

# Optimal configuration of heat engines for maximum efficiency with generalized radiative heat transfer law

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Optimal configuration of a class of endoreversible heat engines with generalized radiative heat transfer law [ $q \propto \Delta(T^n)$ ] has been determined by this paper. The optimal cycle that maximizes the efficiency of the engines with fixed input energy has been obtained using optimal-control theory, and the differential equations are solved by Taylor series expansion. It is shown that the optimal cycle for maximum efficiency has eight branches including two isothermal branches, four maximum-efficiency branches and two adiabatic branches. The interval of each branch has been obtained, as well as the solutions of the temperatures of heat reservoirs and working fluid. Numerical examples are given for the optimal configurations with  $n = -1, n=1, n=2, n=3$  and  $n=4$ , respectively. The results obtained are compared with each other and with those results obtained for maximum power output.

**Keywords:** Generalized radiative heat transfer law; endoreversible heat engine; maximum efficiency; optimal configuration; finite time thermodynamics; generalized thermodynamic optimization.

En este artículo se determina la configuración óptima de una clase de motor térmico endoreversible con la ley generalizada de transferencia de calor radiativa [ $q \propto \Delta(T^n)$ ]. El ciclo óptimo que maximiza la eficiencia de los motores con una entrada de energía dada se obtiene usando la teoría del control óptimo y las ecuaciones diferenciales son resueltas mediante la expansión en series de Taylor. Se muestra que el ciclo óptimo para máxima eficiencia tiene ocho ramas, incluyendo dos ramas isotérmicas, cuatro de máxima eficiencia y dos adiabáticas. Se muestra el intervalo para cada rama, así como la temperatura del recipiente calorífico y del fluido de trabajo. Los ejemplos numéricos se muestran para la configuración óptima con  $n = -1, n=1, n=2, n=3$  y  $n=4$ . Los resultados obtenidos son comparados unos con otros, y con éstos se obtiene la potencia máxima de salida.

**Descriptores:** Leyes de transferencia de calor; motores endoreversibles; máxima eficiencia; configuración óptima; termodinámica de tiempos finitos.

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## 1. Introduction

There are two standard problems in finite time thermodynamics: One is to determine the objective function limits and the relations between objective functions for the given thermodynamic system, and another is to determine the optimal thermodynamic process for the given optimization objectives [1-23]. Gutowicz-Krusion *et al.* [24] proved that in all acceptable cycles, an endoreversible Carnot cycle can produce maximum power, *i.e.* the Curzon-Ahlborn (CA) cycle [25] is the optimal configuration with only First and Second Law constraints. Rubin [26,27] analyzed the optimal configurations of endoreversible heat engines with Newton's heat transfer law [ $q \propto \Delta(T)$ ] and different constraints, and derived the optimal configuration of the engines. The optimal configuration with fixed duration for maximum power output is a six-branch cycle, and the optimal configuration with fixed energy input for maximum efficiency is an eight-branch cycle [26]. The results were extended to a class of heat engines with fixed compression ratio, and an optimal eight-branch cycle configuration of the engines for maximum power output was derived [27]. Since then, many researchers have found various optimal configurations for various systems and devices using optimal-control theory in the last three decades.

In general, heat transfer is not necessarily linear and also obeys other laws; heat transfer law has significant influence on the configuration and performance of heat engine cycles [28-37]. Song *et al.* [38,39] and Li *et al.* [40] determined the optimal configurations of endoreversible heat engines for maximum efficiency objective and maximum power output objective with linear phenomenological heat transfer law [ $q \propto \Delta(T^{-1})$ ] [38,40] and those for maximum power output objective [39] with radiative heat transfer law [ $q \propto \Delta(T^4)$ ]. In this paper, based on Refs. 26 and 38, the optimal configuration of endoreversible heat engines for maximum efficiency objective is obtained with fixed duration and a universal heat transfer law, *i.e.* generalized radiative heat transfer [ $q \propto \Delta(T^n)$ ] in the heat transfer processes between working fluid and heat reservoirs. The generalized radiative heat transfer law [ $q \propto \Delta(T^n)$ ] includes some special cases. When  $n = 1$ , the heat transfer obeys Newtonian law; when  $n = -1$ , the heat transfer obeys linear phenomenological law used in irreversible thermodynamics, in which case the heat transfer coefficients are the so-called kinetic coefficients by Callen [41], and they should be negative; when  $n=2$ , the heat transfer is applicable to radiation propagated along a one-dimensional transmission line [28,29], and the heat transfer coefficient in this case is equal

to  $[\pi^2 k^2/6h]$ , where  $h$  is the Planck's constant and  $k$  is the Stefan-Boltzmann constant; when  $n=3$ , the heat transfer is applicable to radiation propagated along a two-dimensional surface [28,29]; when  $n = 4$ , the heat transfer obeys radiative law if all the bodies are black, and the heat transfer coefficient in this case is related to the Stefan-Boltzmann constant. The optimal cycles that maximize the efficiency of the engines are obtained using optimal-control theory. It is shown that the optimal cycle for maximum efficiency has eight branches including two isothermal branches, four maximum-efficiency branches and two adiabatic branches. The interval of each branch has been obtained, as well as the solutions of the temperatures of heat reservoirs and working fluid. Numerical examples for the optimal configurations with  $n = -1, n=1, n=2, n=3$  and  $n=4$  are provided, respectively.

## 2. Heat engine model

The following assumptions are made for the heat engine model:

- 1) The engine is an endoreversible one;
- 2) The heat conductivity between working fluid and reservoirs is  $\rho$  which subject to

$$0 \leq \rho \leq \rho_0 \quad (1)$$

- 3) The heat transfer between working fluid and reservoirs obeys a universal heat transfer law, the heat flux is

$$\dot{q} = \rho(T_R^n - T^n)\text{sign}(n) \quad (2)$$

where  $T$  is the absolute temperature of the working fluid,  $T_R$  is a constant temperature of each heat reservoir, and

$$T_L \leq T_R \leq T_H \quad (3)$$

where  $T_L$  and  $T_H$  are the lower and upper limits of the temperature of heat reservoirs, respectively.  $\text{sign}(n)$  is a sign function,  $\text{sign}(n)=1$  if  $n>0$  and  $\text{sign}(n)=-1$  if  $n < 0$ .

- 4) The work done by the engine in one cycle is given by

$$W = \int_0^\tau P\dot{V}dt \quad (4)$$

where  $P$  and  $V$  are the pressure and volume of the working fluid, the time derivative of  $V$  is denoted by  $\dot{V}$ , and  $\tau$  is the cycle period of the engine. The working fluid is an ideal gas.

In terms of the first law of thermodynamics for an ideal gas one has

$$C_V\dot{T} + C_V(\gamma - 1)\dot{T}/V = \dot{q} \quad (5)$$

where  $C_V$  is the isometric heat capacity of the gas,  $\gamma$  is the ratio of isobaric heat capacity to  $C_V$ , and the time derivative of  $T$  is denoted by  $\dot{T}$ .

Substituting Eq. (2) into Eq. (5) and defining some new variables yields:

$$\dot{T} = -CT + \hat{K}(T_R^n - T^n)\text{sign}(n) \quad (6)$$

$$\beta = (\gamma - 1) \ln \frac{V}{V_0} \quad (7)$$

$$\dot{\beta} = C \quad (8)$$

where  $\hat{\rho} = \rho/C_V$ ,  $V_0$  is a constant reference volume, and  $C$  is the change rate of cylinder volume.

In terms of these variables, Eq. (4) becomes:

$$W = C_V \int_0^\tau CTdt \quad (9)$$

Apparently, deriving  $\beta$  and  $C$  is convenient for solving the problem one confronts by using optimal control theory. One also has

$$Q_1 = C_V \int_0^\tau \hat{K}(T_R^n - T^n)\text{sign}(n)\theta \times [(T_R^n - T^n)\text{sign}(n)]dt \quad (10)$$

where  $\theta(x)$  is a Heaviside step function,  $\theta = 1$  if  $x > 0$  and  $\theta=0$  if  $x < 0$ , and  $Q_1$  is the work be supplied to system, *i.e.* input energy. The cycle efficiency is given by  $\eta = W/Q_1$ .

The problem now is to determine parameters  $\hat{\rho}(t)$ ,  $T_R(t)$  and  $C(t)$  so that the work output is a maximum with the fixed  $Q_1$ . It is required that  $C$  be restricted such that

$$-C_m \leq C \leq C_M \quad (11)$$

where  $C_m$  and  $C_M$  are arbitrary constant positive numbers. Then the optimal cycle of the engines for maximum-efficiency can be derived by the model.

## 3. Optimization procedure

To determine the optimal configuration for maximum-efficiency with fixed cycle time and input energy,  $Q_1$  is the same as in determining the optimal configuration for maximum  $W - \mu Q_1$ , where  $\mu$  is an ordinary Lagrange multiplier. The derivation of the optimal solution is complicated by the  $\mu$  term. One can show that  $\mu = \partial W_{\max}/\partial Q_1$ , *i.e.*,  $\mu$  is a measure of the sensitivity of  $W_{\max}$  with respect to small changes in the constraint,  $Q_1=\text{const}$ . One can see that  $\mu$  can be both positive and negative and  $\mu$  vanishes for the maximum power output, as expected [26]. To simplify the subsequent discussion,  $\mu \geq 0$  is assumed.

The Hamiltonian function in terms of Eqs. (6), (7), (11) and  $W - \mu Q_1$  is (where  $T$  and  $\beta$  are called state variables and  $T_R$ ,  $\hat{\rho}$  and  $C$  are called control variables) is:

$$H = CT - \mu \hat{\rho} (T_R^n - T^n) \text{sign}(n) \\ \times \theta[(T_R^n - T^n) \text{sign}(n)] + \psi_1 F_1 + \psi_2 F_2 \quad (12)$$

where

$$F_1 = -CT + \hat{K} (T_R^n - T^n) \text{sign}(n) \quad (13)$$

$$F_2 = C \quad (14)$$

Taking  $(W - \mu Q_1)/C_V$  as the performance index, it is convenient to rewrite Eq. (12) as

$$H = [(1 - \Psi_1)T + \Psi_2]C \\ + (\Psi_1 - \mu\theta)\hat{\rho}(T_R^n - T^n) \text{sign}(n) \quad (15)$$

The equations of adjoint variables are given by

$$\dot{\psi}_1 = -\partial H/\partial T = -C(1 - \psi_1) \\ + n\hat{\rho}T^{n-1}(\psi_1 - \mu\theta) \quad (16)$$

$$\dot{\psi}_2 = -\partial H/\partial \beta = 0 \quad (17)$$

where

$$0 \leq \hat{\rho} \leq \rho_0/C_V = \hat{\rho}_0 \quad (18)$$

### 3.1. Application of maximum principle

Define

$$\Delta H = H[\vec{x}^*(t), \vec{u}^*(t), \vec{\psi}^*(t)] \\ - H[\vec{x}^*(t), \vec{u}, \vec{\psi}^*(t)] \quad (19)$$

where  $\vec{u}$  is an admissible solution. The asterisk of symbols state the solutions are optimal. For a maximum, it is required that  $\Delta H \geq 0$ , and thus

$$\Delta H = [(1 - \psi_1^*)T^* + \psi_2^*](C^* - C) \\ + [\psi_1^* - \mu^*\theta((T_R^{*n} - T^{*n}) \text{sign}(n))] \\ \times \hat{\rho}^*(T_R^{*n} - T^{*n}) \text{sign}(n) \\ - [\psi_1^* - \mu\theta((T_R^n - T^{*n}) \text{sign}(n))] \\ \times \hat{\rho}(T_R^n - T^{*n}) \text{sign}(n) \geq 0 \quad (20)$$

where  $0 \leq \hat{\rho} \leq \rho_0/C_V = \hat{\rho}_0 C_V = \hat{\rho}_0$ ,  $T_L \leq T_R \leq T_H$  and  $-C_m \leq C \leq C_M$ .

Now one can consider various possible cases separately.

First set  $\hat{\rho} = \hat{\rho}^*$  and  $T_R = T_R^*$ ; then the second term of Eq. (20) vanishes. To assure  $\Delta H \geq 0$  it is required that

$$C^* = \begin{cases} C_M, & \text{if } (1 - \psi_1^*)T^* + \psi_2^* > 0 \\ -C_m, & \text{if } (1 - \psi_1^*)T^* + \psi_2^* < 0 \\ \text{undetermined,} & \text{if } (1 - \psi_1^*)T^* + \psi_2^* = 0 \end{cases} \quad (21)$$

The last possibility corresponds to what is called the ‘‘singular control problem’’, and for the problem it is easy to obtain that along the singular part of the trajectory  $C^*$  is constant.

Next set  $C = C^*$  and  $T_R = T_R^*$ ; then Eq. (20) yields

$$\Delta H = [\psi_1^* - \mu^*\theta((T_R^{*n} - T^{*n}) \text{sign}(n))] \\ \times (T_R^{*n} - T^{*n}) \text{sign}(n) (\hat{\rho}^* - \hat{\rho}) \geq 0 \quad (22)$$

Correspondingly, it is required that

$$\hat{\rho}^* = \begin{cases} \hat{\rho}_0^* & \text{if } [\psi_1^* - \mu^*\theta((T_R^{*n} - T^{*n}) \text{sign}(n))] \\ & \times (T_R^{*n} - T^{*n}) \text{sign}(n) > 0 \\ 0 & \text{if } [\psi_1^* - \mu^*\theta((T_R^{*n} - T^{*n}) \text{sign}(n))] \\ & \times (T_R^{*n} - T^{*n}) \text{sign}(n) > 0 \\ \text{undetermined} & \text{if } [\psi_1^* - \mu^*\theta((T_R^{*n} - T^{*n}) \\ & \times \text{sign}(n))] (T_R^{*n} - T^{*n}) \text{sign}(n) = 0 \end{cases} \quad (23)$$

The last possibility still corresponds to the singular control problem, because it would call for  $\Psi_1 = \mu = 1$ , further resulting in  $H^* = 0$ ; since this is not in agreement with the request for  $H^* > 0$ , one should exclude this possibility.

Finally, set  $C = C^*$  and  $\hat{\rho} = \hat{\rho}^*$ ; then Eq. (20) yields

$$\Delta H = \begin{cases} \hat{K}^*(\psi_1^* - \mu^*)(T_R^{*n} - T_R^n) \text{sign}(n) \\ \text{if } T_R^* > T^* \\ \hat{K}^*\psi_1^*(T_R^{*n} - T_R^n) \text{sign}(n) \\ \text{if } T_R^* \leq T^* \end{cases} \quad (24)$$

From Eq. (23), one can see that  $\hat{\rho}^* = \hat{\rho}_0$  requires  $\Psi_1^* > \mu^*$  if  $T^* > T^*$  and  $\Psi_1^* < 0$  if  $T_R^* < T^*$ . Consider  $T_R$  in the interval  $[T^*, T_H]$ ; for  $T_R^* > T^*$ , one can find  $T_R^* = T_H$  in order that  $\Delta H > 0$ . It then follows that  $\Delta H \geq 0$  for  $T_R^*$  in the interval  $[T_L, T^*]$ , since  $\mu^* \geq 0$ . In a similar fashion, when  $T_R^* < T^*$ ,  $\Delta H \geq 0$ , provided  $T_R^* = T_L$ . Therefore, one has

$$T_R^* = \begin{cases} T_H & \text{if } \psi_1^* > \mu^* \\ T_L & \text{if } \psi_1^* < 0 \end{cases} \quad (25)$$

and for  $\hat{\rho}^* \neq 0$ , one has  $T_H > T^* > T_L$ .

This problem could be changed into finding the optimal configuration for maximum-power output. If  $\mu^* > 0$ ,  $\hat{\rho}^* = 0$  and  $C^* = C$ , Eq. (20) can be changed into

$$\Delta H = -[\psi_1^* - \mu^*\theta((T_R^n - T^{*n}) \text{sign}(n))] \\ \times \hat{\rho}(T_R^n - T^{*n}) \text{sign}(n) \geq 0 \quad (26)$$

From Eqs. (25) and (26), one can get  $T^* > T_H$  if  $\psi_1^* > \mu^*$ , and  $T^* < T_L$  if  $\psi_1^* < 0$ , so that both cases are impossible. And if  $0 \leq \psi_1^* \leq \mu^*$ ,  $T^*$  is possible between  $T_H$  and  $T_L$ . From the above argument, one can get that adiabatic branches may take place when  $\mu^* > 0$ . If  $\mu^* = 0$ , the <sup>1</sup> adiabatic branch vanished [26, 38-40]. The problem would be more complex if  $\mu < 0$ , when<sup>2</sup> the optimal configuration may be without adiabatic branches.

**3.2. Optimal solutions**

It is easy to find all possible optimal solutions now; one can obtain the optimal trajectories by solving the canonical functions. All functions below are optimal. It is convenient to eliminate the asterisk from the symbols.

- (1) Adiabatic branches:  $\hat{\rho} = 0, C = C_M$  or  $C = -C_m$

$$T(t)=T(t_0)e^{-C(t-t_0)}, \beta(t)=\beta(t_0) + C(t - t_0),$$

$$\psi_1(t)=1-[1-\psi_1(t_0)]e^{C(t-t_0)}, \quad \psi_2 = \text{const} \quad (27)$$

$$H = \{[1 - \psi_1(t)]T + \psi_2\} C$$

$$= \{[1 - \psi_1(t_0)]T(t_0) + \psi_2\} C \quad (28)$$

where  $H$  and  $\psi_2$  are constants as required and the value of  $C$  is determined by Eq. (21);  $t_0$  is the initial time of the branches.

- (2) Maximum-efficiency branches:  $\hat{\rho} = \hat{\rho}_0, T_R = T_H$  or  $T_R = T_L, C = C_M$  or  $C = -C_m$

$$\dot{T} = -CT + \hat{\rho}(T^{-1} - T_R^{-1}),$$

$$\beta(t) = \beta(t_0) + C(t - t_0) \quad (29)$$

$$\dot{\psi}_1 = -C(1 - \psi_1) + n\hat{\rho}_0$$

$$\times [\psi_1 - \mu\theta((T_R^n - T^n)\text{sign}(n))]T^{n-1},$$

$$\psi_2(t) = \text{const} \quad (30)$$

where the value of  $C$  is determined by Eq. (21) and the value of  $T_R$  is determined by Eq. (25), and  $t_0$  is the initial time of the branches. Analytical solutions are rare, thus one has to count on numerical techniques to obtain all the solutions.

- (3) Isothermal branches:  $\hat{\rho} = \hat{\rho}_0, T_R = T_H$  or  $T_R = T_L, (1 - \psi_1)T + \psi_2 = 0$

$$T = T_r, \beta(t) = \beta(t_0) + C_r(t - t_0),$$

$$C_r = \frac{\hat{\rho}_0(T_R^n - T_r^n)\text{sign}(n)}{T_r} \quad (31)$$

$$\psi_1 = \frac{T_R^n + T_r^n [n\mu\theta((T_R^n - T_r^n)\text{sign}(n)) - 1]}{T_R^n - (1 - n)T_r^n},$$

$$\psi_2 = \frac{-n[1 - \mu\theta((T_R^n - T_r^n)\text{sign}(n))]T_r^{n+1}}{T_R^n - (1 - n)T_r^n} \quad (32)$$

where  $T_r$  is a constant. This is a singular case that has not been analyzed. It is easy to prove that  $C, T$  and  $\psi_1$  are constants by differentiating  $(1 - \psi_1)T + \psi_2 = 0$  and eliminating the derivative of time using canonical functions.

The subscript  $r$  used above corresponds to  $R$  in  $T_R$ , i.e.  $r = h$  if  $T_R = T_H$  and  $r = l$  if  $T_R = T_L$ . The value

of  $T_R$  is determined by Eq. (25). It is easy to show that

$$H = \hat{\rho}_0$$

$$\times \frac{(T_R^n - T_r^n)^2 [1 - \mu\theta((T_R^n - T_r^n)\text{sign}(n))]}{T_R^n - (1 - n)T_r^n} \text{sign}(n) \quad (33)$$

From the fact that  $\psi_2$  and  $H$  are constants, one has

$$\frac{(T_H^n - T_h^n)^2(1 - \mu)}{T_H^n - (1 - n)T_h^n} = \frac{(T_L^n - T_l^n)^2}{T_L^n - (1 - n)T_l^n},$$

$$\frac{T_h^{n+1}(1 - \mu)}{T_H^n - (1 - n)T_h^n} = \frac{T_l^{n+1}}{T_L^n - (1 - n)T_l^n} \quad (34)$$

$$C_h = \frac{\hat{\rho}_0(T_H^n - T_h^n)}{T_h} \text{sign}(n),$$

$$C_l = \frac{\hat{\rho}_0(T_L^n - T_l^n)}{T_l} \text{sign}(n) \quad (35)$$

Thus there are eight different solutions to the equations above which are presented by  $1^\pm, 2_H^\pm, 2_L^\pm, 3_H$  and  $3_L$ , where the symbol “+” refers to  $C = C_M$  and the symbol “-” to  $C = -C_m$ ; the subscripts  $H$  and  $L$  correspond to the subscript of  $T_R$ . If  $\mu = 0$ , the analytical results are the same as the optimal solutions of endoreversible heat engines for maximum-power output objective [26, 39, 40].

In order to determine the actual optimal trajectory, one must examine the constancy  $H$  and the continuity of the state variables and costate variables at switchings between pairs of optimal solutions.

**3.3. Switching**

The surfaces in state variable phase space across which optimal-control variables change discontinuously are called “switching surface” in optimal control theory. The switchings of the problem are summarized in Table I.

If there is a switching between branches  $1^+$  and  $1^-$ , Eq. (21) requires that  $(1 - \psi_1)T + \psi_2$  vanish, then Eq. (28) leads to  $H = 0$ , so there cannot be a switching between branches  $1^+$  and  $1^-$ . Similarly, there is no switching between branches  $3_H$  and  $3_L$  because of the continuity of  $\psi_1$ .

However, there may be switchings between  $T_R = T_H$  and  $T_R = T_L$  in case (2), in which  $C$  remains constant and  $\psi_1$  passes through zero at the switching time.  $-C_m$  and  $C_M$ , in

TABLE I. Switchings

	1	2	3
1	①	② or ③	①
2	② or ③	④	④
3	①	④	①

① Forbidden switchings

② Allowed switchings:  $\Delta C = 0$  and  $\Psi_1 = 0$

③ Allowed switchings:  $\Delta C = 0$  and  $\Psi_1 = \mu$

④ Allowed switchings:  $\Delta T_R = 0, (1 - \Psi_1)T + \Psi_2 = 0$

which  $T_R$  remains constant and  $(1 - \psi_1)T + \psi_2$  vanishes. But  $T_R$  and  $C$  cannot change continuously because of  $H \neq 0$ .

A switching between  $1^+$  and  $2_H^+$  is allowed by increasing  $\psi_1$  and making it across  $\mu$ , and a switching between  $1^+$  and  $2_L^+$  is allowed by decreasing  $\psi_1$  and making it across zero. The switching condition of each branch is shown below:

- The switching condition between  $3_H$  and  $2_H^+$  is  $[1 - \psi_1(t_1)]T(t_1) + \psi_2(t_1) = 0$ ;
- The switching condition between  $2_H^+$  and  $1^+$  is  $\psi_1(t_2) = \mu$ ;
- The switching condition between  $1^+$  and  $2_L^+$  is  $\psi_1(t_{2'}) = 0$ ;
- The switching condition between  $2_L^+$  and  $3_L$  is  $[1 - \psi_1(t_3)]T(t_3) + \psi_2(t_3) = 0$ ;
- The switching condition between  $3_L$  and  $2_L^-$  is  $[1 - \psi_1(t_4)]T(t_4) + \psi_2(t_4) = 0$ ;
- The switching condition between  $2_L^-$  and  $1^-$  is  $\psi_1(t_{5'}) = 0$ ;
- The switching condition between  $1^-$  and  $2_H^-$  is  $\psi_1(t_5) = \mu$ .

### 3.4. Optimal controls and trajectory

The case discussed above is an autonomous system, *i.e.*, invariant with respect to time translation, so one may choose any point along optimal trajectory as a starting point. Here one assumes that it begins from branch  $3_H$ , *i.e.*  $T_R = T_H$  and  $T = T_h$ , etc. For  $0 \leq t \leq t_1$ , the only allowed switching is to a branch  $2_H^+$ , *i.e.*  $T_R = T_H$ ,  $C = C_M$ , etc. For  $t_1 \leq t \leq t_2$ ,  $\psi_1$  decreases and  $(1 - \psi_1)T + \psi_2$  increases from zero; thus the only possible transition occurs at  $t_2$  when  $\psi_1 = 0$ , and one can obtain the branch  $2_L^+$ . Notice that there is an adiabatic branch  $1^+$  between  $t_2$  and  $t_{2'}$ , which is different from the optimal configuration for maximum power configuration [26,39,40].

From  $t_{2'}$  to  $t_3$ ,  $\psi_1$  continues to decrease and so does  $(1 - \psi_1)T + \psi_2$  until it vanishes, at which time another switch becomes possible. At  $t_3$ , one begins an isothermal branch  $3_L$  which lasts until time  $t_4$  when one switches to branch  $2_L^-$ . Along this branch,  $(1 - \psi_1)T + \psi_2$  decreases from zero while  $\psi_1$  increases until it reaches zero at  $t_{5'}$ . There is also an adiabatic branch  $1^-$  between  $t_{5'}$  and  $t_5$ . Then one switches to branch  $2_H^-$  until  $(1 - \psi_1)T + \psi_2$  returns to zero at the end of the cycle  $t_6 = \tau$ .

Consequently, one can obtain the solution to the problem. Since  $\psi_2$  is constant throughout the cycle, one records it only once. The state and costate variables are given as the following:

**For**  $0 \leq t \leq t_1$ :

$$T = T_h, C = C_h, \beta = C_h t, T_R = T_H, \hat{\rho} = \hat{\rho}_0 \quad (36)$$

$$\psi_1(t) = \frac{T_H^n + T_h^n(n\mu - 1)}{T_H^n - (1 - n)T_h^n},$$

$$\psi_2(t) = \frac{n(\mu - 1)T_h^{n+1}}{T_H^n - (1 - n)T_h^n} \quad (37)$$

**For**  $t_1 \leq t \leq t_2$ :

Expanding  $T(t)$  with a Taylor series at  $t_1$  gives

$$T(t) = T(t_1) + \dot{T}(t_1)(t - t_1) + O(t - t_1) \quad (38)$$

Taking off high-order infinitely-small  $O(t - t_1)$  gives

$$T(t) \approx T(t_1) + \dot{T}(t_1)(t - t_1) \quad (39)$$

From the continuity of  $T(t)$ , one can obtain  $T(t_1) = T_h$ , and from

$$\dot{T}(t) = -C_M T(t) + \hat{\rho}_0 [T_H^n(t) - T^n(t)] \text{sign}(n) \quad (40)$$

*i.e.*

$$T(t) \approx T_h + [-C_M T_h + \hat{\rho}_0 (T_H^n - T_h^n) \text{sign}(n)](t - t_1) \quad (41)$$

Expanding  $\psi(t)$  with Taylor series at  $t_1$  gives

$$\psi_1(t) = \psi_1(t_1) + \dot{\psi}_1(t_1)(t - t_1) + O(t - t_1) \quad (42)$$

From the continuity of  $\psi_1(t)$ , one can obtain

$$\psi_1(t) = \frac{T_H^n + T_h^n(n\mu - 1)}{T_H^n - (1 - n)T_h^n},$$

and from

$$\dot{\psi}_1(t) = -C_M [1 - \psi_1(t)] + n\hat{\rho}_0 T^{n-1}(t) [\psi_1(t) - \mu] \quad (43)$$

one can obtain

$$\psi_1(t) \approx \frac{T_H^n + T_h^n(n\mu - 1)}{T_H^n - (1 - n)T_h^n} + \frac{[-nC_M T_h^n + n\hat{\rho}_0 T_h^{n-1} (T_H^n - T_h^n) \text{sign}(n)](1 - \mu)}{T_H^n - (1 - n)T_h^n} \times (t - t_1) \quad (44)$$

Combining all the analysis above yields

$$T(t) \approx T_h + [-C_M T_h + \hat{K}_0(T_H^n - T_h^n)\text{sign}(n)](t - t_1) \quad (45)$$

$$\psi_1(t) \approx \frac{T_H^n + T_h^n(n\mu - 1)}{T_H^n - (1-n)T_h^n} + \frac{[-nC_M T_h^n + n\hat{\rho}_0 T_h^{n-1}(T_H^n - T_h^n)\text{sign}(n)](1-\mu)}{T_H^n - (1-n)T_h^n} \times (t - t_1) \quad (46)$$

$$\beta = C_M(t - t_1) + C_h t_1, \quad T_R = T_H, \quad \hat{\rho} = \hat{\rho}_0, \quad C = C_M \quad (47)$$

From above process, the details of each branch can be obtained. The other branches of the maximum-efficiency cycle are presented in detail below:

**For**  $t_2 \leq t \leq t_{2'}$ :

$$T(t) = T(t_2)e^{-C_M(t-t_2)}, \quad \beta = C_M(t - t_1) + C_h t_1, \quad \psi_1(t) = 1 - (1 - \mu)e^{C_M(t-t_2)}, \quad C = C_M, \quad \hat{\rho} = 0 \quad (48)$$

where

$$T(t_2) \approx T_h + [-C_M T_h + \hat{\rho}_0 \times (T_H^n - T_h^n)\text{sign}(n)](t_2 - t_1) \quad (49)$$

**For**  $t_{2'} \leq t \leq t_3$ :

$$T(t) \approx T(t_2)e^{-C_M(t_2'-t_2)} + \{-C_M T(t_2) \times e^{-C_M(t_2'-t_2)} + \hat{\rho}_0(T_L^n - T^n(t_2)) \times e^{-nC_M(t_2'-t_2)}\}\text{sign}(n)\}(t - t_{2'}) \quad (50)$$

$$\beta = C_M(t - t_1) + C_h t_1, \quad \psi_1(t) \approx -C_M(t - t_{2'}), \quad T_R = T_L, \quad \hat{\rho} = \hat{\rho}_0, \quad C = C_M \quad (51)$$

**For**  $t_3 \leq t \leq t_4$ :

$$T = T_l, \quad \beta = C_M(t_3 - t_1) + C_h t_1 + C_l(t - t_3) \quad (52)$$

$$\psi_1 = \frac{T_L^n - T_l^n}{T_L^n - (1-n)T_l^n}, \quad T_R = T_L, \quad \hat{\rho} = \hat{\rho}_0, \quad C = C_l \quad (53)$$

**For**  $t_4 \leq t \leq t_{5'}$ :

$$T(t) \approx T_l + [C_M T_l + \hat{\rho}_0(T_L^n - T_l^n)\text{sign}(n)](t - t_4) \quad (54)$$

$$\psi_1(t) \approx \frac{T_L^n - T_l^n}{T_L^n - (1-n)T_l^n} + \frac{nC_M T_l^n + n\hat{\rho}_0(T_L^n - T_l^n)T_l^{n-1}\text{sign}(n)}{T_L^n - (1-n)T_l^n}(t - t_4) \quad (55)$$

$$\beta = C_l(t_4 - t_3) + C_M(t_3 - t_1) + C_h t_1 - C_m(t - t_4), \quad T_R = T_L, \quad \hat{\rho} = \hat{\rho}_0, \quad C = -C_m \quad (56)$$

**For**  $t_{5'} \leq t \leq t_5$ :

$$T(t) = T(t_{5'})e^{C_m(t-t_{5'})}, \quad \beta = C_M(t_3 - t_1) + C_h t_1 + C_l(t_4 - t_3) - C_m(t - t_4), \quad \psi_1(t) = 1 - e^{-C_m(t-t_{5'})}, \quad \hat{\rho} = 0, \quad C = -C_m \quad (57)$$

where

$$T(t_{5'}) \approx T_l + [C_M T_l + \hat{\rho}_0 \times (T_L^n - T_l^n)\text{sign}(n)](t_{5'} - t_4) \quad (58)$$

**For**  $t_5 \leq t \leq t_6 = \tau$ :

$$T(t) = T(t_{5'})e^{C_m(t_5-t_{5'})} + \{C_M T(t_{5'})e^{C_m(t_5-t_{5'})} + \hat{\rho}_0[T_H^n - T^n(t_{5'}) \times e^{nC_m(t_5-t_{5'})}]\text{sign}(n)\}(t - t_5) \quad (59)$$

$$\beta = C_M(t_3 - t_1) + C_h t_1 + C_l(t_4 - t_3) - C_m(t - t_4), \quad \psi_1 \approx \mu + C_m(1 - \mu)(t - t_5), \quad T_R = T_H, \quad \hat{\rho} = \hat{\rho}_0, \quad C = -C_m \quad (60)$$

From the continuity of the cycle, one can obtain

$$T(t_6) = T(0), \quad \psi_1(t_6) = \psi_1(0), \quad \beta(t_6) = \beta(0) = 0 \quad (61)$$

From the switching condition, one can obtain

$$\psi_1(t_2) = \mu, \quad \psi_1(t_{2'}) = 0, \quad \psi_1(t_{5'}) = 0, \quad \psi_1(t_5) = \mu \quad (62)$$

From the fixed cycle duration one can obtain

$$t_1 + (t_4 - t_3) = \tau - (\tau - t_5) - (t_5 - t_{5'}) - (t_{5'} - t_4) - (t_3 - t_{2'}) - (t_{2'} - t_2) - (t_2 - t_1) \quad (63)$$

From the fixed input energy, one can obtain

$$Q_1 = C_V \int_0^\tau \hat{\rho}(T_R^n - T^n)\text{sign}(n)\theta \times [(T_R^n - T^n)\text{sign}(n)]dt = \text{const} \quad (64)$$

TABLE II. Parameters vs.  $Q_1$  with linear phenomenological heat transfer law

$T_H = 1000 \text{ K}, T_L = 400 \text{ K}, C_M = 8 \text{ s}^{-1}, C_m = 38 \text{ s}^{-1}, C_V = 5 \text{ kJ}/(\text{kgK}), \hat{\rho}_0 = 10^7 \text{ kgK}^2/\text{s}, \tau = 1 \text{ s}$									
	$Q_1 = 8000 \text{ kJ}$			$Q_1 = 10000 \text{ kJ}$			$Q_1 = 12000 \text{ kJ}$		
	$\Delta t(\text{s})$	T(K)	$\beta$	$\Delta t(\text{s})$	T(K)	$\beta$	$\Delta t(\text{s})$	T(K)	$\beta$
$t_1$	0.2862	730.00	1.5362	0.3944	730.00	2.1167	0.5025	730.00	2.6971
$t_2$	0.0887	540.03	2.2460	0.0887	540.03	2.8264	0.0887	540.03	3.4069
$t_{2'}$	0.0034	525.44	2.2734	0.0034	525.44	2.8538	0.0034	525.44	3.4343
$t_3$	0.1850	467.95	3.7530	0.0949	467.95	3.6128	0.0048	467.95	3.4726
$t_4$	0.4246	467.95	0.4590	0.4065	467.95	0.4590	0.3885	467.95	0.4590
$t_{5'}$	0.0056	547.45	0.2455	0.0056	547.45	0.2455	0.0056	547.45	0.2455
$t_5$	0.0007	562.64	0.2181	0.0007	562.64	0.2181	0.0007	562.64	0.2181
$t_6$	0.0057	730.00	0	0.0057	730.00	0	0.0057	730.00	0
$\mu$		0.0270			0.0270			0.0270	
$P$		2578.3 kW			3235.3 kW			3892.2 kW	
$\eta$		0.3223			0.3235			0.3244	

TABLE III. Parameters vs.  $Q_1$  with Newton's heat transfer law

$T_H = 1000 \text{ K}, T_L = 400 \text{ K}, C_M = 12.5 \text{ s}^{-1}, C_m = 3 \text{ s}^{-1}, C_V = 5 \text{ kJ}/(\text{kgK}), \hat{\rho}_0 = 20 \text{ kg/s}, \tau = 1 \text{ s}$									
	$Q_1 = 6450 \text{ kJ}$			$Q_1 = 6500 \text{ kJ}$			$Q_1 = 6550 \text{ kJ}$		
	$\Delta t(\text{s})$	T(K)	$\beta$	$\Delta t(\text{s})$	T(K)	$\beta$	$\Delta t(\text{s})$	T(K)	$\beta$
$t_1$	0.3080	931.53	0.4269	0.3082	931.53	0.4272	0.3084	931.53	0.4276
$t_2$	0.0219	706.18	0.7011	0.0219	706.18	0.7014	0.0219	706.18	0.7017
$t_{2'}$	0.0248	518.16	1.0107	0.0248	518.16	1.0110	0.0248	518.16	1.0113
$t_3$	0.0139	431.73	1.1844	0.0139	431.73	1.1841	0.0138	431.73	1.1839
$t_4$	0.5440	431.73	0.4308	0.5438	431.73	0.4308	0.5436	431.73	0.4308
$t_{5'}$	0.0555	468.42	0.2642	0.0555	468.42	0.2642	0.0555	468.42	0.2642
$t_5$	0.0275	508.66	0.1818	0.0275	508.66	0.1818	0.0275	508.66	0.1818
$t_6$	0.0210	931.53	0	0.0212	931.53	0	0.0215	931.53	0
$\mu$		0.4630			0.4630			0.4630	
$P$		2042.8 kW			2043.2 kW			2044.6 kW	
$\eta$		0.3119			0.3142			0.3165	

Combining Eqs. (61), (62), (63), and (64) with the equations of  $T_h$ ,  $T_l$ ,  $C_h$ , and  $C_l$ , one can solve out numerical solutions for all the intervals of each branch, *i.e.*  $t_1$ ,  $(t_2 - t_1)$ ,  $(t_{2'} - t_2)$ ,  $(t_3 - t_{2'})$ ,  $(t_4 - t_3)$ ,  $(t_{5'} - t_4)$ ,  $(t_5 - t_{5'})$ , and  $(\tau - t_5)$ , as well as  $T_h$ ,  $T_l$ ,  $C_h$ ,  $C_l$  and the Lagrange multiplier  $\mu$ . Substituting them into equations in  $T(t)$ , one can solve out numerical solutions for all temperatures of the working fluid, *i.e.*  $T(t_1)$ ,  $T(t_2)$ ,  $T(t_{2'})$ ,  $T(t_3)$ ,  $T(t_4)$ ,  $T(t_{5'})$ ,  $T(t_5)$ , and  $T(t_6)$ . The equations are presented in detail below:

$$T_h \approx T(t_{5'})e^{C_m(t_5 - t_{5'})} + \{C_m T(t_{5'})e^{C_m(t_5 - t_{5'})} + \hat{\rho}_0 [T_H^n - T^n(t_{5'})e^{nC_m(t_5 - t_{5'})}] \text{sign}(n)\} (t_6 - t_5) \quad (65)$$

$$\mu + C_m(1 - \mu)(t_6 - t_5) \approx \frac{[T_H^n + T_h^n(n\mu - 1)]}{(T_H^n - (1 - n)T_h^n)} \quad (66)$$

$$C_M(t_3 - t_{2'}) + C_M(t_{2'} - t_2) + C_M(t_2 - t_1) + C_h t_1 + C_l(t_4 - t_3) - C_m(t_6 - t_5) - C_m(t_5 - t_{5'}) - C_m(t_{5'} - t_4) = 0 \quad (67)$$

$$\mu \approx \frac{T_H^n + T_h^n(n\mu - 1)}{T_H^n - (1 - n)T_h^n} + \frac{[-nC_M T_h^n + n\hat{\rho}_0 T_h^{n-1}(T_H^n - T_h^n)\text{sign}(n)](1 - \mu)}{T_H^n - (1 - n)T_h^n} \times (t_2 - t_1) \quad (68)$$

$$1 - (1 - \mu)e^{C_M(t_2 - t_2)} = 0 \quad (69)$$

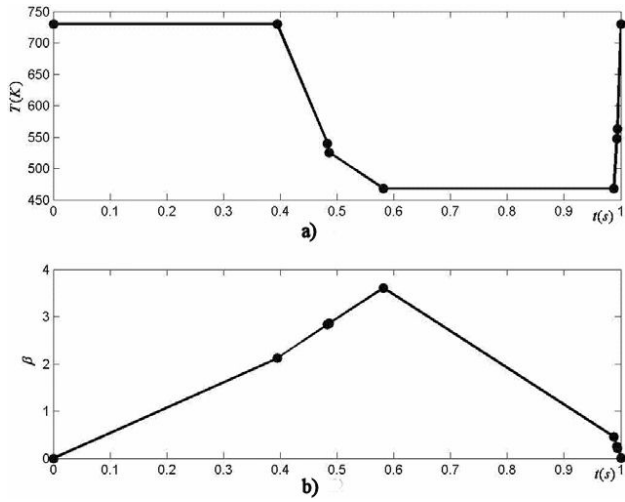


FIGURE 1. State variables for maximum efficiency objective with fixed input energy  $Q_1=10000$  kJ and linear phenomenological heat transfer law.

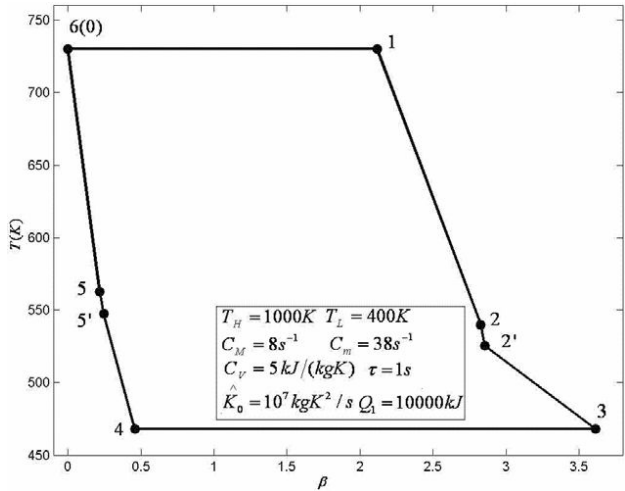


FIGURE 2. The cycle of eight branches for maximum efficiency objective with fixed input energy  $Q_1=10000$  kJ and linear phenomenological heat transfer law.

$$0 \approx \frac{T_L^n - T_l^n}{T_L^n - (1-n)T_l^n} + \frac{nC_m T_l^n + n\hat{\rho}_0(T_L^n - T_l^n)T_l^{n-1} \text{sign}(n)}{T_L^n - (1-n)T_l^n} (t_{5'} - t_4) \quad (70)$$

$$1 - e^{-C_m(t_5 - t_{5'})} = \mu \quad (71)$$

$$t_1 + (t_4 - t_3) = \tau - (\tau - t_5) - (t_5 - t_{5'}) - (t_{5'} - t_4) - (t_3 - t_{2'}) - (t_{2'} - t_2) - (t_2 - t_1) \quad (72)$$

$$Q_1 = C_V \int_0^\tau \hat{\rho}(T_R^n - T^n) \text{sign}(n) \times [(T_R^n - T^n) \text{sign}(n)] dt = \text{const} \quad (73)$$

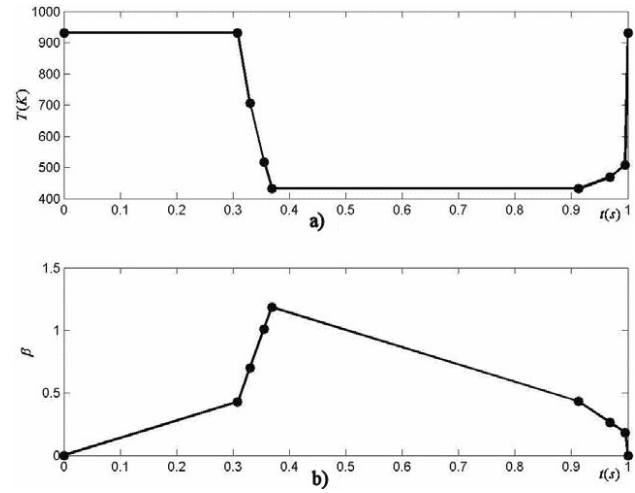


FIGURE 3. State variables for maximum efficiency objective with fixed input energy  $Q_1=6500$ kJ and Newton's heat transfer law.

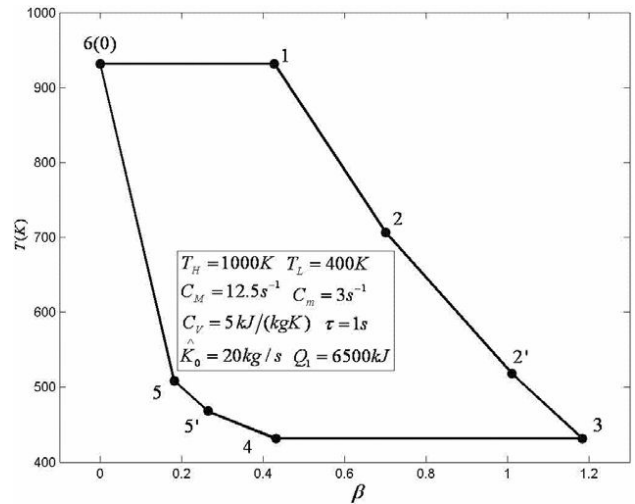


FIGURE 4. The cycle of eight branches for maximum efficiency objective with fixed input energy  $Q_1=6500$ kJ and Newton's heat transfer law.

$$C_h = \frac{\hat{\rho}_0(T_H^n - T_h^n)}{T_h} \text{sign}(n) \quad (74)$$

$$C_l = \frac{\hat{\rho}_0(T_L^n - T_l^n)}{T_l} \text{sign}(n) \quad (75)$$

$$\frac{(T_H^n - T_h^n)^2(1-\mu)}{T_H^n - (1-n)T_h^n} = \frac{(T_L^n - T_l^n)^2}{T_L^n - (1-n)T_l^n} \quad (76)$$

$$\frac{T_h^{n+1}(1-\mu)}{T_H^n - (1-n)T_h^n} = \frac{T_l^{n+1}}{T_L^n - (1-n)T_l^n} \quad (77)$$

Maximum efficiency is

$$\eta = \frac{W}{Q_1} = \frac{C_V \int_0^\tau CT dt}{Q_1} \quad (78)$$



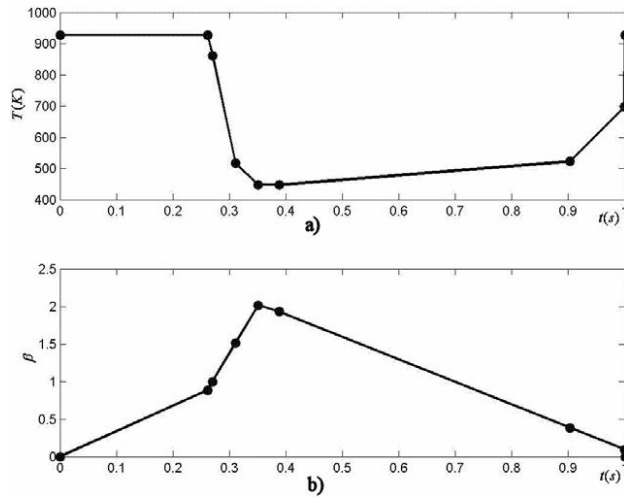


FIGURE 5. State variables for maximum efficiency objective with fixed input energy  $Q_1=6500\text{kJ}$  and square heat transfer law.

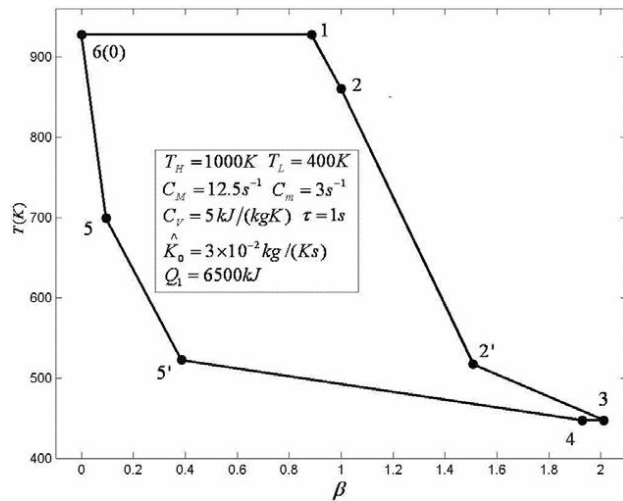


FIGURE 6. The cycle of eight branches for maximum efficiency objective with fixed input energy  $Q_1=6500\text{kJ}$  and square heat transfer law.

## 4. Numerical Examples

Now, numerical examples for the optimal configurations of the heat engines for different heat transfer laws are provided. The results include all special heat transfer laws, such as  $n = -1$ ,  $n = 1$ ,  $n = 2$ ,  $n = 3$  and  $n = 4$ .

### 4.1. Optimal configuration with $n = -1$

In this case,  $T_H=1000\text{K}$ ,  $T_L=400\text{K}$ ,  $C_M=8\text{s}^{-1}$ ,  $C_m=38\text{s}^{-1}$ ,  $\hat{\rho}_0=10^7\text{kgK}^2/\text{s}$ ,  $\tau=1\text{s}$ ,  $C_V=5\text{kJ}/(\text{kgK})$  and the mass of working fluid of  $1\text{kg}$  are set. Also,  $\text{sign}(n) = -1$  holds. Based on the method mentioned above, one can obtain all the numerical solutions. Table II lists the process-times of each branch, the values of the state variables at switchings, the maximum efficiency  $\eta$  and the corresponding power output  $P$ .

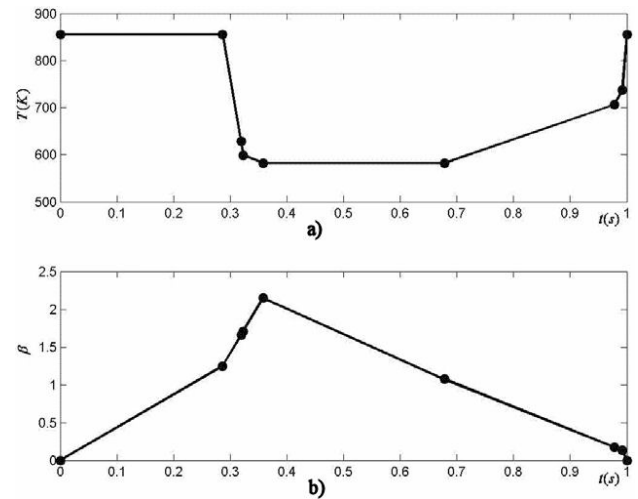


FIGURE 7. State variables for maximum efficiency objective with fixed input energy  $Q_1=6500\text{kJ}$  and cubic heat transfer law.

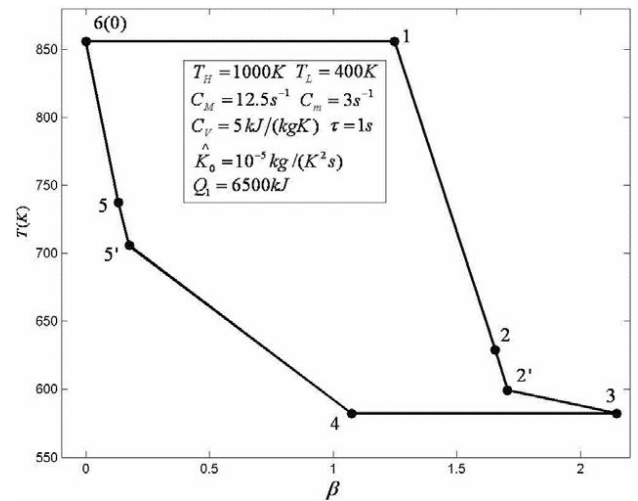


FIGURE 8. The cycle of eight branches for maximum efficiency objective with fixed input energy  $Q_1=6500\text{kJ}$  and cubic heat transfer law.

Figure 1 shows state variables for the efficiency objective with  $Q_1=10000\text{kJ}$ . Figure 2 shows the cycle of eight branches for the maximum efficiency objective with  $Q_1=10000\text{kJ}$ .

In this example, time is mostly spent on two isothermal branches. The change in  $Q_1$  has an obvious influence on the intervals of two isothermal branches and the  $2_L^+$  branch, and has little influence on the intervals of other branches, and the temperatures of working fluid at the switchings, and the Lagrange multiplier  $\mu$ . It is shown that, with the increase in  $Q_1$ , the power output  $P$  and the maximum efficiency  $\eta$  increase, as well as the interval of  $3_H$  branch, and the interval of  $3_L$  branch decreases.

### 4.2. Optimal configuration with $n=1$

In this case,  $T_H=1000\text{K}$ ,  $T_L=400\text{K}$ ,  $C_M=100\text{s}^{-1}$ ,  $C_m=100\text{s}^{-1}$ ,  $\hat{\rho}_0=20\text{kg}/\text{s}$ ,  $\tau=1\text{s}$ ,  $C_V=5\text{kJ}/(\text{kgK})$ , and the

TABLE IV. Parameters vs.  $Q_1$  with square heat transfer law

$T_H = 1000 \text{ K}, T_L = 400 \text{ K}, C_M = 12.5 \text{ s}^{-1}, C_m = 3 \text{ s}^{-1}, C_V = 5 \text{ kJ}/(\text{kgK}), \hat{\rho}_0 = 3 \times 10^{-2} \text{ kg}/(\text{Ks}), \tau = 1 \text{ s}$									
	$Q_1 = 6450 \text{ kJ}$			$Q_1 = 6500 \text{ kJ}$			$Q_1 = 6550 \text{ kJ}$		
	$\Delta t(\text{s})$	$T(\text{K})$	$\beta$	$\Delta t(\text{s})$	$T(\text{K})$	$\beta$	$\Delta t(\text{s})$	$T(\text{K})$	$\beta$
$t_1$	0.2770	929.31	0.8864	0.2611	927.23	0.8859	0.2632	927.90	0.8999
$t_2$	0.0069	877.51	0.9725	0.0091	860.30	0.9992	0.0112	844.45	1.0403
$t_{2'}$	0.0419	519.56	1.4966	0.0407	517.16	1.5081	0.0411	505.31	1.5539
$t_3$	0.0409	444.99	2.0073	0.0403	447.01	2.0119	0.0366	447.01	2.0117
$t_4$	0.0211	444.99	1.9629	0.0375	447.01	1.9280	0.0085	447.01	1.9925
$t_{5'}$	0.5153	545.23	0.4169	0.5142	522.36	0.3855	0.5397	526.02	0.3736
$t_5$	0.0964	728.07	0.1277	0.0970	698.76	0.0946	0.1015	713.26	0.0691
$t_6$	0.0126	929.31	0	0.0143	927.23	0	0.0142	927.90	0
$\mu$		0.5942			0.5446			0.5424	
$P$		2520.3 kW			2565.7 kW			2592.1 kW	
$\eta$		0.3848			0.3978			0.3988	

mass of working fluid of 1 kg are set. Table III lists the process-times of each branch, the values of the state variables at switchings, the maximum efficiency  $\eta$  and the corresponding power output  $P$ . Figure 3 shows state variables for the maximum efficiency objective with  $Q_1=6500 \text{ kJ}$ . Figure 4 shows the cycle of eight branches for the maximum efficiency objective with  $Q_1=6500 \text{ kJ}$ .

4.3. Optimal configuration with  $n=2$

In this case,  $T_H=1000 \text{ K}, T_L=400 \text{ K}, C_M=8 \text{ s}^{-1}, C_m = 8 \text{ s}^{-1}, \hat{\rho}_0=3 \times 10^{-2} \text{ kg}/\text{Ks}, \tau=1 \text{ s}, C_V=5 \text{ kJ}/(\text{kgK})$ , and the mass of working fluid of 1 kg are set. Table IV lists the process-times of each branch, the values of the state variables at switchings, the maximum efficiency  $\eta$  and the corresponding power output  $P$ . Figure 5 shows state variables

for the maximum efficiency objective with  $Q_1=6500 \text{ kJ}$ . Figure 6 shows the cycle of eight branches for the maximum efficiency objective with  $Q_1=6500 \text{ kJ}$ .

In this example, time is mostly spent on two isothermal branches. It is shown that, with the increase in  $\hat{\rho}_0$ , there is little influence on the temperatures of working fluid at the switchings and intervals of each branch, and the maximum power output per cycle  $P$  increases, but the corresponding efficiency  $\eta$  decreases.

4.4. Optimal configuration with  $n=3$

In this case,  $T_H=1000 \text{ K}, T_L=400 \text{ K}, C_M=100 \text{ s}^{-1}, C_m=100 \text{ s}^{-1}, \hat{\rho}_0 = 10^{-5} \text{ kg}/(\text{K}^2\text{s}), \tau=1 \text{ s}, C_V=5 \text{ kJ}/(\text{kgK})$ , and the mass of working fluid of 1 kg are set. Table V lists

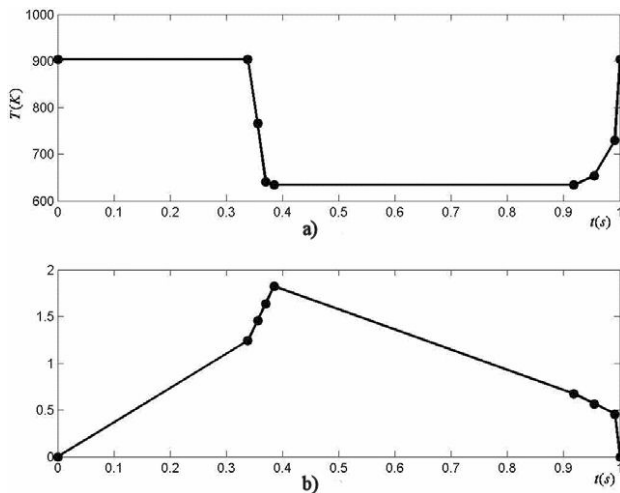


FIGURE 9. State variables for maximum efficiency objective with fixed input energy  $Q_1=6500 \text{ kJ}$  and radiative heat transfer law.

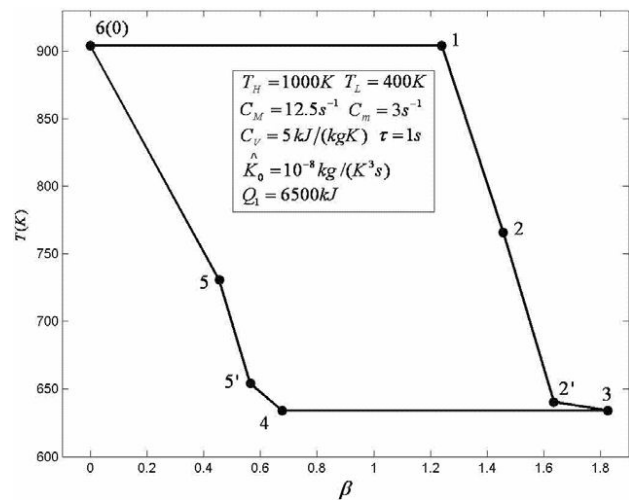


FIGURE 10. The cycle of eight branches for maximum efficiency objective with fixed input energy  $Q_1 = 6500 \text{ kJ}$  and radiative heat transfer law.

TABLE V. Parameters vs.  $Q_1$  with cubic heat transfer law

$T_H = 1000 \text{ K}, T_L = 400 \text{ K}, C_M = 12.5 \text{ s}^{-1}, C_m = 3 \text{ s}^{-1}, C_V = 5 \text{ kJ}/(\text{kgK}), \hat{\rho}_0 = 10^{-5} \text{ kg}/(\text{K}^2 \text{ s}), \tau = 1 \text{ s}$									
	$Q_1 = 6450 \text{ kJ}$			$Q_1 = 6500 \text{ kJ}$			$Q_1 = 6550 \text{ kJ}$		
	$\Delta t(s)$	$T(K)$	$\beta$	$\Delta t(s)$	$T(K)$	$\beta$	$\Delta t(s)$	$T(K)$	$\beta$
$t_1$	0.2803	853.96	1.2367	0.2865	855.72	1.2484	0.2913	857.27	1.2550
$t_2$	0.0329	626.56	1.6485	0.0326	628.87	1.6557	0.0299	647.71	1.6284
$t_{2'}$	0.0054	585.53	1.7163	0.0038	599.401	1.7037	0.0074	590.68	1.7206
$t_3$	0.0424	586.38	2.2463	0.0355	582.30	2.1469	0.0319	574.87	2.1194
$t_4$	0.3332	586.38	1.1016	0.3195	582.30	1.0740	0.3153	574.87	1.0826
$t_{5'}$	0.2873	696.41	0.2396	0.2998	705.95	0.1747	0.2916	710.40	0.2079
$t_5$	0.0112	720.17	0.2061	0.0147	737.73	0.1306	0.0260	767.94	0.1300
$t_6$	0.0072	853.96	0	0.0076	855.72	0	0.0132	857.27	0
$\mu$		0.3604			0.3717			0.4098	
$P$		1992.7 kW			2040.0 kW			2239.0 kW	
$\eta$		0.3042			0.3138			0.3471	

TABLE VI. Parameters vs.  $Q_1$  with radiative heat transfer law

$T_H = 1000 \text{ K}, T_L = 400 \text{ K}, C_M = 12.5 \text{ s}^{-1}, C_m = 3 \text{ s}^{-1}, C_V = 5 \text{ kJ}/(\text{kgK}), \hat{\rho}_0 = 10^{-8} \text{ kg}/(\text{K}^3 \text{ s}), \tau = 1 \text{ s}$									
	$Q_1 = 6450 \text{ kJ}$			$Q_1 = 6500 \text{ kJ}$			$Q_1 = 6550 \text{ kJ}$		
	$\Delta t(s)$	$T(K)$	$\beta$	$\Delta t(s)$	$T(K)$	$\beta$	$\Delta t(s)$	$T(K)$	$\beta$
$t_1$	0.3378	904.00	1.2385	0.3382	904.00	1.2400	0.3386	904.00	1.2414
$t_2$	0.0166	771.66	1.4458	0.0173	765.71	1.4566	0.0181	759.82	1.4673
$t_{2'}$	0.0143	645.35	1.6246	0.0143	640.38	1.6354	0.0143	635.45	1.6461
$t_3$	0.0152	634.00	1.8146	0.0152	634.00	1.8254	0.0152	634.00	1.8361
$t_4$	0.5324	634.00	0.6658	0.5324	634.00	0.6766	0.5324	634.00	0.6873
$t_{5'}$	0.0370	654.08	0.5548	0.0370	654.07	0.5656	0.0370	654.06	0.5763
$t_5$	0.0369	730.72	0.4440	0.0369	730.63	0.4549	0.0369	730.54	0.4657
$t_6$	0.0151	904.00	0	0.0158	904.00	0	0.0165	904.00	0
$\mu$		0.0137			0.0137			0.0137	
$P$		3139.3 kW			3166.8 kW			3193.8 kW	
$\eta$		0.4867			0.4872			0.4876	

the process-times of each branch, the values of the state variables at switchings, the maximum efficiency  $\eta$  and the corresponding power output  $P$ . Figure 7 shows state variables for maximum efficiency objective with  $Q_1=6500 \text{ kJ}$ . Figure 8 shows the cycle of eight branches for the maximum efficiency objective with  $Q_1=6500 \text{ kJ}$ . The result is similar to that with  $n=2$ .

#### 4.5. Optimal configuration with $n=4$

In this case,  $T_H=1000 \text{ K}$ ,  $T_L=400 \text{ K}$ ,  $C_M=8 \text{ s}^{-1}$ ,  $C_m=5 \text{ s}^{-1}$ ,  $\hat{\rho}_0 = 10^{-8} \text{ kg}/(\text{K}^3 \text{ s})$ ,  $\tau=1 \text{ s}$ ,  $C_V=5 \text{ kJ}/(\text{kgK})$ , and the mass of working fluid of  $1 \text{ kg}$  are set. Table VI lists the process-times of each branch, the values of the state variables at switchings, the maximum efficiency  $\eta$  and the corresponding power output  $P$  with radiative heat transfer law. Figure 9 shows state variables for the maximum effi-

ciency objective with  $Q_1=6500 \text{ kJ}$ . Figure 10 shows the cycle of eight branches for maximum efficiency objective with  $Q_1=6500 \text{ kJ}$ .

In this example, time is also mostly spent on two isothermal branches. The change in  $Q_1$  has just a little influence on the intervals of each branch, the temperatures of working fluid at the most switchings and the Lagrange multiplier  $\mu$ . It is shown that with the increase in  $Q_1$ , the power output  $P$  and the maximum efficiency  $\eta$  increase, and the temperatures of working fluid at  $t_2$  and  $t_2'$  decrease.

#### 4.6. Comparisons for several special heat transfer laws

For more clearer observation, one just chooses three heat transfer laws, *i.e.* the linear phenomenological heat transfer law ( $\hat{\rho}_0=10^7 \text{ kgK}^2/\text{s}$ ), Newton's heat transfer law ( $\hat{\rho}_0=20 \text{ kg/s}$ ) and the radiative heat transfer law

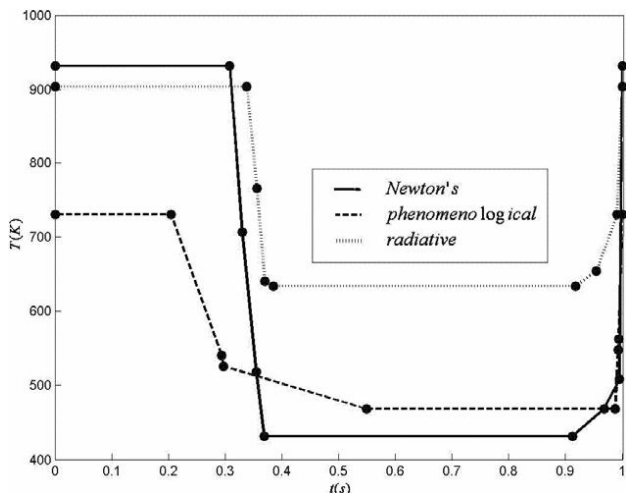


FIGURE 11. Optimal temperature of working fluid vs. time for maximum efficiency objective with three heat transfer laws.

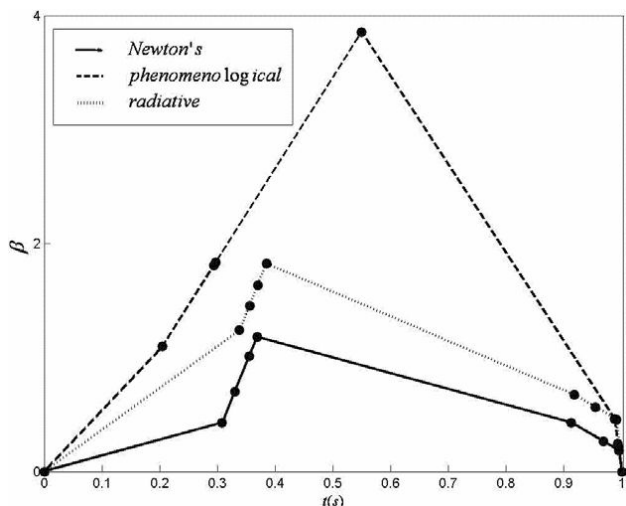


FIGURE 12. Optimal temperature of working fluid vs. time for maximum efficiency objective with three heat transfer laws.

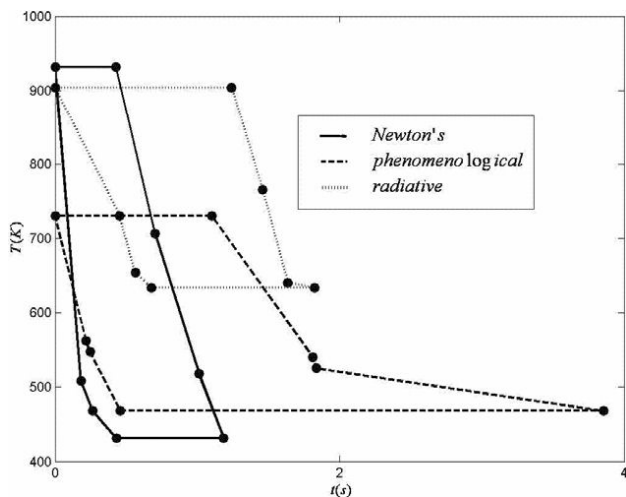


FIGURE 13. The cycles of eight branches for maximum efficiency objective with three heat transfer laws.

( $\hat{\rho}_0=10^{-8} \text{ kg}/(\text{K}^3 \text{ s})$ ), to compare. Figure 11 shows temperature of work fluid versus time with three heat transfer laws and  $Q_1=6500 \text{ kJ}$ . Figure 12 shows the relative volume of the working fluid versus time with three heat transfer laws and  $Q_1=6500 \text{ kJ}$ . Figure 13 shows the cycles of eight branches for maximum efficiency objective with three heat transfer laws and  $Q_1=6500 \text{ kJ}$ . From Figs. 11 and 12, one can see that time is mostly spent on two isothermal branches. From Fig. 13, one can see that the optimal configurations with three heat transfer laws are obviously different from each other, so the heat transfer law has an obvious influence on the optimal configurations of the heat engines.

### 5. Conclusion

The optimal configuration of an endoreversible heat engine for maximum efficiency objective with fixed cycle period, fixed input energy and generalized radiative heat transfer law has eight branches including two isothermal branches, two adiabatic branches and four maximum efficiency branches which give maximum efficiency.

The similarities and differences of optimal cycles among five heat transfer laws are given as follows: the optimal cycles for five heat transfer laws all contain two isothermal branches, two adiabatic branches and four maximum-efficiency branches; for five different heat transfer laws, the temperatures of their isothermal branches are different, and the process paths of four maximum efficiency branches are different as well; the process-time is different for different branches under five different optimal configuration. Since both the process-path and the process-time are different under five different optimal configurations, the maximum efficiencies for the five configurations are different.

The results obtained can also be compared with those results of Refs. [26, 38-40] for the maximum power output objective. One can see that, because of the introduction of the Lagrange multiplier  $\mu$ , the problem becomes complex, the optimal configuration changes from six branches to eight branches, and the process-path and the process-time are also different from the case of six branches. The optimal configuration is the same as the case of six branches with the same heat transfer law if and only if  $\mu = 0$ .

A first-order item of the Taylor series expansion was used in this paper, and the solutions would be more exact if high-order items of the Taylor series were used. In the calculations, there is only one solution for the problem if the initial values of the parameters for the calculations are located in the reasonable practical ranges.

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