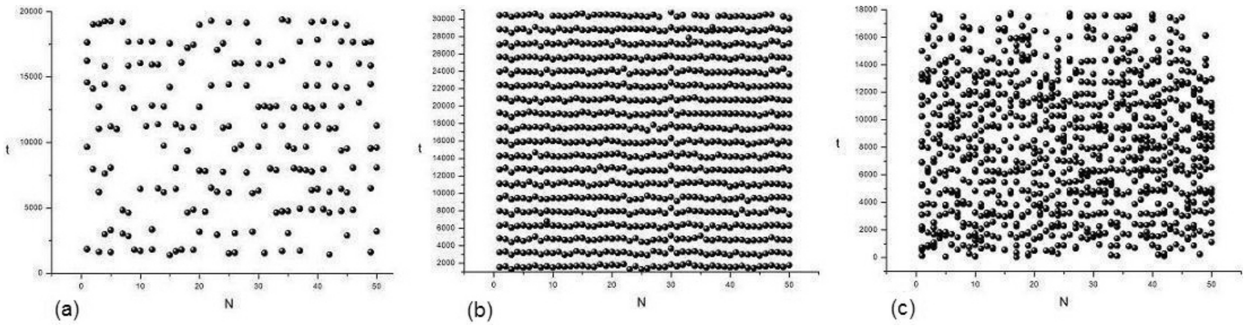

FIGURE 1. Transition times for the *N* subsystems, for *D= 0.02, 0.04 and 0.07* from left to right.

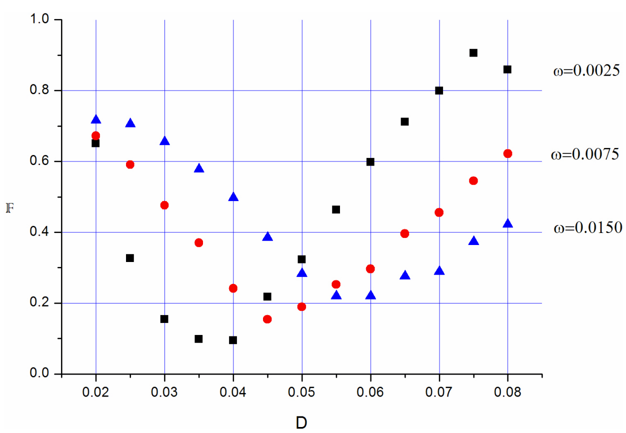

FIGURE 2. Fractional fluctuation $\mathbb{F}$ vs. noise intensity D, for three values of ω.

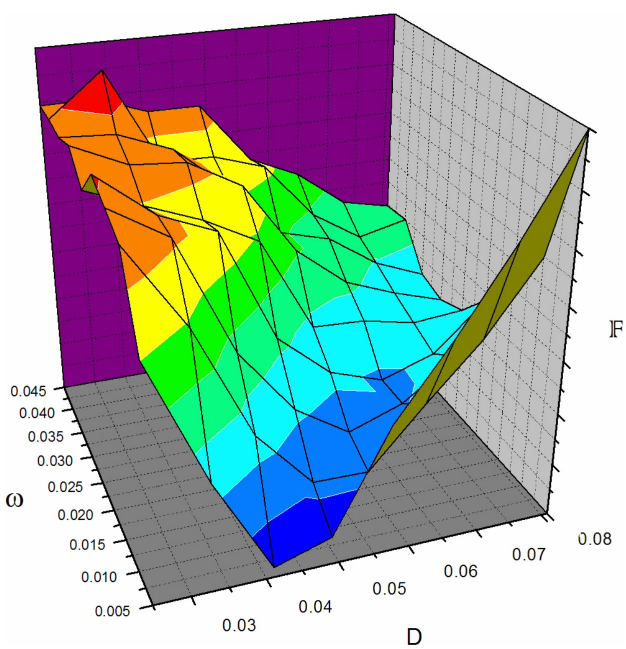

FIGURE 3. 3D plot, $\mathbb{F}$ vs. D and ω.

we only consider the over-damped (non inertial) regime, and consequently this equation will be:

$$\dot{x} = x - x^3 + A_0 \sin(\omega t) + \xi(t), \tag{1}$$

where x is the coordinate of the particle and the noise $\xi(t)$ has zero mean and a correlation

$$\langle \xi(t)\xi(t')\rangle = 2D\,\delta(t - t'). \tag{2}$$

Then we consider N subsystems of this type, and we focus on the *transition times* (t_{tr}) of each subsystem, *i.e.*, the times when each of these subsystems jumps from the left-side minimum over the barrier, reaches the opposite minimum, and then goes back to the left-side minimum. This "double transition" could then be compared to the period of the applied force, especially in the linear response regime. The synchronization of these transition times can be considered a collective behavior that is particularly important in activation processes, and therefore our goal here is to determine the dependence of this synchronization on the values of the noise instensity and the frequency of the external field.

In order to appreciate graphically that the noise may have a constructive effect on the synchronization of the set of bistable subsystems, we solved Eq.(1) numerically for $N = 50$ subsystems by means of a fourth-order Runge Kuta algorithm, using the same initial condition $x(0) = -1$ (the left side minimum) and plotted the sequence of transition times for each of the subsytems (all of them measured with respect to the same initial time). Here we fixed an external frequency $\omega = 0.005$, and then we repeated these calculations for three different values of D $(0.02, 0.04, 0.07)$. In Fig. 1 we show the transition times of the N subsystems (plotted in N columns) for the three chosen values for D.

What we see in these plots shows that there is a value of D for which there is better synchronization of the N bistable subsystems, Fig. 1b. In these graphs we observe that the synchronization assumes a qualitative ordering effect. In order to quantify and characterize more precisely this synchronization phenomenon, it is necessary to define an adequate measure of the synchronization. Here we recall that the phenomenon of stochastic resonance, very extensively studied in bistable systems [10, 11], is usually detected by measuring the *spectral amplification*, which in the linear-response regime is defined as [11]:

$$\eta(D,\omega) = \left[\frac{A(D,\omega)}{A_0}\right]^2. \tag{3}$$

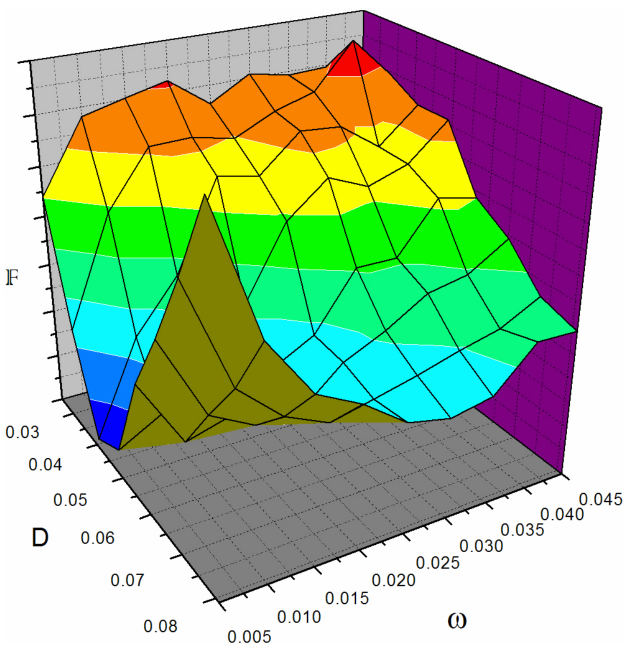

FIGURE 4. 3D plot, $\mathbb{F}$ vs. D and ω.

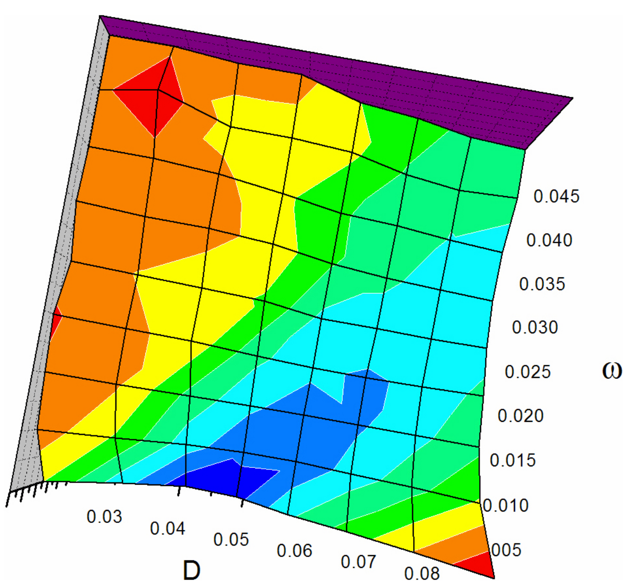

FIGURE 5. Level plot in the plane (D,ω).

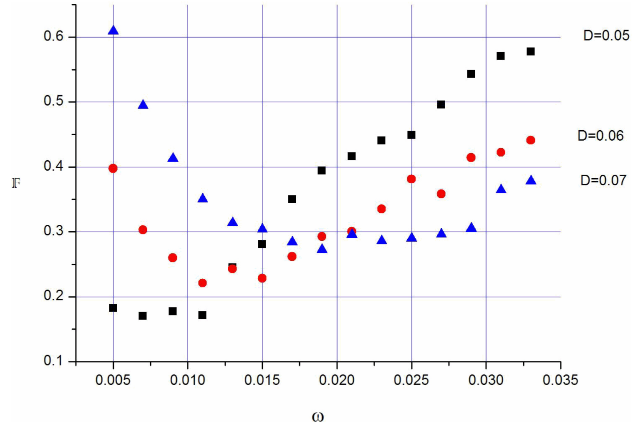

FIGURE 6. Fractional fluctuation $\mathbb{F}$ vs. frequency ω, for three values of D.

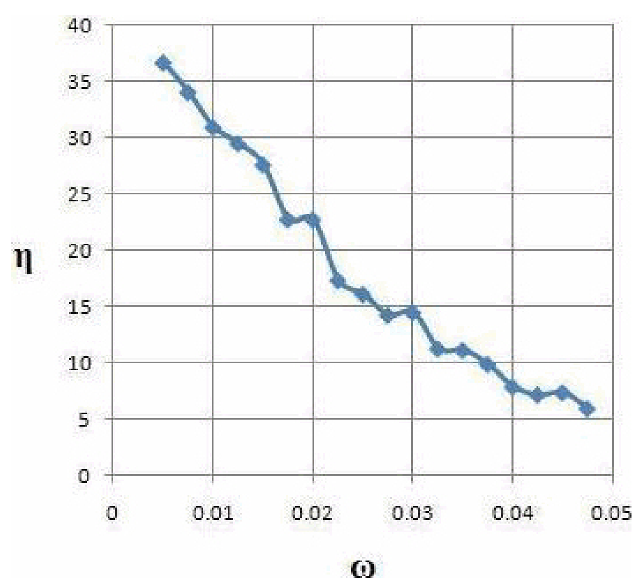

FIGURE 7. Spectral amplification $\eta(D,\omega)$ as defined from Eq.(3). The value of $A(D,\omega)$ was obtained by averaging the position of the particles.

Here A_0 is the amplitude of the periodic forcing, and $A(D,\omega)$ is the amplitude of the mean position of the particle $\langle x(t)\rangle = A\sin(\omega t + \phi_0)$, where the average is taken over an ensemble of noise realizations. However, if we are interested in the synchronization of the *transition times* of a *set* of bistable subsystems, the use of the above spectral amplification is not adequate, since this quantity might have significant non-zero values even when no transitions occur at all. However, a good measure of the synchronization of the transition times of a set of bistable elements is the *fractional fluctuation* ($\mathbb{F}$) of the sequence of *all* the transition times, t_{tr}, (of all the subsystems), which is defined as follows:

$$\mathbb{F} = \frac{\sqrt{\langle {t_{tr}}^2\rangle - \langle t_{tr}\rangle^2}}{\langle t_{tr}\rangle}. \tag{4}$$

This quantity depends on the noise intensity and the frequency of the external field, and in Fig. 2 we can see the graphs of $\mathbb{F}(D)$ for three different values of ω.

This figure shows that for each value of ω, $\mathbb{F}(D)$ presents a well-defined minimum which corresponds to the optimal synchronization, and the position of the minimum depends on the frequency of the external field. In order to appreciate the dependence of $\mathbb{F}$ on both parameters, D and ω, in Figs. 3 and 4 we present the surface $\mathbb{F}(D,\omega)$ as seen from two different angles. Fig. 5 shows the surface $\mathbb{F}(D,\omega)$ as seen from above, and therefore we can appreciate the shape of the level curves.

These figures clearly show that the intersection of the surface $\mathbb{F}(D,\omega)$ with a plane $\omega = \omega_0$ yields a curve $\mathbb{F}(D,\omega_0)$ with a clear minimum. In fact, in Fig. 5 we can see that a plane $\omega = \omega_0$, with $\omega_0 < 0.02$, intersects the lowest portion of the surface $\mathbb{F}(D,\omega)$. The existence of this minimun is the analog of the maximum of $\eta(D,\omega_0)$, which is found when the stochastic resonance phenomenon occurs in a single bistable system. On the other hand, if we intersect the surface $\mathbb{F}(D,\omega)$ with a plane $D = D_0$ we also obtain curves $\mathbb{F}(\mathbb{D}_0,\omega)$ with well defined minima for some values of D_0, thus enabling us to select the frequency which leads to the best synchronization of the set of bistable systems. In Fig. 6 we can appreciate the shape of three curves of this type.

It is important to emphasize that the existence of the minimum of the function $\mathbb{F}(\mathbb{D}_0,\omega)$ has no counterpart when the function $\eta(D_0,\omega)$ is considered (see Fig. 7). Indeed, this function decreases monotonically with ω (when a bistable system is considered), and therefore, if we are interested in the synchronization of a set of bistable subsystems, the knowledge of $\eta(D,\omega)$ does not allow us to choose an optimal frequency for a given value of the noise intensity. This fact implies that the optimal conditions for the synchronization phenomenon studied in this work differs from the conditions which optimize the stochastic resonance of a single bistable system. As mentioned above, the fractional fluctuation $\mathbb{F}(D,\omega)$ is indeed an adequate measure to quantify the synchronization of the transition times of a set of bistable elements. The geometrical structure of the function $\mathbb{F}(D,\omega)$ clearly shows the region (within the space D-ω) where the synchronization is optimal, and therefore it contains useful information for controlling a collection of bistable subsystems.

In closing this letter, it is worth mentioning the following points. It is essential to emphazise that the $3D$ plots in Figs. 3, 4 and 5 show that, in order to achieve good synchronization, an interplay between D and ω should be mantained. That is, if ω increases, D should also be increased. With regard to the influence of the noise intensity, we can appreciate in Fig. 1 that for very small values of D the transitions are scarce and not coupled to the frequency. Therefore, synchronization is not favoured in this case. We can also see that for high values of D, the transitions occur very frequently since the noise dominates the transitions and the synchronization also becomes very poor. However, for intermediate values of D the system tends to the linear response regime (the *rows* that appear in Fig.1b are an average of one period apart) and it is here that the synchronization can be enhanced depending, as mentioned above, on the frequency values.

Here we assumed that the interaction between the bistable subsystems was negligible in comparison to the effects of the external force and the noise; obviously this restricts the usefulness and predictive value of the information given by $\mathbb{F}(D,\omega)$. Therefore, it would be important to study how this function is modified when different interaction mechanisms are taken into account. It should also be pointed out that in the present letter we have considered the effect of white noise only. The effect of colored noise is significantly more complicated since it incorporates an additional parameter, the correlation time of the noise, and this study is in progress. As a final remark, we would like to point out that it would be interesting to investigate how the present analysis should be modified if the *non-inertial* assumption is removed. Although the behavior of a set of bistable subsystems with inertia might have some similarities when colored noise is considered, this relationship merits further investigation.

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