Controlled Lagrangian approach to the stabilization of the inverted pendulum system

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A controlled Lagrangian approach is presented for the stabilization of an inverted pendulum mounted on a cart. The stabilization strategy consists in forcing the closed-loop system to behave as an Euler-Lagrange system, with a fixed inertia matrix. For carrying it out, it is necessary to adequately shape the potential and kinetic energies of the closed-loop system. The idea behind this procedure is to make an energy-balance between the overall energy of the pendulum system and the dissipation energy produced by the action of the control force. The resulting closed-loop system is locally asymptotically stable about its unstable equilibrium point with a very large attraction domain.

Keywords: Euler-Lagrange system, energy balance, Lyapunov method.

En este trabajo se presenta el problema de la estabilización de un péndulo invertido sobre un carro mediante el lagrangiano controlado. La estrategia de estabilización consiste en forzar al sistema en lazo cerrado a que se comporte como un sistema Euler-Lagrange, donde la matriz característica de inercia es constante. Para lograr esto, es necesario modelar adecuadamente las energías potencial y cinética del sistema en lazo cerrado. La idea fundamental de este procedimiento es hacer un balanceo de energía entre la energía de todo el sistema y la energía disipada mediante los movimientos horizontales del carro. El sistema obtenido en lazo cerrado es localmente estable asintóticamente alrededor de su punto de equilibrio inestable, con un dominio de atracción muy grande.

Descriptores: Sistema Euler-Lagrange, balanceo de energía, método de Lyapunov.

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1. Introduction

The controlled Lagrangian approach is a useful method that allows us to stabilize a broad class of physical systems that can be described by the Euler-Lagrange motion equations. Loosely speaking, it consists in looking for an external input that forces the closed-loop system[i] to follow a convenient Euler-Lagrange system with a suitable stability property. For instance, in some cases, it is useful for the closed-loop system to be asymptotically stable around one unstable equilibrium point. In other cases, it is necessary for the system to follow periodic orbits or simply diminish the effect of undesirable vibrations. An advantage to this method is that the original system can be seen as an energy transformation device, where the action of the controller may be interpreted, in terms of energy, as another system interconnected with the process, in order to modify as desired the behavior of the overall energy. In general, it is desirable for the total energy of the closed-loop system to go to zero or to one positive constant, depending on the requirements of the problem (see Refs. 1 to 8). Here, we are focusing on the asymptotic stabilization of an inverted pendulum mounted on a cart with a restricted domain of attraction. This mechanical device consists of a free vertical rotating pendulum with a pivot point mounted on a cart. The cart can be moved horizontally by means of a horizontal force. The stabilization problem consists of bringing up the pendulum to the upright vertical position, with the cart resting at the origin. This control problem has been dealt with from two different approaches. The first approach is based on the mathematical foundations of the nonlinear theory, and the second is based on soft-computing control methods. In regard to the first approach we mention the following works. In Ref. 4 the authors solve the stabilization problem based on the controlled Lagrangian method in conjunction with some symmetry properties that the system satisfies. However, they cannot guarantee regulation of the cart position. A similar work, with similar tools, was presented in Refs. 1 and 10, with the advantages of guaranteeing, for a large domain of stability, the asymptotic convergence of all the state variables. In Ref. 11 the authors solve the stabilization problem by using a Lyapunov-based approach. To this end, they shape a candidate Lyapunov function that would make it possible to derive the stabilizing controller. Related to the second approach we only refer the reader to Refs. 12 to 15, not without mentioning that, in general, these works have the advantage of been able to control the device without the need to know either the full parametric physical model or all the state variables; however, they cannot ensure asymptotic stability for the whole state, and computing the corresponding domain of attraction is quite difficult.

Our approach focuses mainly on proposing a simple control strategy for bringing the pendulum to the top position and the cart to the zero position simultaneously, by forcing the closed-loop system to behave as a locally stable Euler-Lagrange system, with a constant inertia matrix. The main contribution of this work is building the closed-loop potential and kinetic energies for the entire system by solving two restricted equations, which are very easy to solve in comparison to other energy-based approaches (see the previous works of Refs. 1, 3, and 10). On the other hand, the new potential energy (related to the total energy of the system) allows us to determine the domain of attraction of the closed-loop system. As a matter of fact, the latter can be adjusted adequately to keep the two position variables moving inside a certain admissible set.

The rest of the paper is organized as follows. Section 2 presents the dynamic model of the **IPC**. In Sec. 3, a suitable target system is presented in order to shape the needed potential and kinetic energies. Section 4 depicts the stability analysis of the closed-loop system and Sec. 5 presents some computer simulations, while the conclusions are given in Sec. 6.

2. The inverted pendulum cart system

Consider the traditional inverted pendulum mounted on a cart (**IPC**) (see Fig. 1), which is described by the following normalized set of differential equations:

$$\begin{aligned} \cos\theta\ddot{q} + \ddot{\theta} - \sin\theta &= 0, \\ (1+\delta)\ddot{q} + \cos\theta\ddot{\theta} - \dot{\theta}^2\sin\theta &= f, \end{aligned} \tag{1}$$

where q is the cart normalized displacement, θ is the angle that the pendulum forms with the vertical, f is the force applied to the cart, acting as a control input, and δ is a structural parameter related to the mass of the cart and the pendulum, respectively [9]. As the damping force in the actuated coordinate q can be easily compensated, we do not include this term. After applying the following feedback,

$$f = \cos\theta\sin\theta - \dot{\theta}^2\sin\theta + v(2+\sin^2\theta+\delta),$$

into system (1), we obtain

$$\begin{aligned} \ddot{\theta} &= \sin\theta - \cos\theta v, \\ \ddot{q} &= v. \end{aligned} \tag{2}$$

Evidently, the above system may be expressed as:

$$\ddot{\mathbf{x}} = -F(\theta) + G(\theta)v, \tag{3}$$

where

$$F(\theta) = \begin{bmatrix} -\sin\theta \\ 0 \end{bmatrix}, \quad G(\theta) = \begin{bmatrix} -\cos\theta \\ 1 \end{bmatrix}, \tag{4}$$

and $\mathbf{x}$ stands for $\mathbf{x}^T = (\theta, q)$. Note that the system (2) has two equilibrium points, when $v = 0$ and $\theta \in [0, 2\pi]$, one being an unstable equilibrium point $(\theta, \dot{\theta}, q, \dot{q}) = (0,0,0,0)$ and the other being a stable equilibrium point $(\theta, \dot{\theta}, q, \dot{q}) = (\pi,0,0,0)$.

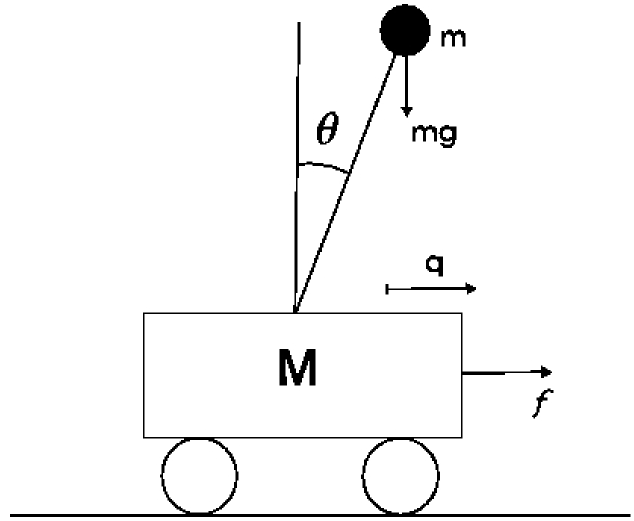

FIGURE 1. The inverted pendulum cart system.

Problem statement: The objective is to bring the pendulum to its unstable upright position with the cart resting at the origin by using the control Lagrangian approach, restricting the pendulum angle and the cart position to movement inside the admissible set. This set can be synthesized by analyzing the domain of attraction of the obtained closed-loop system.

3. Control Lagrangian approach

Under the assumption that the pendulum angle position is initialized over the horizontal plane, we attempt to asymptotically stabilize the pendulum about its unstable equilibrium point. For that purpose we propose a simple controlled Lagrangian approach to solve it. Intuitively, we wished to find a controller v that would transform the original system (3) into another nonlinear Euler-Lagrange system, with some desired stability properties. That is, we are looking for a control law v such that the closed-loop system can be written in the form

$$M_d\ddot{\mathbf{x}} = -K_d(\mathbf{x})\dot{\mathbf{x}} - \nabla_x V_d(\mathbf{x}), \tag{5}$$

where M_d is strictly symmetric positive definitive, $K_d(\mathbf{x})$ is symmetric and positive semi-definitive, and $V_d(\mathbf{x})$ is a local positive function with a local minimum. We refer to system (5) as a “target system”. For our convenience, function $V_d(\mathbf{x})$ is selected in such a way that $\nabla_x V_d(0) = 0$ and $\nabla_x^2 V_d(0) > 0$. That is to say, $V_d(\mathbf{x})$ is a locally strictly convex function around the origin. Evidently, the two systems (3) and (5) are equivalent systems, for a given control law v if the solution of both systems are the same. That is, $(\mathbf{x}(t), v(t))$ is a solution of (3), if and only if , $\mathbf{x}(t)$ is a solution of (5).

Therefore, the two systems (3) and (5) are equivalent, if the following equality is satisfied:

$$-F(\theta) + G(\theta)v = -M_d^{-1}\left(K_d(\mathbf{x})\dot{\mathbf{x}} + \nabla_x V_d(\mathbf{x})\right) \tag{6}$$

It should be noticed that if G is invertible[ii], then we can obtain directly the desired controller v, for any given K_d, and

V_d. However, G is not invertible because it is a single column. That is, control v only acts in the range space of G. From this, we have the following constraint equation:

$$0 = G^{\perp} M_d^{-1} \nabla_x V_d(\mathbf{x}) - G^{\perp} F(\theta) + G^{\perp} M_d^{-1} K_d(\mathbf{x})\dot{\mathbf{x}}. \quad (7)$$

This is followed by multiplying both sides of (6) by the annihilator of G.[iii] Consequently, if the unknown functions K_d and V_d are obtained for a given F, then control v can be computed directly by

$$v = \frac{-G^T M_d^{-1} \left(K_d(\mathbf{x})\dot{\mathbf{x}} + \nabla_x V_d(\mathbf{x})\right) + G^T F(\theta)}{1 + \cos^2\theta}. \quad (8)$$

Clearly, the above restricted equation can be split into two restricted partial differential equations, given by:

$$0 = G^{\perp}\left[M_d^{-1} \nabla_x V_d(\mathbf{x}) - F(\theta)\right], \quad (9)$$

and

$$0 = G^{\perp} M_d^{-1} K_d(\mathbf{x})\dot{\mathbf{x}}, \quad (10)$$

where the control v can be obtained via relation (8).

Property: The proposed Euler-Lagrange system (5) is a dissipative system with respect to the total stored energy function defined as:

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}\dot{\mathbf{x}}^T M_d \dot{\mathbf{x}} + V_d(\mathbf{x}), \quad (11)$$

since the time derivative of E, with respect to the trajectories of (5), leads to:

$$\dot{E}(\mathbf{x}, \dot{\mathbf{x}}) = \dot{\mathbf{x}}^T M_d \ddot{\mathbf{x}} + \dot{\mathbf{x}}^T \frac{\partial V_d(\mathbf{x})}{\partial x} = -\dot{\mathbf{x}}^T K_d(\theta)\dot{\mathbf{x}}.$$

Due to the fact that $K_d(\theta) \geq 0$ for all $\theta \in I_s \subset (-\pi/2, \pi/2)$, then we have that $\dot{E}$ is semi-definite negative.

Remark 1: *The closed-loop system, given by (3) and (8), is stable if and only if the target system (5) is stable. This was the main reason for selecting the target system as we did. Thus, we can use E as a candidate Lyapunov function to analyze the stability of the closed-loop system.*

Let us find the unknown matrices M_d, K_d and the unknown function V_d that satisfy the two restricted conditions. To do so, we introduce the following lemma:

Lemma 1: *Tacking M_d^{-1} and $K_d(\theta)$ as*

$$M_d^{-1} = \begin{bmatrix} 1 & -\mu_2 \\ -\mu_2 & \mu_3 \end{bmatrix};$$
$$K_d(\theta) = \begin{bmatrix} \frac{(\mu_2 - \mu_3\cos\theta)k_2(\theta)}{1 - \mu_2\cos\theta} & k_2(\theta) \\ k_2(\theta) & \frac{(1-\mu_2\cos\theta)k_2(\theta)}{\mu_2 - \mu_3\cos\theta} \end{bmatrix}, \quad (12)$$

where $\mu_2 > 1$, $\mu_3 > \mu_2^2$, $k_2(\theta) > 0$ and

$$V_d(\mathbf{x}) = \frac{1}{\mu_2}\ln(-1 + \mu_2) - \frac{1}{\mu_2}\ln(-1 + \mu_2\cos\theta) + \frac{k_p}{2} g(s^2), \quad (13)$$

where g is any smooth function and

$$s = q + \frac{\mu_3}{\mu_2}\theta + \frac{2(\mu_3 - \mu_2^2)}{\mu_2\sqrt{-1 + \mu_2^2}} \times \tanh^{-1}\left(\frac{1 + \mu_2}{\sqrt{-1 + \mu_2^2}}\tan\frac{\theta}{2}\right). \quad (14)$$

Then, Eqs. (9) and (10) are simultaneously fulfilled, for all $\theta \in I_\mu = (-\theta_\mu, \theta_\mu)$, by

$$\theta_\mu = \cos^{-1}\left(\frac{1}{\mu_2}\right). \quad (15)$$

Function g can be selected almost any way we wish. For example, we can introduce any saturation function such as $g(s) = \tanh(s)$ or $g(s) = s^2$. Here, we use the simple quadratic function.

Comment 1: As $\nabla_x V_d(0) = 0$ and $\nabla_x^2 V_d(0) > 0$, then the set formed by $V_d(x) \leq \alpha$ (with $\alpha > 0$ and small enough) is a convex set.[iv] On the other hand, if the closed-loop kinetic energy function is a globally strictly convex function, then the set $E(x, \dot{\mathbf{x}}) \leq \alpha$ defines a compact set. This property is important in order to apply the LaSalle theorem. Figure 2 shows the level curves defined by the obtained $V_d(x)$. Notice that when $\alpha \geq 3$, the set $\left\{x \in R^2 : V_d(x) \leq 3\right\}$ is no longer either a convex set or a compact set. On the other, when $\alpha \leq 1$, the set $\left\{x \in R^2 : V_d(x) \leq 1\right\}$ is a compact set. The physical meaning of it is that any solution that fulfills $E(x, \dot{x}) \leq 1$ will always remain inside this compact set.

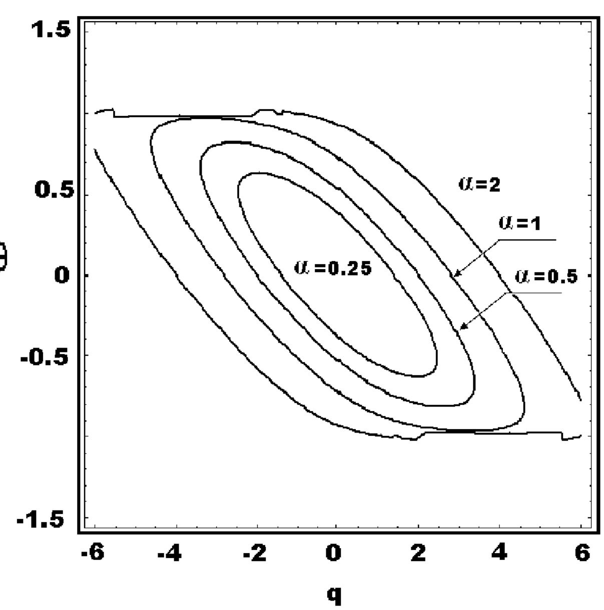

FIGURE 2. Level curves for $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 1.0$ and $\alpha = 3$.

4. Closed-loop stability analysis

From **Property 1** and **Remark 1**, we find that system (5) is stable in the sense of Lyapunov, since

$$\dot{E}(\mathbf{x},\dot{\mathbf{x}}) = -\dot{\mathbf{x}}^T K_d(\theta)\dot{\mathbf{x}}. \tag{16}$$

Thus, to guarantee the asymptotic stability of the closed-loop system, we need to use LaSalle's invariance theorem. First of all, we need to ensure that angle θ belongs to the set I_μ.[v] For that purpose, it is sufficient that the initial condition $(\mathbf{x}_0, \dot{\mathbf{x}}_0)$ with $\theta_0 \in I_\mu$ belonging to a neighborhood of the origin such that

$$E(\mathbf{x}_0, \dot{\mathbf{x}}_0) < V_d(\theta_\mu, 0) = C_\mu, \tag{17}$$

where θ_μ was defined previously.

Remark 2 : *The above inequality defines a stability region for the closed-loop system. That is, if the initial condition fulfills the inequality* $E(x_0, \dot{\mathbf{x}}_0) < C_\mu$, with $\theta_0 \in I_\mu$, *then necessarily* $\theta(t) \in I_\mu$. *According to this fact, we can define a compact set* Ω *as:*[vi]

$$\Omega = \{(\mathbf{x},\dot{\mathbf{x}}) : E(\mathbf{x},\dot{\mathbf{x}}) < C_\mu\} \tag{18}$$

The set Ω *has the property that all solutions of the closed-loop system (5) that begin in* Ω *always remain in* Ω.

Continuing with the stability analysis, in order to apply LaSalles's Theorem, we must define the following invariant set:

$$S = \{(\mathbf{x},\dot{\mathbf{x}}) \in \Omega : -\dot{\mathbf{x}}^T K_d(\theta)\dot{\mathbf{x}} = 0\}. \tag{19}$$

Now, let M be the largest invariant set in S. LaSalle's theorem guarantees that every solution starting in a compact set Ω approaches M as $t \to \infty$ [16]. Therefore, we need to compute the largest invariant set M in S.

Let us then compute the largest invariant set M in S. To do so, we first rewrite S as

$$S = \left\{ (\mathbf{x},\dot{\mathbf{x}}) \in \Omega : -k_2(\theta)\beta(\theta)\left(\dot{\theta} + \frac{1}{\beta(\theta)}\dot{q}\right)^2 = 0 \right\}, \tag{20}$$

where the free function $k_2(\theta)$ is different from zero, and

$$\beta(\theta) = \frac{(\mu_2 - \mu_3 \cos\theta)}{1 - \mu_2 \cos\theta}.$$

Note that on the set S, we must have that $\theta \in I_\mu$, so that $\beta(\theta) > 0$, in S (recalling that $\mu_3 > \mu_2^2$). Therefore, from the definition of S (20), we have that

$$\dot{\theta} + \frac{1}{\beta(\theta)}\dot{q} = 0, \quad \text{with} \quad \beta(\theta) > 0, \quad \text{on the set} \quad S.$$

That is, on the set S the variables $\dot{\theta}$ and $\dot{q}$ do not change their sign. Now, if the variables $\dot{\theta}$ and $\dot{q}$ are different from zero and have the same sign inside of the set S, then (θ, q) tends to go outside the invariant set Ω. But this case is a contradiction because we have assumed that $(\mathbf{x},\dot{\mathbf{x}}) \in \Omega$. Therefore, we have that $\dot{\mathbf{x}} = 0$ and also $\mathbf{x}$ is a fixed constant vector on the set S. Let us define $\mathbf{x} = \overline{\mathbf{x}}$.[vii] Then $\overline{\mathbf{x}}$ is one of the two equilibrium points of the system (3). In other words, $\overline{\mathbf{x}} = (0,0)$ or $\overline{\mathbf{x}} = (\theta = \pi, q = 0)$. But from definitions of the invariant set Ω, given in (18), necessarily $\overline{\mathbf{x}} = 0$. Hence, $M = 0$. That is, the largest invariant set M contained inside the set S is constituted by the single equilibrium point $(\mathbf{x} = 0, \dot{\mathbf{x}} = 0)$. According to LaSalles's theorem, all the closed-loop solutions starting in Ω asymptotically converge towards the largest invariant set M, which is given by $(\mathbf{x} = 0, \dot{\mathbf{x}} = 0)$. ■

In summary, we present the main proposition of this paper:

Proposition 1: *Consider the system (3) in closed-loop with (8), where* M_d, K_d *and* V_d *are selected according to Lemma 1. Then, the closed-loop system is locally asymptotically stable with its domain of attraction defined by the set* Ω *(18).*

The locally exponential stability of the closed-loop system around the origin can be easily probed by simple linearization, but for space limitations we omit this demonstration. Nevertheless, we can say that the closed-loop system is robust with respect to small un-modeled dynamics. That is, even in the case when the damping force is small enough and the system is initialized close to the origin, the system stills achieving the desired unstable equilibrium point. It can be seen in the numerical simulations presented in the following section.

Remark 3: *If the two position variables are initialized inside of* Ω *(see* ***Remark 2****) with zero velocities, then we can tuning the control parameters ensuring that the cart position and the pendulum angle position remain inside of certain admissible set* $Q \subset \Omega$, *where*

$$Q = \{\mathbf{x} = (\theta, q) \subset \Omega : |\theta| < \theta_\mu < \pi/2 \text{ and } |q| < \overline{q}\}.$$

Of course $\overline{q}$ *must be selected according to the physical restriction on the cart movement. In other words, it is possible to bring all the states to the upright unstable position, restricting the angle position and the cart movement to confinement within the admissible set* Q.

5. Simulation results

To test the performance of the obtained control law we carried out some numerical simulations using the MATLAB™ system. The controller parameters were fixed as $\mu_2 = 2$, $\mu_3 = 5$, $k_2 = 1$ and $k_p = 0.25$, and the initial conditions were set as $\theta_0 = -1.05$[rad], $\dot{\theta}_0 = 0.1$[rad/sec], $q_0 = 3$ and $\dot{q}_0 = 0$. To show the damping force effect we simulated the system once again under the same initial conditions, but we includes the linear term "0.1 $\dot{\theta}$" in the non-actuated coordinate. Figures 3 and 4 show the closed-loop response of each state when damping is not present and when damping is

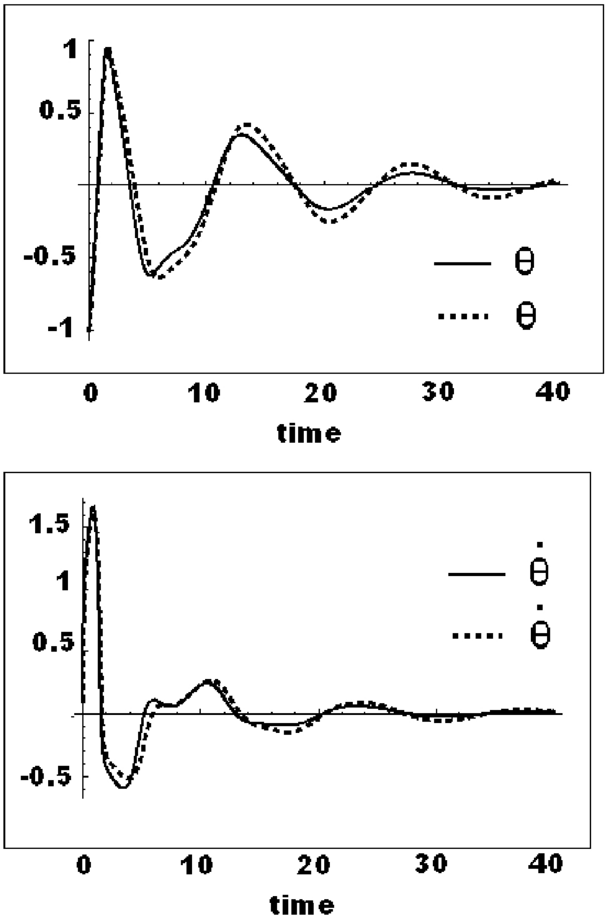

FIGURE 3. Closed-loop behavior of the variables θ and $\dot{\theta}$, for two values of β. Continuous line and dotted line indicate that $\beta = 0$ and $\beta = 0.1$, respectively.

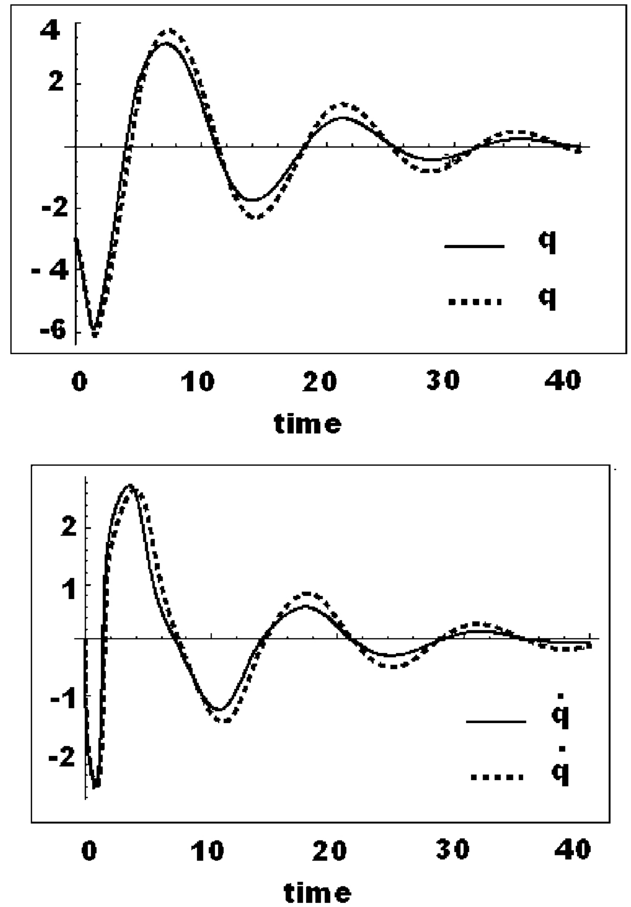

FIGURE 4. Closed-loop behavior of the position and velocity of the cart, for two values of β. Continuous line and dotted line indicate that $\beta = 0$ and $\beta = 0.1$, respectively.

present, respectively. In the first case the closed-loop response is represented by a continuous line and the second case is represented by a dotted line. Also we can see in these figures that the damping force effect produces wider oscillations around the origin. As we can see our strategy is quite robust with respect to the dissipation force, because the resulting closed-loop system is locally exponentially stable around.

6. Conclusions

The controlled Lagrangian approach is used for the stabilization of the **IPC** around its unstable equilibrium point, assuming that the pendulum is initialized above the horizontal plane. The idea behind this is to introduce an adequate feedback that allows us to re-write the original system as a stable Euler-Lagrange system with a constant inertia matrix (5). To this end, we need to build adequate potential and kinetic closed-loop energy functions, which are obtained by solving two restricted equations. Afterward, the stabilizing controller is proposed in such a way that the total energy function is a non-increasing function. That is, the obtained control causes the closed-loop system to be dissipative. Physically, the initial pendulum energy is dissipated by the convenient cart horizontal movements, until the pendulum achieves the top position and the cart rests at the origin.

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A Appendix

Proof of Lemma 1: We first check the first restricted conditions related to the potential energy V_d. Substituting M_d^{-1} and $F(\theta)$, defined previously in the first matrix of (12) and the first matrix of (4), respectively, in Eq. (9). We have, after recalling that $G^{\perp} = (1, \cos\theta)$: the following

$$G^{\perp}\left[M_d^{-1}\nabla_x V_d(\mathbf{x}) - F(\theta)\right] = \frac{\partial V_d}{\partial \theta}(1-\mu_2\cos\theta) + \frac{\partial V_d}{\partial q}(-\mu_2+\mu_3\cos\theta)+\sin\theta = 0. \quad \text{(A.1)}$$

We can easily check that the following function

$$V_d(\mathbf{x}) = k_1 - \frac{1}{\mu_2}\ln(-1+\mu_2\cos\theta) + \Phi_p(s), \quad \text{(A.2)}$$

is one solution of the **PDE** given in (A.1), where k_1 is a constant, s is an auxiliary variable given in (14), and Φ_p is any arbitrary function. To guarantee that the potential energy V_d is locally positive definite in a neighborhood of $\mathbf{x}=0$, it is enough that

$$V_d(0)=0, \quad \nabla_x V_d(0)=0, \quad \nabla_x^2 V_d(0) > 0. \quad \text{(A.3)}$$

Applying the above conditions (A.3) into (A.2), we obtain

$$k_1 = \ln(-1+\mu_2)/\mu_2, \Phi_p^{'}(0)=0, \\ \Phi_p^{''}(0)>0, \mu_2>1, \mu_3>\mu_2^2,$$

so that Φ_p may be fixed as

$$\Phi_p(z) = \frac{k_p}{2}z^2,$$

with $k_p>0$. That is, we have validated the expression of V_d, given by (A.2), which is strictly positive and well-defined, if

$$-1+\mu_2\cos\theta > 0.$$

Evidently the above inequality is satisfied, for all $\theta \in (-\theta_\mu, \theta_\mu)$, with θ_μ defined in (15). Consequently, the proposed V_d satisfies the restricted Eq. (9), for all $\theta \in I_\mu$. Now, we proceed to show that the proposed K_d guarantees the second restricted condition. From (10), we can select K_d, provided that

$$G^{\perp} M_d^{-1} K_d(\mathbf{x}) = 0. \quad \text{(A.4)}$$

Substituting M_d^{-1}, previously defined in **Lemma 1**, into the first set of linear equations of (A.4), we have that $d_{11}=\beta=0$ and $d_{22}=0$. In the same manner, we can easily show that the previous defined matrices M_d^{-1} and K_d (both matrices in (12)), fulfill the second set of linear equations of (A.4). Besides, K_d is semi-definite positive, if the free function $k_2(\theta)$ is strictly positive, for all $\theta \in I_\mu$. Indeed, from the two inequalities given in (13), it follows that $K_d(\theta)$ is strictly positive in I_μ. Finally, it is worth mentioning that the two restricted equations have been easily solved almost in algebraic form. Notice that if we employ the methodology based on the matching condition of the controlled Lagrangian, it is necessary to solve three ordinary differential equations related to the kinetic energy shaping, and one nonlinear partial differential equation related to the potential energy [1,3,10].

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i. The original system (or physical plant) interconnected with the control action is referred to the closed-loop system.

ii. Here, we use G to denote $G(\theta)$.

iii. Recalling that $G^{\perp}$ satisfies $G^{\perp}G=0$.

iv. The largest $\alpha>0$, such that $V_d(x)<\alpha$ is a compact set, allows us to have one estimated of the stability domain, since $E(\mathbf{x},\dot{\mathbf{x}}) \leq \alpha$ defines an invariant compact set, due to the fact that E is a non-increasing function.

v. Recalling from Lemma 1 that the signs of E and $\dot{E}$ are well defined, for all $\theta \in I_\mu$. That is we need to guarantee that $\theta(t) \in I_\mu$, for all $t>0$.

vi. This set will be used later to apply LaSalle's invariance Theorem.

vii. We use the symbol $\overline{y}$ to indicate that the variable y is a constant.

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