Stationary oscillations in a damped wave equation from isospectral Bessel functions

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Using the isospectral partners of the Bessel functions derived by Reyes *et al.* [1], we find, on one hand, that these functions show non-typical supersymmetric (SUSY) behavior and, on the other, that the isospectral partner of the classical wave equation is equivalent to that of a damped system whose oscillations do not vanish in time, but show a non-harmonic shape.

Keywords: Supersymmetry; Bessel functions.

Usando las compañeras isoespectrales de las funciones de Bessel obtenidas por Reyes *et al.* [1], encontramos, por un lado, que estas funciones muestran un comportamiento atípico de SUSY, mientras que, por otro lado, la compañera isoespectral de la ecuación de onda clásica es equivalente a la de un sistema amortiguado cuyas oscilaciones no desvanecen con el tiempo, sino que obtienen una forma que no es armónica.

Descriptores: Supersimetría; ecuación de Bessel.

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1. Introduction

In quantum mechanics (QM), the number of problems that can be exactly solved is very limited, and one can only hope to get the approximate solution using a variety of methods, or turn to a modified problem which can be described in terms of the known exact problems. In fact, one of the main virtues of supersymmetric (SUSY) quantum mechanics is that there one can find an infinite number of one-parameter problems which possesses the same spectra as the known exact ones. If SUSY cannot be applied, one can still find isospectral solutions by using the classical factorization method, or the interwinning method, to look for new equations with the old spectra.

In classical mechanics, the only problem that can be directly compared to quantum mechanics is that described by the classical wave equation

$$\left(\nabla^2 - \frac{1}{v^2}\right)\psi(x,y;t) = 0, \qquad (1)$$

but it has been stated before that there is no SUSY partner for this equation [2]. In fact, a factorization of the Bessel equation *a la* Infeld and Hull cannot be found [3]. However, the Bessel equation,

$$\frac{d^2J_n(r)}{dr^2} + \frac{1}{r}\frac{dJ_n(r)}{dr} + \left(1 - \frac{n^2}{r^2}\right)J_n(r) = 0\,, \qquad (2)$$

with $n \geq 0$, which arises from the wave equation after separation of variables, still possesses a factorization in terms of raising and lowering operators defined by the equations [4]

$$A_n^+ J_n(r) = \left(\frac{d}{dr} - \frac{n}{r}\right)J_n(r) = -J_{n+1}(r)\,, \qquad (3)$$

$$A_{n+1}^- J_{n+1}(r) = \left(\frac{d}{dr} + \frac{n+1}{r}\right)J_{n+1}(r) = J_n(r)\,, \qquad (4)$$

respectively.

Following the work of Mielnik [5], who finds a family of potentials that posses the same spectrum as that of the one-dimensional harmonic oscillator, and the work of Piña [6] on the factorization of some special functions found in mathematical physics, Reyes *et al.* [1] are able to derive second-order differential equations that are 'isospectral' to the equations described in the Sturm-Liouville theory. Note that since the 'spectral' parameter n appears in A_n^+ and A_n^-, contrary to SUSYQM [5], it shows up in the partner equation in a more complicated fashion than the original equation.

As one can see, obtaining the isospectral partner of the Bessel equation is the first step toward an isospectral classical wave equation, this being the main purpose of this letter. But, before proceeding in this direction, we first show that the partner functions of the Bessel functions show a unique and unusual SUSY behavior. Then, we show that the isospectral classical wave equation that comes from the isospectral partner of the Bessel equation resembles the problem of damped waves, which nevertheless do not vanish in time, but show non-harmonic shapes due to this damping term.

2. Isospectral Bessel Equation

In Ref. 1, the isospectral partner of the Bessel equation was found to be

$$\frac{d^2\widetilde{J}_{n+1}}{dr^2}+\frac{1}{r}\frac{d\widetilde{J}_{n+1}}{dr}+\left(1-\frac{(n+1)^2}{r^2}\right)\widetilde{J}_{n+1} = -\frac{4n}{r^2}\frac{\left[(2n+1)\gamma r^{2n}+1\right]}{\left(\gamma r^{2n}+1\right)^2}\,\widetilde{J}_{n+1}, \quad (5)$$

where, as is usual in SUSYQM, $n\geq 0$ and the lowest lying eigenvalue is lost. Here, γ is the parameter of the partner functions, with $0\leq\gamma<\infty$.

The term on the right of Eq.(5) corresponds to the extra potential function term in a typical SUSYQM problem, with the difference that in this case, the integral defining this term is exactly solvable. This also happens in the partner Bessel functions, which were found to be

$$\widetilde{J}_{n+1}(r;\gamma)=-J_{n+1}(r)+\frac{2n}{r}\left(\gamma r^{2n}+1\right)^{-1}J_n(r)\,. \quad (6)$$

Notice that the fact that these functions are explicitely found allows one to take a closer look at their properties, and by doing so we are here able to show that there exists a nontypical SUSY behavior for the isospectral Bessel functions, in the following sense.

In Ref. 1 it was thought that the partner functions were not regular at $r=0$. This is not the case, however. One can see that $\widetilde{J}_1=-J_1$, and that for $n>0$ we can use one of the recursion relations among the Bessel functions, namely,

$$\frac{2n}{r}J_n(r)=J_{n+1}(r)+J_{n-1}(r)\,, \quad (7)$$

to find that the partner Bessel functions are regular at $r=0$, since then

$$\widetilde{J}_{n+1}(r;\gamma)=\frac{-\gamma r^{2n}}{\gamma r^{2n}+1}J_{n+1}(r)+\frac{1}{\gamma r^{2n}+1}J_{n-1}(r)\,. \quad (8)$$

Now, this is a very unique feature of the Bessel partner functions. In SUSYQM the partner function of order $n+1$, $\tilde{\psi}_{n+1}(x;\gamma)$, is related to the original function of preceding order $\psi_n(x)$ through a differential operator. Here, the partner function of order $n+1$ is related to two Bessel functions, of orders $n-1$ and $n+1$ (see Fig. 1.) Moreover, one can see that the limits of the parameter γ give

$$\widetilde{J}_{n+1}(r;\gamma=0)=J_{n-1}(r)\,, \quad (9)$$

while

$$\widetilde{J}_{n+1}(r;\gamma=\infty)=-J_{n+1}(r)\,. \quad (10)$$

Therefore, the partner Bessel function $\widetilde{J}_{n+1}(r;\gamma)$, transforms from $J_{n-1}(r)$ to $-J_{n+1}(r)$ as γ goes from 0 to ∞. This is its most unusual characteristic, to be related to a Bessel function of two orders less than its own.

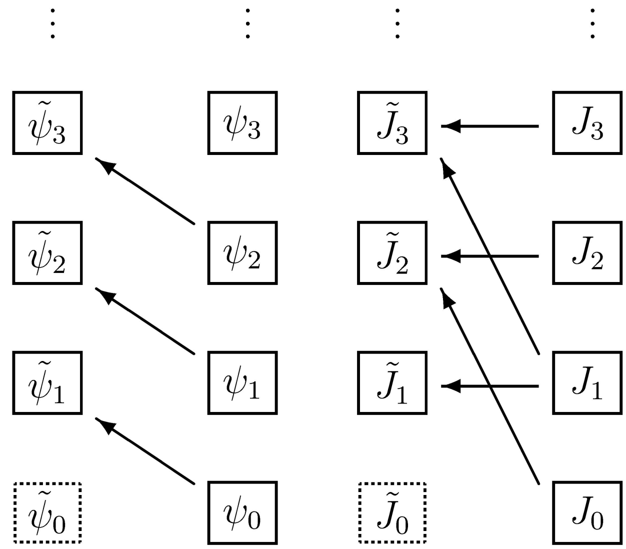

FIGURE 1. Typical SUSY isospectral function generating scheme (left), and the way Bessel isospectral partners are generated (right), in terms of the $n-1$ and $n+1$ orders of the original functions.

3. Stationary oscillations in a damped wave equation

In order to connect the partner Bessel funcions to the wave equation, we begin writing Eq.(5) for kr instead of r:

$$r^2\frac{d^2\widetilde{J}_{n+1}}{dr^2}+r\frac{d\widetilde{J}_{n+1}}{dr}+\left(k^2r^2-(n+1)^2\right)\widetilde{J}_{n+1} = g_{n+1}(kr;\gamma)\widetilde{J}_{n+1}\,, \quad (11)$$

where

$$g_{n+1}(u;\gamma)=-\frac{4n\left[(2n+1)\gamma u^{2n}+1\right]}{\left(\gamma u^{2n}+1\right)^2}\,. \quad (12)$$

Now, we multiply by a function of the polar angle θ, $H(\theta)$, and assume that this function satisfies the equation

$$\frac{d^2H}{d\theta^2}+(n+1)^2H=0\,, \quad (13)$$

in order to write the wave equation

$$\left(\nabla^2-\frac{1}{v^2}\right)\widetilde{\psi}_{n+1}(r,\theta;t;\gamma) = \frac{g_{n+1}(kr;\gamma)}{r^2}\widetilde{\psi}_{n+1}(r,\theta;t;\gamma), \quad (14)$$

where $\widetilde{\psi}_{n+1}(r,\theta;t;\gamma)=\widetilde{J}_{n+1}(kr;\gamma)H(\theta)e^{i\omega t}$.

This is the wave equation derived from the isospectral Bessel functions in Ref. 1, possessing a damping term $g_{n+1}(kr;\gamma)/r^2$ which, though singular at $r=0$, allows for stationary solutions whose radial part is the partner Bessel function (8). It is the kind of equation that may help in our understanding of neuronal activity, where such equations appear [7].

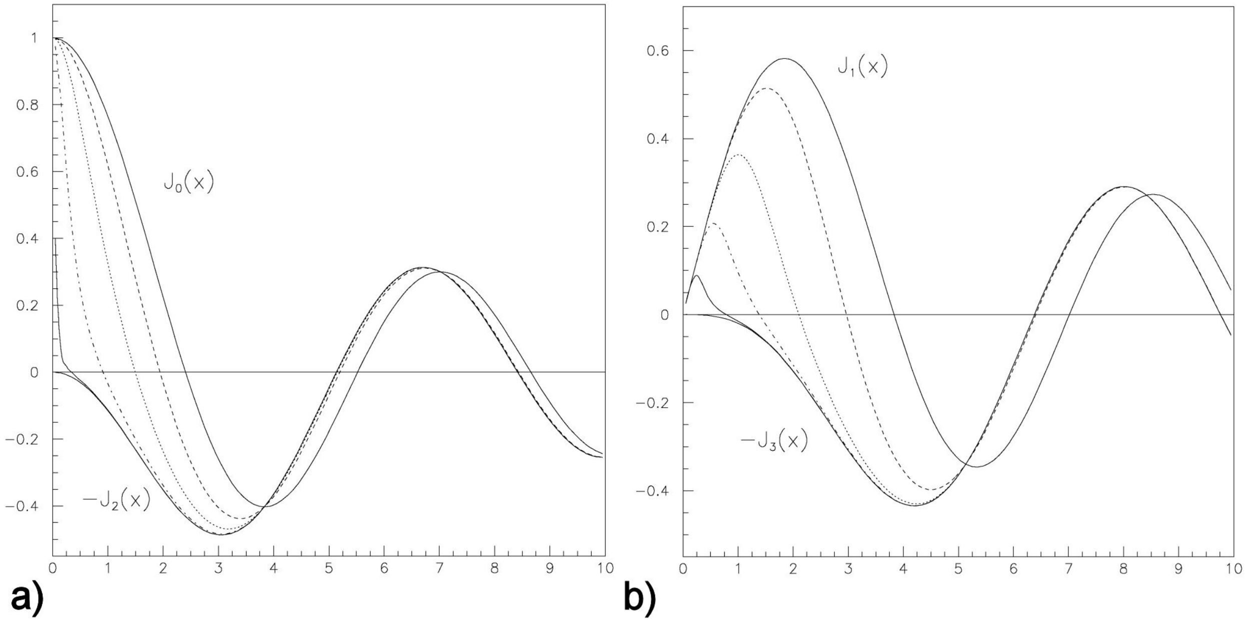

FIGURE 2. Partner Bessel functions: $\widetilde{J}_2$ (left), evolving from J_0, $\gamma = 0$, to $-J_2$, $\gamma = \infty$, and $\widetilde{J}_3$, evolving from J_1 to $-J_3$ (right). Note how the damping term affects the typical harmonic shapes of the Bessel functions, except when $\gamma = 0$ and $\gamma = \infty$.

One may ask then, how is this damping term reflected in the stationary waves? The answer comes from our discussion above, about the way $\widetilde{J}_{n+1}$ is determined by J_{n-1} and J_{n+1}. Notice that for $\gamma = 0$, the damping term reduces to $4n/r^2$ and is absorbed into the ordinary Bessel equation (2), and that for $\gamma = \infty$ the damping term reduces to zero. For other values of γ, the damping term modifies the Bessel functions, making them lose their typical harmonic shapes, as can be seen in Fig. 2, where drastic changes in the harmonic shapes are seen as γ increases towards infinity, especially for r close to zero.

4. Conclusion

In this letter we have shown that it is possible to find an isospectral partner of the classical wave equation. By using the isospectral partners of the Bessel functions from Ref. 1, we have shown that the arising wave equation contains a damping term, whose action does not destroy the stationary waves, but makes them acquire non-harmonic shapes. Since the Bessel equation cannot be supersymmetrized [2], this may be the only way one can find an isospectral partner of the wave equation.

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