A parameterized family of single-double-triple-scroll chaotic oscillations

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We present a system with three equilibrium points which exhibit a single, double or triple scroll oscillation without introducing more equilibrium points. The study is based on one parameter of the nonlinear function which is the bifurcation parameter. With this bifurcation parameter it is possible to control the eigenvalues of the equilibrium points and consequently the type of oscillation.

Keywords: Chaotic oscillator; nonlinear circuit; bifurcation parameter; fixed points.

Se presenta un sistema con tres puntos de equilibrio que muestra oscilaciones de uno, dos y tres enroscados sin agregar más puntos de equilibrio. El estudio se basa sobre un parámetro de la función no lineal, siendo este el parámetro de bifurcación. Haciendo uso de este parámetro de bifurcación es posible controlar los eigenvalores de los puntos de equilibrio y consecuentemente el tipo de oscilación.

Descriptores: Oscilador caótico; circuito no lineal; parámetros de bifurcación; puntos fijos.

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1. Introduction

Utilization of chaotic signals in communication systems, biomedical engineering, radars, decoding algorithms are some of the engineering applications of dynamical systems with a chaotic behavior [1]. Synchronization phenomena between two coupled chaotic oscillators are being studied intensively. Different types of synchronization have been reported in regard to potential applications in different areas. For example, in Ref. 2 the concept of phase synchronization is used for the analysis of noisy nonstationary bivariate data with application to magnetoencephalography.

Recently, the design and the analysis of the multi-scroll chaotic attractors have been extensively studied. The first multiscroll oscillators [3] were originally derived from Chua's circuit by introducing a nonlinear resistor with multiple breakpoints, where the number of scroll attractor was controlled by adding fixed points. Yalcin *et al.* [4] physically created the 3 and 5 scroll chaotic attractors in a generalized Chua's circuit. Several multiscroll chaotic oscillations studies have appeared recently, using different oscillators and techniques. These techniques can be summarized in three categories; hysteresis multiscroll chaotic attractors [5], saturated multiscroll chaotic attractors via switching [6], and multilevel-logic pulse-excitations [7]. In the same spirit, the Rössler system has been modified to produce a double scroll by adding one fixed point [8].

The goal of this paper is to present a numerical simulation and theoretical analysis of the nonlinear function that produces single, double or triple chaotic scroll oscillations in a chaotic generator without introducing more fixed points into the system. The chaotic generator always remains three fixed points. We present a study of the sensitivity of the scroll structure to perturbation in the system parameters. In principle, this should provide some insight into the robustness of the multiple-scroll geometry to small changes in the circuit parameters.

This paper is organized in the following way. Section 2 contains the chaotic generator and the nonlinear converter. Sec. 3 describes the relationship between the fixed points of the chaotic generator with the bifurcation parameter in order to get single, double or triple scroll attractors. Finally, Sec. 4 presents the conclusions.

2. Chaotic generator

The electronic circuit in Fig. 1a is called a Chaotic Generator(CG), which consists of a nonlinear converter (NC) and a linear feedback equipped with a low-pass filter RC', and a resonator circuit rLC. The dynamics of the CG is well modelled in the following set of differential equations:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z - x - \delta y,\\ \dot{z} &= \gamma\left[kf(x) - z\right] - \sigma y,\end{aligned} \tag{1}$$

where $x(t)$ and $z(t)$ are voltages across the capacitors C and C', respectively, and $y(t) = J(t)(L/C)^{1/2}$ is the current through the coil. The output of the nonlinear converter is $kf(x)$. The unit time is given by $\tau = \sqrt{LC}$. The other parameters of the model depend on the physical values of the circuit elements: $\gamma = \sqrt{LC}/RC'$, $\delta = r\sqrt{C/L}$ and $\sigma = C/C'$. The schematic diagram of the NC is shown in Fig. 1b. The NC is well modelled by the following equation [9]:

$$F(x) = kf(x), \tag{2}$$

where

$$f(x) = \begin{cases} \dfrac{[(1-b_1)(1-w)R_3 - b_1R_1]x + R_1V_D}{(R_1+(1-w)R_3)a}, & \text{if} \quad x > V_D. \\[2ex] \left(\dfrac{1-b_2}{a}\right)x, & \text{if} \quad |x| \le V_D \\[2ex] \dfrac{[(1-b_1)wR_3 - b_1R_1]x - R_1V_D}{(R_1+wR_3)a}, & \text{if} \quad x < -V_D \end{cases}. \tag{3}$$

TABLE I. By linearizing system (1) at equilibria $S_{1,2}$.

b_1	$\Lambda_{S_{1,2}} :=$	kf'	$\sum \Lambda$
0.4896	$\{0.3700 + 1.5488i, 0.3700 - 1.5488i, -1.3065\}$	-7.2368	-1.3567

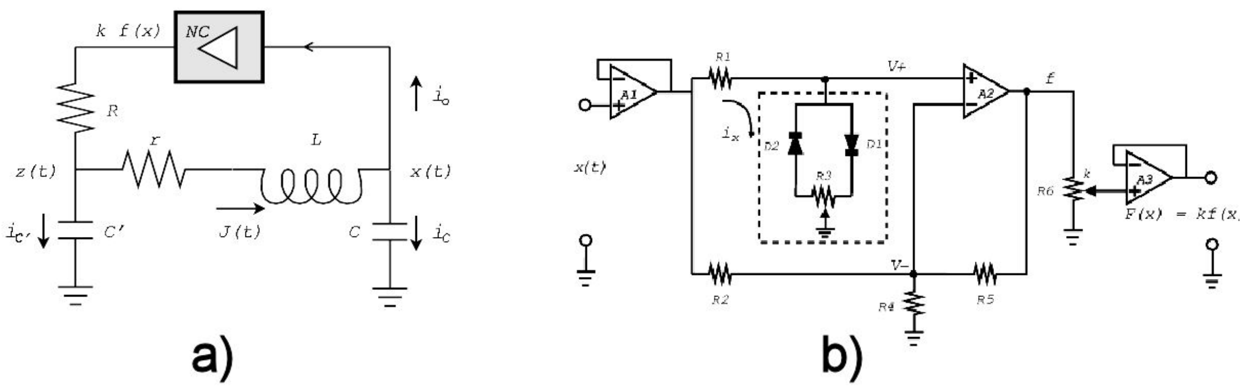

FIGURE 1. (a) The circuit diagram of a nonlinear chaotic oscillator and component values employed to build the CG experimentally: C'=100.2 nF, C=201.0 nF, L=63.8 mH, r=138.9 Ω and R=1018 Ω. (b) Schematic diagram of the nonlinear converter NC and $R1$=2.7 KΩ, $R2$=$R4$=7.5 KΩ, $R3$=50 Ω, $R5$=177 KΩ $R6$=2 KΩ, A1 and A2 are Op. Amp. TL082, A3 is an Op. Amp. LF356N, and D1 and D2 are Silicon diode 1N4148.

where

$$a = \frac{R_2||R_4}{R_5 + R_2||R_4}$$

and

$$b_{1,2} = \frac{R_5||R_4}{R_2 + R_5||R_4}.$$

Our study is based on the b_2 bifurcation parameter. It is possible to obtain none, single, double and triple scroll attractors simply by changing the value of the b_2 parameter and always leaving three equilibrium points. This is discussed in the next section.

3. Saddle hyperbolic stationary points

The equilibria of system (1) can easily be found by solving the system. The trivial equilibrium is at $x = y = z = 0$ and the others given symmetrically; the equilibria $(\pm x, 0, \pm x)$ are defined at the intersection of $f(x)$ and $(1/k)x$. The k parameter belongs to the closed interval $[0, 1]$, given by R_6 potentiometer (see Fig. 1b).

By linearizing system (1) at equilibria $S_0 = (0, 0, 0)$ and $S_{1,2} = (\pm x, 0, \pm x)$, one obtains the following characteristic polynomial:

$$g(\lambda) = \lambda^3 + (\delta + \gamma)\lambda^2 + (\delta\gamma + \sigma + 1)\lambda + (1 - kf')\gamma. \tag{4}$$

The roots of the characteristic polynomial (4), in order to ensure that system (1) be dissipative, as in Lorenz and Chua systems, require that their sum be a negative quantity. Due to the fact that the bifurcation parameter is b_2, the equilibria $S_{1,2} = (\pm x, 0, \pm x)$ remain independent of the value of b_2 parameter and only the equilibrium point $S_0 = (0, 0, 0)$ changes with this parameter. So, begin by analyzing the equilibria $S_{1,2}$ followed by S_0.

For equilibria $S_{1,2} = (\pm x, 0, \pm x)$ to be unstable, thereby yielding a possibility for chaos, the coefficients of the characteristic polynomial (4) must satisfy the following conditions, which are obtained according to the Routh criterion:

$$\delta + \gamma > 0 \qquad \delta\gamma + \sigma + 1 > 0 \qquad (1 - kf')\gamma > 0. \tag{5}$$

TABLE II. By linearizing system (1) at equilibrium S_0.

b_2	$\Lambda_{S_0} :=$	kf’	$\sum \Lambda$
0.4896	$\{1.1821, -1.2694 + 2.1609i, -1.2694 - 2.1609i\}$	7.6875	-1.3567
0.8000	$\{0.5239, -0.9403 + 1.8387i, -0.9403 - 1.8387i\}$	3.0125.	-1.3567
1.5000	$\{0.3386 + 2.1312i, 0.3386 - 2.1312i, -2.0012\}$	-7.5313	-1.3567

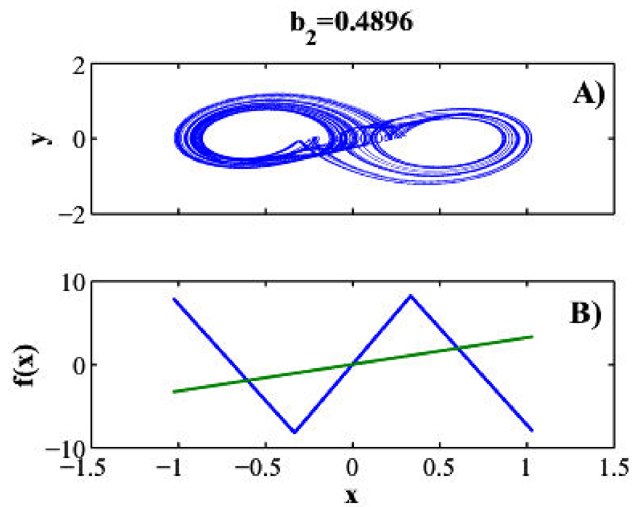

FIGURE 2. Double scroll oscillations: A) Projection of the attractor on the xy-plane. B) The response curve $f(x)$ of the nonlinear converter and the line x/k for $k = 0.3125$.

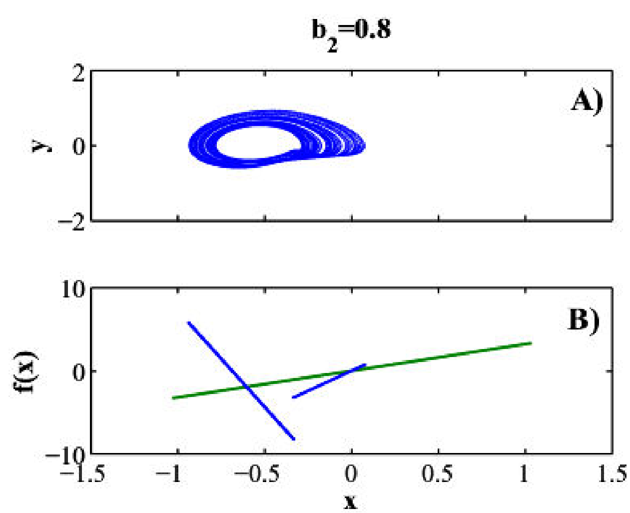

FIGURE 3. Single scroll oscillations: A)Projection of the attractor on the xy-plane, the response curve $f(x)$ of the nonlinear converter and the line x/k for $k = 0.3125$.

There are no positive real values; due to the fact that the coefficients of the characteristic polynomial (4) are all positive, $kf' < 0$. From (5), one can see that it satisfies the following condition: one of the three eigenvalues of Eq. (4) is a negative real value. The other two are complex with a real positive part. In Table I, the specific eigenvalues are given for the parameter $k = 0.3125$. These eigenvalues have several implications. First, the two eigenvalues $\Lambda_{S_{1,2}}$ are responsible for the steady outward slide, and the last eigenvalue is attracting. Second, system (1) is dissipative ($\sum \Lambda < 0$) and these equilibria would be surrounded by chaotic scroll oscillations. The two equilibrium points $S_{1,2} = (\pm x, 0, \pm x)$ remain identical since the b_2 parameter bifurcation only changes the equilibrium point $S_0 = (0, 0, 0)$.

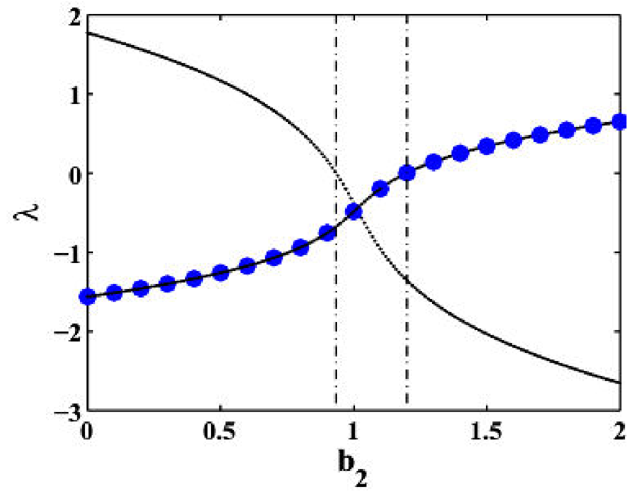

FIGURE 4. Eigenvalues of the trivial equilibrium point against b_2 bifurcation parameter. The three eigenvalues are negative for values between vertical dashed lines b_2 equals to 0.93361 and 1.1997.

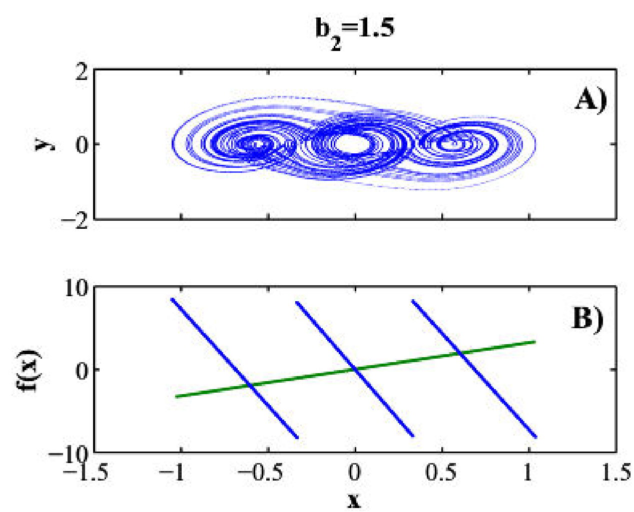

FIGURE 5. Triple scroll oscillations: A) Projection of the attractor on the xy-plane, the response curve $f(x)$ of the nonlinear converter and the line x/k for $k = 0.3125$.

For equilibrium point $S_0 = (0, 0, 0)$, the first two coefficients of the characteristic polynomial (4) remain positive;

$$\delta + \gamma > 0 \qquad \delta\gamma + \sigma + 1 > 0,$$

meanwhile the third coefficient can be controlled with the b_2 parameter, and presents the following conditions:

$$(1 - kf')\gamma > 0 \qquad \text{or} \qquad (1 - kf')\gamma < 0. \quad (6)$$

The first condition (6) has two modes for certain values of the b_2 parameter. When S_0 is induced by a similar behavior to $S_{1,2}$, there is oscillation around the equilibrium point S_0. As a consequence, the system (1) presents the triple-scroll chaotic oscillations. When the real parts of the complex eigenvalues are negative, system (1) does not oscillate. The second condition must be satisfy the following: One of the three roots of Eq. (4) must be positive, and the other two must be negative. It is noted that the second condition (6) is necessary for system (1) to generate chaos and present double-scroll oscillations. Within this condition, there is a region where bi-stable chaos occurs producing only single scroll oscillation. One natural question is what happen when $(1 - kf')\gamma = 0$? This occurs when $b_2 = 1 - a/k$ and an infinite number of equilibrium points appear. The b_2 bifurcation parameter serves as a control parameter to induce in the system (1) single, double or triple oscillations, and even to guarantee that the system (1) does not oscillate.

From Eq. (3), if we have that $f' = (1 - b_2)/a$, then the condition $(1 - kf')\gamma < 0$ is always satisfied.

$$b_2 < 1 - a/k \approx 0.9336$$

This guarantees that there is one positive Lyapunov exponent. In Table II, eigenvalues are given for different value from the b_2 bifurcation parameter. Notice that in the first two values of the b_2 parameter (0.4896 and 0.8000), the second condition (6) is satisfied by $(1 - kf') < 0$. There is one positive root (Λ_{S_0}) and system (1) is dissipative ($\sum \Lambda < 0$). There is no oscillation around S_0. In Fig. 2a, the projection of the chaotic attractor on the plane($x - y$) is represented by the case $b_2 = 0.4896$. The chaotic attractor presents double scroll oscillations around $S_{1,2}$. Fig. 2b illustrates the nonlinear function $f(x)$ and the line $(1/k)x$ against the x state. In the intersections of these two functions are defined geometrically the fixed points of the system.

For $b_2 = 0.8000$, the chaotic attractor presents a single scroll oscillation around $S_1 = (-x, 0, -x)$. In Fig. 3a, the projection of the chaotic attractor is shown on the xy-plane. In Fig. 3b, the nonlinear function and the line $(1/k)x$ against x state is illustrated. For this value of the b_2 parameter, there is another attractor around $S_2 = (x, 0, x)$. The oscillations around S_1 and S_2 depend on the initial condition.

If the b_2 bifurcation parameter continues increasing, it will obtain the first condition of (6), and the equilibrium point $S_0 = (0, 0, 0)$ becomes a sink fixed point until two complex eigenvalues appear with positive real part for

$$b_2 > 1 - \frac{a}{k}[1 - \frac{p^2}{\gamma}(\delta + \gamma)] \approx 1.1997,$$

where $p^2 = \delta\gamma + 1$. In Fig. 4, the eigenvalues of the equilibrium $S_0 = (0, 0, 0)$ are illustrated in contrast to the b_2 bifurcation parameter. The line with the big circles corresponds to the pair of complex eigenvalues and the other corresponds to the real eigenvalue. The left vertical line corresponds to $b_2 \approx 0.9336$ and the right vertical line to $b_2 \approx 1.1997$. When the value of the b_2 bifurcation parameter is between these two vertical lines, system (1) does not oscillate because of the real part of the three eigenvalues is negative. For values of b_2 greater than $b_2 \approx 1.1997$, the three equilibria S_0, S_1 and S_2 have two complex eigenvalues with a positive real part, which are responsible for the steady outward slide, and the last eigenvalue is attracting. For $b_2 = 1.5000$, system (1) presents triple scroll oscillations. In Fig. 5a, the projection of the chaotic attractor is shown on the xy-plane, where it is possible to observe triple scroll oscillations around the equilibria S_0, S_1 and S_2. In Fig. 5b, the nonlinear function and the line $(1/k)x$ is shown in contrast to the x state. In Table II, the last row corresponds to $b_2 = 1.5000$, system (1) is dissipative ($\sum \Lambda < 0$) and the eigenvalues are given for Λ_{S_0}.

4. Conclusions

We have introduced a b_2 bifurcation parameter which controls the number of generated scrolls. We have presented the projections of the attractor on the xy-plane for specific values of the b_2 bifurcation parameter where single, double and triple scroll oscillations occur and there always remain three equilibria. This is an approach to controlling the number of scrolls in this chaotic generator.

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