A hamiltonian control approach for the stabilization of the angular velocity of a rigid body system controlled by two torques

C. Aguilar-Ibáñez*
*Centro de Investigación en Computación, Instituto del Politécnico Nacional,
Av. Juan de Dios Bátiz s/n ,Esq. Miguel Othón de M.,
Unidad Profesional Adolfo López Mateos, Col. Nueva Industrial Vallejo,
Apartado Postal 75476 México, D.F. 07738, México,
FAX: +(52) 55 5586-2936.*

M.S. Suárez-Castañón
Escuela Superior de Cómputo del Instituto Politécnico Nacional.

F. Guzmán-Aguilar
Escuela Superior de Física y Matemáticas del Instituto Politécnico Nacional.

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We present a Hamiltonian control approach for the stabilization of a rigid body system that is controlled by two torques. The stabilization strategy consists in solving a feasible matching condition in order to derive a feedback controller which forces the closed-loop system to be globally asymptotically stable.

Keywords: Control of rigid body system; nonlinear control; Lyapunov stability.

Presentamos un enfoque de control Hamiltoniano para la estabilización de un sistema de cuerpo rígido que es controlado por dos torques. La estrategia de control consiste en resolver una condición de acoplamiento conveniente con el fin de derivar un controlador de retroalimentación que haga al sistema en lazo cerrado global y asintóticamente estable.

Descriptores: Control de un sistema de cuerpo rígido; control no-lineal; estabilidad de Lyapunov.

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1. Introduction

The controlled Hamiltonian approach is a useful method that allows us to stabilize a broader class of physical systems, that can be described by means of Hamiltonian equations. Roughly speaking, it consists in finding an external input that forces the closed-loop system[i] to follow another suitable Hamiltonian system with some stability properties. In some cases, it is convenient for the closed-loop system to be asymptotically stable around one unstable equilibrium point. In other cases, it is necessary for the desired system to follow periodic orbits or simply diminish the effect of undesirable vibration. In general, it is desirable for the total energy of the closed-loop system to go to zero or to a positive constant, depending on the requirements of the problem. One advantage to this method is that the original system can be seen as an energy transformation device, where the action of the controller may be interpreted, in terms of energy, as another system interconnected to the process to be controlled, in order to modify, as desired, the behavior of the target system (see Refs. 1 and 2). And this advantage allows us to see the control as a dissipator of the total energy of the system. While a survey of this topic is beyond the scope of this article, we refer the reader to see Ref. 3.

In this work we deal with the stabilization of the angular velocity of a rigid body system controlled by two torques using the energy-based control approach. This problem is important because it has a great number of applications to several engineering fields, such as the control of spacecrafts and satellite systems [4]. When a rigid body system is controlled by three torques, the problem is solved. However, when only one or two torques are available, we have an under-actuated mechanical system, because it has fewer actuators than degrees-of-freedom. As a result, many controlling strategies used for controlling fully-actuated systems cannot be directly applied to control this mechanical device. Also, this system cannot be input-output linearized by means of static feedback and it is not locally controllable around the origin [5, 6]. This fact makes it especially difficult to carry out some controlled maneuvers such as regulation at a point or tracking a trajectory [5]. On the other hand, a complete solution for the angular velocity stabilization and the tracking problem exists when the rigid body has three independent controllers. Sira *et al*. [7] proposed a redundant dynamical sliding mode control scheme for controlling a rigid body system, with the advantage of its being robust with respect to external perturbations. In Refs. 8 and 9 the regulation problem is solved by means of a PD-like control law, whereas in Ref. 10 the Energy-Casimir method is used to solve the stabilization around the origin. Brockett in Ref. 11 and Aeyels in Ref. 12 showed that the asymptotical stabilization of the angular velocity could be achieved by two independent controllers. A similar problem was addressed by Refs. 13 and 14, where the stabilization problem for a single torque is handled. In Ref. 15, the authors proposed time-varying feedback controllers to regulate the altitude of a rigid spacecraft with two inputs. In Ref. 16, the authors present a robust control strategy in order to attenuate the effect of external disturbances, with two independent torques. Reference 17 was devoted to the stabilization of the angular velocity of an Euler's system via variable structure based controllers. In Ref. 18, the author presents a control strategy for the stabilization of the angular velocity with two torques. The proposed strategy consists in transforming the original system into a discontinuous one by applying a discontinuous coordinate transformation, which achieves asymptotic stability with exponential convergence rates. While a survey of this topic is beyond the scope of this paper, we refer the reader to Refs. 19 and 20, for a detailed treatment of it.

In this paper we present a solution for the stabilization of the angular velocity of a rigid body system that is controlled by two independent actuators. Our control strategy consists in solving a feasible energy matching condition that allows us to build the total energy of the desired closed-loop system in such a way that it is globally asymptotically stable at the origin. Having satisfied this condition, we derive the state feedback control laws that asymptotically stabilize the rigid body system at the origin. The main contribution of this paper is in proposing and solving, in a very simple way, a suitable energy matching condition that allows us to obtain the two stabilizing controllers that render the system asymptotically stable at the origin. We must emphasize that this control problem is of considerable practical interest, since the designed state feedback laws can stabilize the system at the origin, even when one of the actuators of the rigid body system fails.

The remainder is organized as follows: Sec. 2 presents Euler's equations of the body system. Section 3 discusses the obtaining of the two stabilizing controllers by solving a convenient matching condition. Then, the convergence of the closed-loop system is guaranteed by applying the well-known LaSalle's invariance theorem. In Sec. 4 we evaluate the controllers' performance through some computer simulations. Finally, Sec. 4 contains the concluding remarks. The proof of Lemma 1 is found in the Appendix.

2. The rigid body

Consider a rigid body which is controlled by means of two torque inputs applied to two principal axes. Let w_1, w_2 and w_3 be the angular velocity components with respect to the principal axes, and denote by J_1, J_2 and J_3 the moments of inertia of the rigid body about the principal body axes. Let us assume that the two inputs are about the first two principal axes. The Euler equations for the rigid body system are given by [5]

$$J_1\dot{w}_1 = (J_2 - J_3)w_2w_3 + \tau_1$$
$$J_2\dot{w}_2 = (J_3 - J_1)w_1w_3 + \tau_1$$
$$J_3\dot{w}_3 = (J_1 - J_2)w_2w_3. \tag{1}$$

Here τ_1 and τ_2 are the torques that act as inputs for the system. In order to apply a matching energy controller based approach, we proceed to rewrite the above system as a controlled Hamiltonian system, described by

$$\dot{w} = J^{-1}\left(S(w)\frac{\partial V_0}{\partial w}(w) + Bu\right) \tag{2}$$

where $w = (w_1, w_2, w_3)^T$ is the state, $u^T = (\tau_1, \tau_2)$ is the controller, $J = \text{diag}(J_1, J_2, J_3)$ the inertia matrix, S and B are the internal and external interconnection matrices given by

$$S(w) = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

and V_0 is the total energy of the rigid body system, defined by

$$V_0(w) = \frac{1}{2}w^T Jw.$$

Notice that matrix S is a skew-symmetric matrix, that is, $x^T S(w)x = 0$, for all $x \in R^3$.

The control objective is to find smooth feedback controllers τ_1 and τ_2 that bring all the angular velocities to the rest equilibrium point. That is, we force the closed-loop system to be asymptotically stable at the origin from any initial conditions.

We must emphasize that the linearization of the system (1) about the origin has one uncontrollable eigenvalue at the origin. Hence the resulting linearized system is not stabilizable and can not be exponentially stabilized by a smooth feedback at the origin (see Ref. 21).

3. Control strategy

System (2) suggests the use of the matching control energy approach for the design of the stabilizing feedback control laws, which force the motion, starting from any arbitrary initial conditions $w(0)$, towards the desired resting equilibrium point $w = 0$. Intuitively, this control strategy consists in finding a suitable control u, such that the closed-loop system can be rewritten as a new asymptotic Hamiltonian system (see the previous works of Refs. 3, 22, and 23). To this end, we first introduce the definition of matching energy condition, then we obtain the necessary matching condition, which allows us to explicitly obtain the convenient candidate Lyapunov function and the desired control.

Now, consider a second, autonomous Hamiltonian system described by

$$\dot{w} = (S_d(w) - D)\frac{\partial V_d}{\partial w}(w), \tag{3}$$

where D is a constant positive diagonal matrix, $S_d(w)$ is a skew-symmetric matrix, and $V_d(w)$ is the desired energy function of the closed-loop system, selected in such a way that V_d is strictly positive with a global minimum at the origin. That is, $V_d(w) > 0$ for all $w \in R^3$, with $w \neq 0$ and $V_d(w) = 0$ if and only if $w = 0$. System (3) is the desired closed-loop system or target system. We chose system (3) as the *target system* because it is asymptotically stable, as we shall demonstrate in the next section.

Now we introduce a useful definition: we say that systems (2) and (3) are matched for some convenient control law $u(w)$ if the solutions to both systems are the same^{ii}. That is, ($w(t)$,$u(w(t))$) is a solution for (2) if and only if $w(t)$ is a solution for (3), for all $t \geq 0^{iii}$.

Therefore, systems (2) and (3) are matched if and only if the dynamics of the two systems are equal. Thus, equating the left-hand sides of (2) and (3) we have the following equality:

$$Bu = J\left(S_d(w) - D\right)\frac{\partial V_d}{\partial w}(w) - S(w)\frac{\partial V_0}{\partial w}(w). \quad (4)$$

From the above we have the following set of partial differential constraint equations, which have to be fulfilled for any control law (see Refs. 22 and 23):

$$B^{\perp}\left[S(w)\frac{\partial V_0}{\partial w}(w) - J\left(S_d(w) - D\right)\frac{\partial V_d}{\partial w}(w)\right] = 0, \quad (5)$$

where $B^{\perp}$ is the left annihilator of B. That is, $B^{\perp}B = 0$. Therefore, if variables S_d, D and V_d are known, then control $u(w)$ can be directly computed as

$$u = -\left(B^T B\right)^{-1} B^T \times \left[J\left(S_d(w) - D\right)\frac{\partial V_d}{\partial w}(w) - S(w)Jw\right]. \quad (6)$$

It is worth mentioning that Eq. (5) represents the dynamics of the system that cannot be manipulated or modified, while Eq. (6) represents the dynamics of the system that can be manipulated (or external control), which transforms the original system into a dissipative system with respect to the total energy function.

We summarize the control strategy as follows: we first need to solve the matching energy condition (5), which is directly related to the total energy of target system (3). Afterwards, control u is obtained via (6).

Remark 1: The above energy matching condition allows us to characterize all the energy functions that can be assigned to the target system by fixing the structure of the desired interconnection matrices S_d and D^{iv}. That is, matrices S_d and D can be seen as free parameters, used to achieve the above-mentioned energy matching condition. In general, this is not an easy task because we need to solve a non-linear partial differential equation (PDE). Therefore, there is no one single method to obtain V_d and the solution is not unique. Besides, the solution might not be feasible, that is, the obtained V_d might not be strictly positive or not well-defined for all $w \in R^3$. However, for this particular case it is relatively easy to ensure the desired energy matching condition, as we shall show in the next section.

Comments: We wish to emphasize that there are no explicit conditions for the existence of the solution to the PDE related to the energy-matching condition, as pointed out in Ref. 24. However, in many applications it is possible to ensure these conditions by adequately selecting the needful interconnection matrices S_d and D. Examples of these applications, such as the inverted pendulum, the inertia wheel pendulum and the spherical inverted pendulum, can be found in Refs. 22 and 23.

3.1. Solving the matching condition

The following lemma allows us to shape the stored energy function of the target system:

Lemma 1: *Let* D=diag$\{d_1, d_2, 1\}$, *with* d_1 *and* d_2 *strictly positive constants, and let* S_d *be a skew-symmetric matrix defined by*

$$S_d(w) = \begin{bmatrix} 0 & k & -k_2 - \delta w_2 \\ -k & 0 & -2k_3 w_3 \\ k_2 + \delta w_2 & 2k_3 w_3 & 0 \end{bmatrix}, \quad (7)$$

where $\delta = (J_1 - J_2)/J_3$, *and* k *is an arbitrary constant, and the constants* k_1, k_2 *and* k_3 *are selected according to*

$$\delta k_2(\delta k_2 + k_1 k_3) < 0 \quad \text{with} \quad k_1 > 0. \quad (8)$$

Then, the energy matching condition (5) *is satisfied, for the following:*

$$V_d(w) = \frac{1}{2}(w_1 + k_2 w_3)^2 + f(w_2, w_3) \quad (9)$$

where

$$f(w_2, w_3) = \frac{1}{4}\delta k_2 w_3^2(2w_2 + k_3 w_3^2) + \frac{1}{4}k_1(w_2 + k_3 w_3^2)^2. \quad (10)$$

Furthermore, $V_d(w)$ *is strictly positive with a global minimum at the origin.* Proof is given in the Appendix.

Observe that for any structural parameter δ we can always find k_1, k_2 and k_3 satisfying (8).

3.2. Closed-loop stability analysis

From the definition of the energy matching condition, already discussed in the previous section, it follows that the stability of system (2) in closed-loop with (6) is equivalent to the stability of the desired closed-loop system (3). Therefore, the stability analysis can be carried out using the target system.

Under the condition of Lemma 1, let us take $V_d(w)$ as a candidate Lyapunov function for the target system. Now, computing the time derivative of $V_d(w)$ around the trajectories of system (3) leads to

$$\dot{V}_d(w) = \left(\frac{\partial V_d}{\partial w}\right)^T \left(S_d(w) - D\right)\frac{\partial V_d}{\partial w} = -\left(\frac{\partial V_d}{\partial w}\right)^T D\frac{\partial V_d}{\partial w} = -\sum_{i=1}^{3} d_i\left(\frac{\partial V}{\partial w_i}\right)^2.$$

It is easy to show, by using simple algebraic considerations, that the above expression is strictly negative definitive. That is, V_d and $-\dot{V}_d$ are strictly positive definitive. Therefore, from the Lyapunov theorem (see Ref. 25), the origin of the closed-loop system is globally asymptotically stable.
Summarizing the above discussion, we present the main proposition of this paper:
Proposition 1 *Under the assumption of Lemma 1, the nonlinear system (2) in closed-loop with (6), is globally asymptotically stable.*

4. Numerical simulations

Simulations were performed for system (1) in closed-loop with (6). The physical parameters of the rigid body were selected as if it were a real satellite: J_1=27 kg m^2, J_2=17 kg m^2 and $J_3 = 25$ kg m^2. The initial conditions of the system were fixed as $w_1 = -3$, $w_2 = 20$ and $w_3 = 4$.

In the first experiment, we have fixed the gains of the controller as $d_1 = 35$, $d_2 = 25$, $k_1 = 1$, $k_2 = 3$, $k_3 = -3.5$ and $k = -2$. Figure 1 depicts the state response of the closed-loop system, with its respective controllers τ_1 and τ_2. It can be observed in Fig. 1 how the states converge to zero: w_1 does it almost instantly and it is followed by w_2 and w_3 in that order. Also, it can be seen that initially the rate convergence is fast, but after $t >= 5$ it becomes very slow, and as t is increased, little by little, all the states move closer and closer to zero. This happens because the closed-loop system is asymptotically stable but not locally exponentially stable. That is, we expect that as time goes to infinity, eventually all the states are closer to the origin. This is a disadvantage of the resulting asymptotic convergence of the closed-loop system, compared to other methods such as discontinuous control law [18], where exponential stability is guaranteed except at the origin.

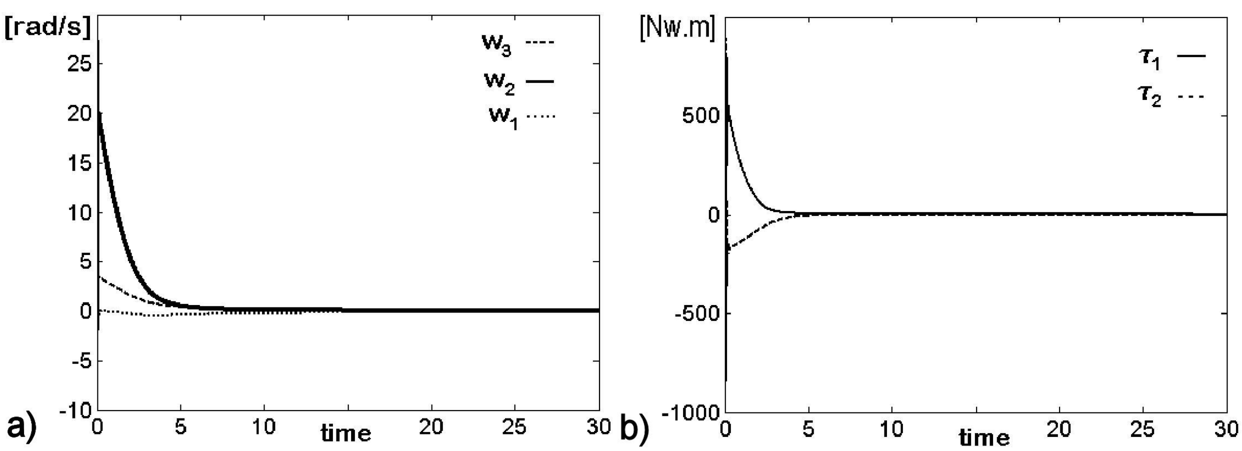

FIGURE 1. Closed-loop response of all the states of the rigid body system.

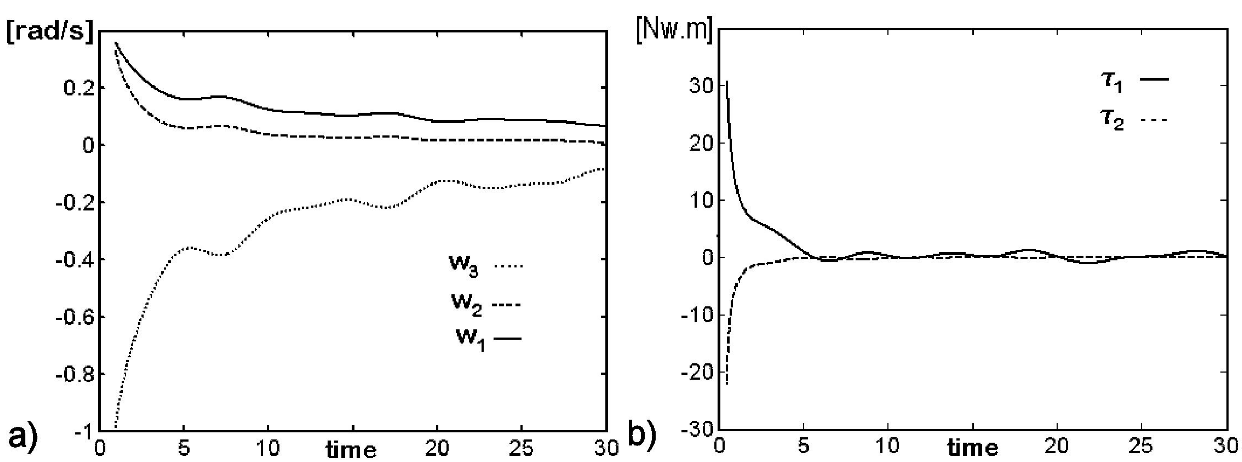

FIGURE 2. Closed-loop robustness of the control strategy when the rigid body system is exposed to external perturbations.

In the second experiment, we set the same initial conditions as in the first experiment. Nevertheless, to show the robustness of the proposed control strategy, we added the external perturbations $s(t)w_i$, $i = 1, 2, 3$ in the direction of the three axes, where $s(t)$ is a sinusoidal function uniformly distributed in $[-1, 1]$. Figure 2 depicts the state response of the closed loop, with its respective two controllers. It is clear that the control strategy is quite effective, even if the system is exposed to external perturbations.

5. Conclusions

An energy control strategy is used to stabilize the angular velocity of a rigid body system, which is controlled by two independent torques. The stabilization strategy is based on solving a feasible energy matching condition, which is directly related to the candidate Lyapunov function of the desired target system. The idea behind it consists in forcing the desired closed-loop system to behave like an asymptotic stable Hamiltonian system (3). To ensure the matching condition, it is necessary to solve a single third-order partial differential equation. Fortunately, the matching condition can be easily solved, as we showed in Lemma 1. The stability analysis is carried out by using the traditional Lyapunov method. The closed-loop performance of the controlled system is seen to be quite satisfactory, as assessed from the numerical simulations.

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Appendix

In this appendix section we show how the matrices D and S_d can be proposed in order to satisfy the matching condition (5). By definition of the desired closed-loop system (3), matrices D and S_d are given respectively, as:

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix},$$

$$S_d(w) = \begin{bmatrix} 0 & X_3 & -X_2 \\ -X_3 & 0 & X_1 \\ X_2 & -X_1 & 0 \end{bmatrix}, \tag{11}$$

where $d_i > 0$ for $i = \{1, 2, 3\}$. For simplicity we let $d_3 = 1$. After substituting the above matrices D and $S_d(w)$ and the values of $S(w)$, J and $B^{\perp}$, defined previously in (3), into the matching condition (5), we have[v]

$$0 = \delta w_1 w_2 + \frac{\partial V}{\partial w_3} + X_1 \frac{\partial V}{\partial w_2} - X_2 \frac{\partial V}{\partial w_1}. \tag{12}$$

To solve the above partial differential equation, we shape the desired positive function V, as we stated previously in (9). This trick was introduced in order to reduce the order of the above partial differential equation, from third to second. Then, substituting V, defined in (9), into relation (12), we obtain the following partial differential equation:

$$0 = w_1(k_2 + \delta w_2 - X_2) + w_3\left(k_2^2 - k_2 X_2\right) + X_1 \frac{\partial}{\partial w_2} f(w_2, w_3) + \frac{\partial}{\partial w_3} f(w_2, w_3).$$

From the above, we must note that it is convenient to eliminate the coefficient of w_1 in order to obtain a feasible $f(w_2, w_3)$. Thus, variable X_2 can be selected as $X_2 = k_2 + \delta w_2$. Also, variable X_1 can be selected as desired. However, in order to get a simple solution, we let $X_1 = -2k_3 w_3$. Thus the above relation turns out to be:

$$0 = -\delta k_2 w_2 w_3 - 2k_3 w_3 \frac{\partial}{\partial w_2} f(w_2, w_3) + \frac{\partial}{\partial w_3} f(w_2, w_3), \tag{13}$$

the solution to which has been given previously in the Lemma (see 10). That is, the obtained matrices D and S_d, and the proposed V, previously defined in the Lemma, satisfy the matching condition (5).

Finally, we need to guarantee the positiveness of the obtained function V. Indeed, the function f (10) can be expressed as a quadratic form given by $z^T Q z$, where $z = (w_2, w_3^2)$ and

$$Q = \begin{bmatrix} k_1 & \delta k_2 + k_1 k_3 \\ \delta k_2 + k_1 k_3 & \delta k_2 k_3 + k_1 k_3^2 \end{bmatrix}.$$

Evidently, $Q > 0$[vi] if and only if

$$\text{det(Q)} = -\delta k_2(\delta k_2 + k_1 k_3) > 0.$$

That is, if the set of constants $\{k_1, k_2, k_3\}$ satisfy inequality (8) then function f, defined previously in (10), is strictly positive definite. Now, it is relatively easy to check that the proposed V, defined by

$$V(w) = 1/2(w_1 + k_2 w_3)^2 + f(w_2, w_3),$$

is a strictly positive-definite function, for any $f(w_2, w_3)$ which is strictly positive-definite.■

$*$. To whom correspondence should be addressed (caguilar@cic.ipn.mx).

i. The original system (or physical plant) interconnected with the control action is referred as closed-loop system.

ii. V_0 and V_d refer the original and the desired energies, respectively.

iii. It is important to emphasize that the initial conditions of both systems, the target (3) and the open-loop (2), are the same. That is because we are forcing the dynamics of both systems to be the same.

iv. Recall that V_0 is given a priori.

v. Recall that $\delta = (J_1 - J_2)/J_3$ and the variables X_1 and X_2 can be selected, as desired.

vi. Recall that matrix $Q = \{q_{ij}\};\ i, j = 1, 2$ is strictly positive definitive, that is $Q > 0$, if and only if $a_{11} > 0$ and $det(Q) > 0$.

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