

SU(1,1) coherent states as Bessel-Gauss states

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The SU(1,1) coherent states of Barut and Girardello can be interpreted as Bessel-Gauss states that are time-dependent solutions of the Schrödinger equation. In this article, some basic properties of these states and their related operators are analyzed.

Keywords: Coherent states; SU(1, 1) Barut-Girardello states.

Los estados coherentes SU(1,1) de Barut y Girardello se pueden interpretar como estados de Bessel-Gauss que son soluciones dependientes del tiempo de la ecuación de Schrödinger. En este artículo, se analizan algunas propiedades básicas de estos estados y sus operadores asociados.

Descriptores: Estados coherentes; estados SU(1,1) de Barut-Girardello.

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1. Introduction

A coherent state [1] can be defined as a Gaussian wave packet evolving without spreading, or as an infinite superposition of the number states of a harmonic oscillator. Generalization to two or more modes can be easily achieved by coupling several linear harmonic oscillators. However, it is advantageous to express two-modes coherent states in a form that reflects the axial symmetry of the problem. This is particularly relevant due to recent interest in the angular momentum of light, associated to electromagnetic fields such as Bessel and Laguerre beams (see *e.g.* Padgett *et al.* [2]), and is also useful in the study of exactly solvable potentials (see, *e.g.*, [3]). In this context, the coherent states of Barut and Girardello [4], associated to the SU(1,1) group, are of great importance. These same states were studied with a different approach by Dodonov, Malkin and Man'ko [5] and have found applications in quantum optics (see, *e.g.*, [6, 7]); they are particularly interesting due to their propagation without spreading, in exact analogy with their optical analogues in the paraxial approximation.

The aim of the present article is to analyze some basic properties of the SU(1,1) coherent states using particular representations that exhibit the symmetries of the problem. In Sec. 2, these states are identified as Gauss-Bessel states, and the associated operators are analyzed in Sec. 3.

2. Bessel-Gauss states

Consider a two dimensional harmonic oscillator with unit mass and frequency. Setting $\hbar = 1$, the Schrödinger equation in polar coordinates takes the form:

$$i \frac{\partial}{\partial t} \psi(r, \phi, t) = \frac{1}{2} \left(-\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial r \phi^2} + r^2 \right) \psi(r, \phi, t). \quad (1)$$

It is well known that the eigenstates of energy and angular momentum, $|n, m\rangle$, are given by (see, *e.g.* [8])

$$\langle r, \phi | n, m \rangle = \left(\frac{n!}{\pi(n+|m|)!} \right)^{1/2} \times e^{im\phi} e^{-r^2/2} r^{|m|} L_n^{|m|}(r^2), \quad (2)$$

where $L_n^{|m|}$ are associated Laguerre polynomials. The energies of these states are $E_{n,m} = 2n + |m| + 1$, that is, $H|n, m\rangle = E_{n,m}|n, m\rangle$, where H is the Hamiltonian of the two-dimensional harmonic oscillator.

As for the SU(1,1) coherent states, they can be defined as

$$|\alpha, m\rangle = g_m(|\alpha|) \sum_{n=0}^{\infty} \frac{\alpha^{2n+|m|+1}}{\sqrt{n!(n+|m|)!}} |n, m\rangle, \quad (3)$$

where $g_m(|\alpha|)$ is a normalization constant (with this definition, $e^{-iHt}|\alpha, m\rangle = e^{-it}|\alpha e^{-it}, m\rangle$). In polar coordinates representation:

$$\begin{aligned} \psi_m(r, \phi, t) &\equiv \langle r, \phi | \alpha, m \rangle \\ &= \frac{1}{\pi^{1/2}} g_m(|\alpha|) \alpha^{|m|} e^{-\frac{1}{2}r^2 + im\phi} r^{|m|} \\ &\times \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(n+|m|)!} L_n^{|m|}(r^2). \end{aligned} \quad (5)$$

Time dependent solutions of the Schrödinger equation can be obtained using the expansion of Bessel functions in terms of Laguerre polynomials. In particular

$$J_{|m|}(2\alpha r) e^{\alpha^2} (\alpha r)^{|m|} = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(n+|m|)!} L_n^{|m|}(r^2), \quad (6)$$

from where it follows that

$$\psi_m(r, \phi, t) = \frac{1}{\pi^{1/2}} g_m(|\alpha|) e^{-i\omega\alpha^2 - \frac{1}{2}r^2 + im\phi} J_m(2\alpha r), \quad (7)$$

(modulo an irrelevant sign, since $J_{-m} = (-)^m J_m$). Since $\alpha = \alpha_0 e^{-it}$, this is precisely the time dependent solution of Dodonov *et al.* [5] (see also [9]).

The orthogonality condition determines $g_m(|\alpha|)$. From the definition of the associated Bessel function $I_m(x)$,

$$I_m(x) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2n+m}, \tag{8}$$

it follows that

$$|g_m(|\alpha|)|^2 = \frac{1}{I_{|m|}(2|\alpha|^2)}, \tag{9}$$

and the expectation value of the Hamiltonian in the state $|\alpha, m\rangle$ turns out to be

$$\langle H \rangle = \frac{x I'_{|m|}(x)}{I_{|m|}(x)} + 1, \tag{10}$$

where $x = 2|\alpha|^2$.

3. Operators

Consider the operator K defined as

$$K = \sum_{m,n} \sqrt{n(n+|m|)} |n-1, m\rangle \langle n, m|. \tag{11}$$

This operator, its hermitian conjugate K^\dagger and the Hamiltonian

$$H = \sum_{m,n} (2n + |m| + 1) |n, m\rangle \langle n, m| \tag{12}$$

satisfy the $SU(1, 1)$ algebra [10]:

$$\begin{aligned} [K, K^\dagger] &= H & (13) \\ [K, H] &= 2K \\ [K^\dagger, H] &= -2K^\dagger. \end{aligned}$$

It follows that the state $|\alpha, m\rangle$ is an eigenstate of the operator K :

$$K|\alpha, m\rangle = \alpha^2 |\alpha, m\rangle, \tag{14}$$

and therefore $\langle K \rangle = \alpha^2$. Puri and Agarwal [6] have shown that these states satisfy the minimum uncertainty condition. Indeed, defining $K = (1/2)(K_1 - iK_2)$ in terms of the hermitian operators K_i , it follows that

$$\langle \Delta K_1^2 \rangle = \langle \Delta K_2^2 \rangle = \langle H. \rangle \tag{15}$$

Thus the states $|\alpha, m\rangle$ have the minimum dispersion.

Notice also that:

$$|\alpha, m\rangle = \frac{1}{\sqrt{I_{|m|}(2|\alpha|^2)}} I_{|m|}(2\alpha K)|0, m\rangle, \tag{16}$$

where $I_{|m|}(2\alpha K)$ must be interpreted as an infinite power series of the operators K . This is analogous to the textbook formula $D(\alpha)|0\rangle = |\alpha\rangle$ for the standard coherent state, where $D(\alpha)$ is the displacement operator.

In order to compare with the results of the previous section, it is convenient to define the following operators [11]:

$$\begin{aligned} A &= \frac{1}{2} e^{i\phi} \left(r + \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right) \\ B &= \frac{1}{2} e^{-i\phi} \left(r + \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \phi} \right). \end{aligned}$$

These operators satisfy the commutation relations

$$\begin{aligned} [A, B] &= 0 = [A, B^\dagger] \\ [A, A^\dagger] &= 1 = [B, B^\dagger], \end{aligned}$$

and therefore one can identify:

$$K = AB, \tag{17}$$

$$K^\dagger = A^\dagger B^\dagger, \tag{18}$$

$$H = A^\dagger A + B^\dagger B + 1. \tag{19}$$

It can be seen by direct substitution, using the recurrence relations of the Bessel functions, that

$$A \psi_m = \alpha \left[\frac{I_{|m|+1}(2|\alpha|^2)}{I_{|m|}(2|\alpha|^2)} \right]^{1/2} \psi_{m+1}, \tag{20}$$

and

$$B \psi_m = \alpha \left[\frac{I_{|m|-1}(2|\alpha|^2)}{I_{|m|}(2|\alpha|^2)} \right]^{1/2} \psi_{m-1}. \tag{21}$$

Thus A and B act as raising and lowering operators for the angular momentum and clearly $AB\psi_m = \alpha^2\psi_m$, in agreement with (14).

Summing up, we have identified a particular representation for the operators of the $SU(1,1)$ group. This representation permits to express the Bessel-Gauss states of Dodonov *et al.* [5] in terms of the coherent states associated to this group, and makes explicit some algebraic relations.

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