

## Lattices with variable and constant occupation density and $q$ -exponential distribution

P. Cavalcante da Silva<sup>a</sup>, G. Corso<sup>b,c</sup>, and L.R. da Silva<sup>c</sup>

<sup>a</sup> *Centro Federal de Educação Tecnológica do Rio Grande do Norte, Tirol, CEP 59015 000 Natal, RN, Brazil,*

<sup>b</sup> *Departamento de Biofísica e Farmacologia, Centro de Biociências, Universidade Federal do Rio Grande do Norte, UFRN - Campus Universitário, Lagoa Nova, CEP 59078 972, Natal, RN, Brazil,*

<sup>c</sup> *Departamento de Física Teórica e Experimental, Universidade Federal do Rio Grande do Norte, UFRN - Campus Universitário, Lagoa Nova, CEP 59078 970, Natal, RN, Brazil.*

Recibido el 7 de noviembre de 2008; aceptado el 4 de diciembre de 2008

In this paper we test the hypothesis that  $q$ -exponential distribution fits better on distributions arising from lattices with a heterogeneous topology than a homogeneous topology. We compare two lattices: the first is the typical square lattice with a constant occupation density  $p$  (the lattice used in standard percolation theory), and the second is a lattice constructed with a gradient of  $p$ . In the homogeneous lattice the occupied number of neighbors of each cell is the same (on average) for the full lattice, otherwise in the  $p$ -gradient lattice this number changes along the lattice. In this sense the  $p$ -gradient lattice shows a more complex topology than the homogeneous lattice. We fit the  $q$ -exponential and the stretched exponential distribution on the cluster size distribution that arises in the lattices. We observe that the  $q$ -exponential fits better on the  $p$ -gradient lattice than on a constant  $p$  lattice. On the other hand, the stretched exponential distribution fits equally well on both lattices.

*Keywords:*  $q$ -exponential distribution; gradient lattices; stretched exponential; topology.

En este trabajo se prueba la hipótesis de que la distribución  $q$ -exponencial se adapta mejor en distribuciones derivadas de redes con una topología heterogénea que en una topología homogénea. Se comparan dos redes: la primera es la típica red cuadrada con una densidad de ocupación constante  $p$  (la red estándar de la percolación), y la segunda es una red construida con un gradiente de ocupación  $p$ . En la red homogénea, el número de vecinos ocupados de cada celda es el mismo (en promedio), pero por otro lado, en la red con  $p$ -gradiente, este número sufre cambio a lo largo de la red. En este sentido, la  $p$ -gradiente red muestra una topología más compleja que la red homogénea. Nos ajustamos la  $q$ -exponencial y la distribución exponencial estirada sobre la distribución de clústers de las redes. Observamos que la  $q$ -exponencial encaja mejor en la red  $p$ -gradiente que en una red con  $p$  constante. Por otro lado, la distribución exponencial estirada encaja bien en ambas redes.

*Descriptores:* Distribución  $q$ -exponencial; redes en gradiente; exponencial estirada; topología.

PACS: 05.50.+q; 02.40.Pe

### 1. Introduction

In recent years, a vehement discussion regarding the foundation of thermodynamics [1]. Several formulations of entropy have been proposed to account for the deviations from Boltzmann statistics that are measured in many experiments [2]. We focus here on the non-extensive (or non-additive, to use a more rigorous terminology) interpretation of the thermodynamics as developed by C. Tsallis [3]. According to this paradigm, the distribution of probabilities that expand Boltzmann distribution,  $p(x)=A_0 \exp(-A_1 x)$ , is the  $q$ -exponential distribution,  $p(x)=A_0(1 - (1 - q)\beta x)^{1/(1-q)}$ , for  $A_0$ ,  $A_1$ ,  $\beta$ , and  $q$  adequate constants. In the limit case of  $q=1$ , the  $q$ -exponential distribution becomes the usual Boltzmann statistics. In this paper we shall not discuss the basis of statistics. We observe that in the literature, most of computational experiments that are well fitted by  $q$ -exponential distribution arises from systems that show a non-trivial topology, for instance: the logistic map [4], Hamiltonian systems in the weak chaotic regime [5], coupled nonlinear dynamical

systems [6], or long-range correlated systems [7]. All these systems are known to show an intricate topology, as we shall discuss below.

Topology is the branch of Mathematics that studies neighborhood properties of sets [8]. In this sense the square lattice, or any regular lattice, has a simple topology since the number of neighbors of each cell is the same all along the entire lattice (except the border of the system). Moreover, a lattice with an average random distribution of neighboring cells also shows a trivial topology. An example of such a system is the regular lattice with randomly empty and occupied cells, where the neighbor of an occupied cell is exclusively another occupied cell. We say that a system shows a non-trivial topology when its distribution of a number of neighbors of cells (or open balls in continuous systems) deviates from a constant or an isotropic random distribution.

In this paper we computationally test the hypothesis that  $q$ -exponential distribution is more adequate to fit distributions arising from systems with a non-homogeneous topology than

systems with a trivial one. In Sec. 2 we introduce the lattice with variable topology that we use to test our hypothesis. In this way we produce lattices with homogeneous (trivial) and heterogeneous (non-trivial) topology. In Sec. 3 we compare the fittings of the  $q$ -exponential and the stretched exponential (an alternative distribution with the same number of parameters) to the data of our lattices. In the last section we conclude the work, give our final remarks and point towards future developments of this article.

## 2. Construction of the gradient lattice

The basic elements in the construction of the gradient lattice are the usual square lattice and the concept of occupation probability  $p$ . The  $p$  variable lattice was initially used in the context of gradient percolation in the eighties [9-11]. The focus of that papers was: to analyze percolation in a heterogeneous lattice, to study the scaling relation of the boundary of the large cluster and to find an optimal value of the critical percolation threshold of the square lattice. In this paper we use the  $p$  variable lattice (gradient lattice or heterogeneous lattice) as a toy model to explore the connection between topology and distribution functions.

To construct the variable  $p$  lattice, we start with an empty square lattice of size  $L$ . We choose two arbitrary occupation probabilities  $p_1$  and  $p_2$  for two opposite boundaries of the lattice and generate a linear gradient of probabilities  $p$  between the limits:  $p_2 < p < p_1$ . In other words, we generate a linear

pattern of lines with a given  $p$ . To fill the lattice  $L^2$ , we sort random numbers  $0 < r < 1$  and use the rule: cells in a  $p$ -line are occupied for  $r < p$ .

Figure 1 shows two sketches of the square lattice with random (but not necessarily isotropic) occupation of probability: in (a) a constant  $p$  lattice with  $p = 0.5$ , similar pictures are very common in textbooks of percolation or statistical physics and in (b), otherwise, shows a  $p$  gradient lattice with  $p_1 = 1$  and  $p_2 = 0$ . To explore statistical properties related to such lattices we use the cluster size distribution that naturally arise from the lattice. This quantity computes the average number of cluster with a given size. The cluster size distribution is very much used in spin lattice and percolation theory. It is well known [12] that this quantity obeys a power law distribution at criticality. In our work we are not interested in critical phenomena that potentially could interfere in our results because criticality impose long range correlations. In this article we decide to work far from the critical point of lattice percolation.

Figure 2 shows the cluster size distribution (non-normalized). We fix  $p_1 = 1$  and test several  $p_2$  as indicated in the figure. In the simulation we use lattice size  $L = 200$ ; the number of samples in the simulation is  $N = 200$ . We see in the picture that the number of clusters increases with the amplitude  $p_1 - p_2$ . This phenomenon is roughly expected since in the limit  $p_1 \rightarrow p_2 = 1$  there is just one single infinite cluster. The case  $p_2 = 0.6$  in the figure is somewhat patho-

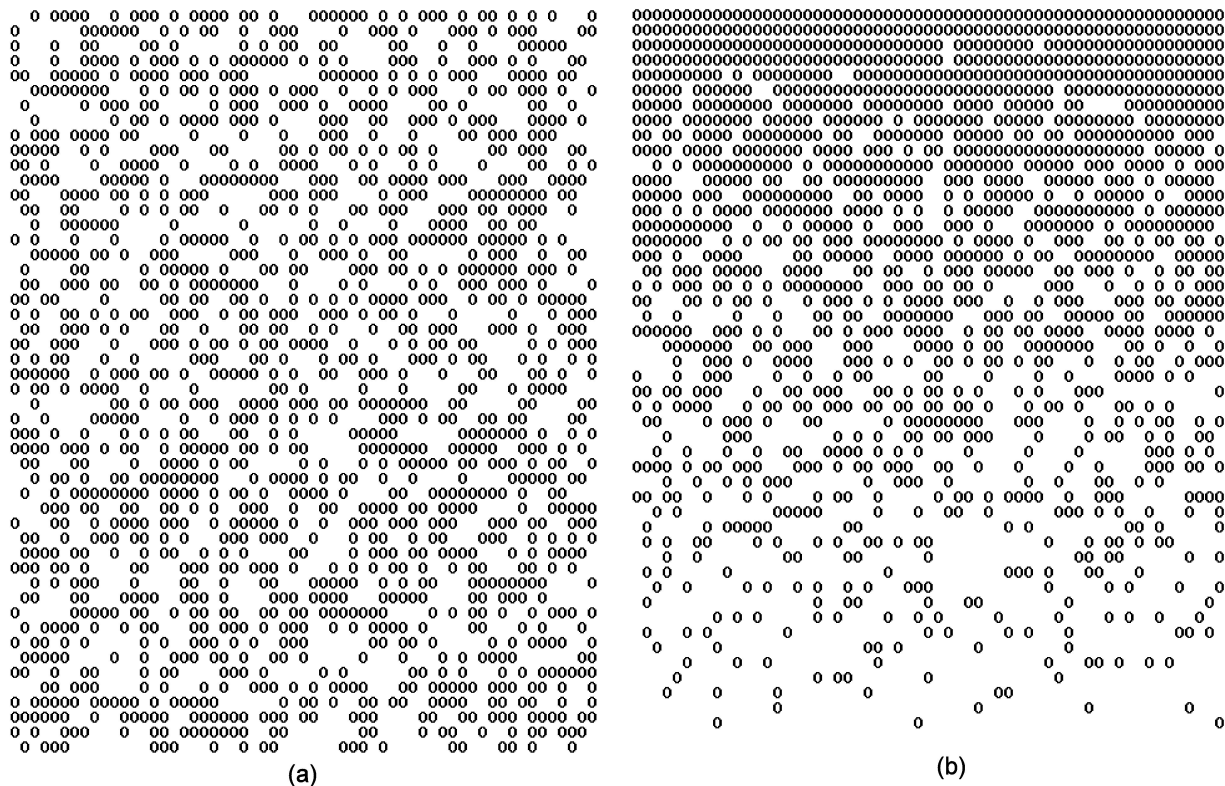


FIGURE 1. Picture of two lattices: (a) a constant  $p$  lattice with  $p = 0.5$  and (b) a  $p$  gradient lattice with  $p_1 = 1$  and  $p_2 = 0$ . The lattice size is  $L = 50$ .

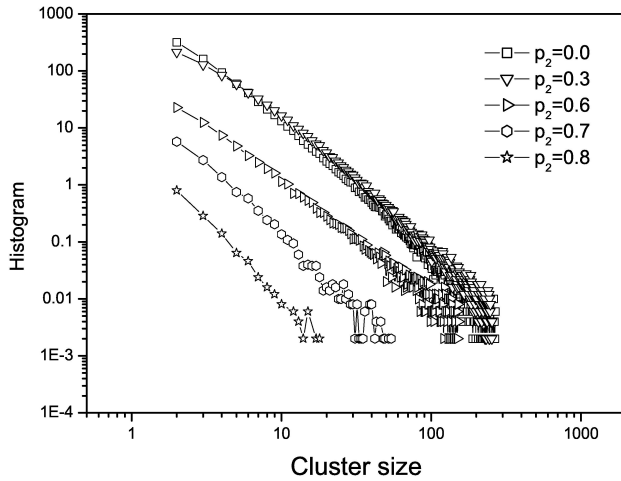


FIGURE 2. Histogram of cluster size distribution for a  $p$  gradient lattice, we use  $p_1 = 1$  and several  $p_2$  as shown in the figure.

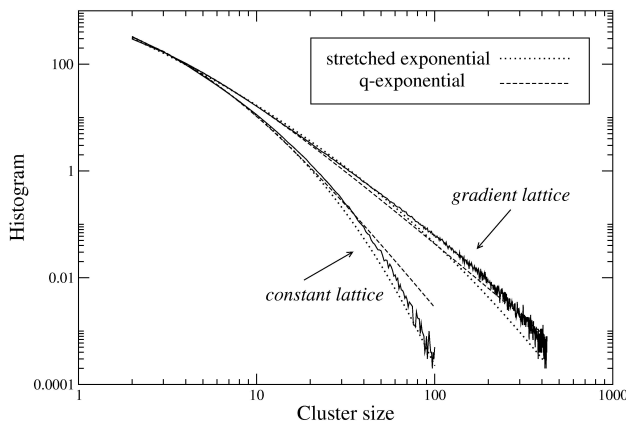


FIGURE 3. Numerical fittings of two histograms of cluster size distribution. We illustrate a  $p = 0.4$  constant lattice and a  $p_1 = 1$  and  $p_2 = 0.2$  gradient lattice. The  $q$ -exponential (dashed lines) and the stretched exponential (dotted lines) fittings are shown in both cases.

logical since it is very close to the critical percolation value of the square lattice  $p_c = 0.5972$ . We see in the figure that the distribution corresponding to this case resembles a power-law distribution as is to be expected at criticality. The best fit for all these curves is explored in the next section.

### 3. Numerical results

In this section we test the  $q$ -exponential and the stretched exponential distributions for the cluster size distribution arising from a  $p$  constant and a  $p$  gradient lattices. The equation for the  $q$ -exponential is:

$$y = A_0 (1 - (1 - A_1)A_2x)^{1/(1-A_1)} \tag{1}$$

and the equation for the stretched exponential is:

$$y = A_0 \exp(-A_1x^{A_2}), \tag{2}$$

where  $A_0$ ,  $A_1$  and  $A_2$  are the fitting parameters of the distribution. We use the stretched distribution to compare to the fitting of the  $q$ -exponential. In fact both distributions have three real parameters  $A_i$ ,  $1 \leq i \leq 3$ . Usually in the literature on  $q$ -exponential distribution and Tsallis statistics the notation  $A_1=q$  and  $A_2=\beta$  is employed..

Figure 3 shows the data corresponding to a  $p$  constant lattice  $p=0.4$  and a  $p$  variable lattice ( $p_1=1$  and  $p_2=0.2$ ). We use in the simulation lattice size  $L=200$  and number of samples  $N=10000$ . We have performed a numerical exploration of the system from  $L=50$  to  $L=2000$ ; we choose  $L=200$  because it fits the compromise between numerical precision and reasonable time-machine. The respective fittings are shown in the figure:  $q$ -exponential (dashed lines) and the stretched exponential (dotted lines). We observe in this figure that the  $q$ -exponential distribution fits better in a  $p$  gradient than a  $p$  variable lattice. In what follows we discuss in detail the adjustment of the data to the two distributions.

We estimate the goodness of the fitting using the Root Mean Square deviation **RMS** done by

$$RMS = \sqrt{\frac{\sum_{i=1}^N (x_i - f_i)^2}{N}}$$

for  $x_i$  the data corresponding to the cluster size distribution,  $f_i$  the artificial data from the fitted curve and  $N$  the number of points of the data. The best fit corresponds to the smallest **RMS**. Tables I and II summarize the information of the fitting problem. We indicate the **RMS** and the best fitting parameters  $A_1$  and  $A_2$  for constant  $p$  lattice ( $p_1=1$  and  $p_2=0.0, 0.1, 0.2, 0.3$ , and  $0.4$ ) and for variable  $p$  lattice ( $p=0.35, 0.4, 0.45, 0.5$ , and  $0.6$ ). Table I corresponds to the fitting of the  $q$ -exponential distribution and Table II to the stretched exponential distribution.

TABLE I. Table for the  $q$ -exponential distribution. The first 5 lines correspond to  $p$  variable lattice, and the first column shows the limits of the interval of occupation probability of the gradient. The last five lines correspond to a  $p$  constant lattices, and the first column indicates the constant  $p$  value. The remaining columns show: the number of points of the histogram,  $N$ , the root mean square deviation, **RMS**, and the fitting parameters  $A_1$  and  $A_2$ .

Interval	N	RMS(%)	A1	A2
1.0 - 0.0	353	0.94	1.38	0.54
1.0 - 0.1	379	0.59	1.36	0.63
1.0 - 0.2	245	0.45	1.36	0.86
1.0 - 0.3	500	1.19	1.38	1.07
1.0 - 0.4	551	1.61	1.42	1.18
0.35	37	3.48	1.13	0.51
0.40	65	3	1.18	1.00
0.45	99	3.38	1.24	1.02
0.50	197	2.95	1.32	0.91
0.60	806	0.29	1.44	0.42

TABLE II. Table for the stretched exponential. The first 5 lines correspond to  $p$  variable lattice, and the first column shows the limits of the interval of occupation probability of the gradient. The last five lines correspond to  $p$  constant lattices, and the first column indicates the  $p$  value. The other columns show: the number of points of the histogram,  $N$ , the root mean square deviation, **RMS**, and the fitting parameters  $A_1$  and  $A_2$ .

Interval	N	RMS(%)	A1	A2
1.0 - 0.0	353	0.34	12.38	0.13
1.0 - 0.1	379	0.29	11.05	0.14
1.0 - 0.2	245	0.43	7.69	0.18
1.0 - 0.3	500	0.44	5.74	0.21
1.0 - 0.4	551	0.32	6.13	0.18
0.35	37	0.58	2.21	0.57
0.40	65	0.5	2.67	0.57
0.45	99	0.33	2.91	0.4
0.50	197	0.49	4.14	0.29
0.60	806	0.47	19.36	0.08

The  $q$ -exponential fitting is explored in Table I. We observe in this table that the **RMS** is 2 to 3 times larger for a gradient lattice than a constant one. The case constant  $p = 0.6$  lattice is anomalous as discussed before. Table II, on the other hand, shows that the stretched exponential distribution fits as well in  $p$  gradient as in  $p$  constant lattices. There is no significant differences between these two sets.

#### 4. Final remarks

In this work we test the hypothesis that the  $q$ -exponential distribution is adequate to fit quantities arising from non-trivial topology. We use an appropriate rule for cell probability occupation of the square lattice in order to induce variable topology in lattices. We employ the cluster size distribution of the constant and variable lattices to perform our tests. The main conclusion of this paper is that the **RMS** deviation of the  $q$ -exponential fitting is smaller for a  $p$  gradient lattice than for a  $p$  constant lattice. We use the stretched exponential distribution as a null model to compare our results. The **RMS** of

the stretched exponential fitting is roughly the same for a gradient lattice as for a homogeneous lattice. As opposed to the  $q$ -exponential distribution, the stretched exponential has no feeling to detect topological differences between a  $p$  gradient and a  $p$  constant lattice.

In this work we tested only two distributions of probabilities, the stretched exponential and the  $q$ -exponential distribution. Our aim in this paper is to test the  $q$ -exponential fitting for heterogeneous topologies, and we did the test against the stretched exponential distribution. Another possibility would be to use more sophisticated distributions like the one suggested in Ref. 13. The cited distribution is a natural extension of the  $q$ -exponential distribution with one more parameter. However, the best fit comparison between two distributions with different numbers of parameters is not an easy task. In a future work we intend to explore this point in more detail.

This paper follows an alternative trend in the context of Tsallis thermodynamics and  $q$ -exponential distribution. Instead of searching for a thermodynamic justification for the use of  $q$ -exponential distribution, we explore the relation between topology and fitting of  $q$ -exponential distribution. Based on our results, we conjecture that the  $q$ -exponential is an adequate fitting distributions to model systems showing a non-trivial topology. We explore this concept in a lattice system where we can change the topology by playing with the number of neighbors. The same framework should be tested in non-linear dynamical systems, where it is possible to tune the non-linear parameter and change the topology of the system. In the context of dynamic systems, the term symbolic dynamics is used instead of topology, but the ground idea is always to describe transition rules among subsets of the system. In this context, systems with simple symbolic dynamics (ergodic systems) follow a Boltzmann distribution and systems with more evolved symbolic dynamics should follow a  $q$ -exponential fitting.

#### Acknowledgements

The authors gratefully acknowledge the financial support of Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil.

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