

The Saturn, Janus and Epimetheus dynamics as a gravitational three-body problem in the plane

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Using a coordinate system given by the principal axis of inertia, as determined by an angle, and also two distances related to the principal moments of inertia and an auxiliary angle as coordinates, we consider the Three Body Problem, interacting through gravitational forces in a plane. The dynamics of the triple Saturn-Janus-Epimetheus has been considered in these coordinates as an adiabatic perturbation of the classical equilateral triangle Lagrange solution and of the collinear Euler solution. The co-orbital motion remembering the Saturn-Janus-Epimetheus behavior is then developed theoretically based on numerical and experimental evidence.

Keywords: Three-body problem; Saturn; Janus; Epimetheus.

Se usa un sistema de coordenadas dado por los ejes principales de inercia, determinados por un ángulo, y además, dos distancias relacionadas a los momentos principales de inercia y un ángulo auxiliar. Consideramos al problema de tres cuerpos, que interactúan a través de fuerzas gravitacionales en un plano. La dinámica de la terna Saturno-Jano-Epimeteo se ha descrito en estas coordenadas como una perturbación adiabática de las soluciones clásicas triangular equilátera de Lagrange y la solución colineal de Euler. El movimiento co-orbital semejante al comportamiento de Saturno-Jano-Epimeteo se desarrolla teóricamente basados en evidencia numérica y experimental.

Descriptores: Problema de tres cuerpos; Saturno; Jano; Epimeteo.

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1. Introduction

Voyager 1 flight confirmed the existence of two minor satellites of Saturn: Janus and Epimetheus, providing a first estimate of their masses as well as their orbital elements. A detailed history of this discovery was given by Aksnes [1].

These two moons became familiar after the *Voyager 1* travel in space. They received a strong attention from different researchers.

Some of them consider their orbits from a numerical point of view Janus and Epimetheus roughly share the same mean orbit of a radius of 151440 km from Saturn and a frequency near 518 deg day^{-1} [2], they are called coorbital. One of these moons is faster than the other by the small amount of velocity produced by a difference of radius of 50 km. Each 4 years [3] the two moons become 15000 km apart, but their mutual gravitational interaction prevents collision, and they switch orbits, the one in the interior orbit becoming exterior and viceversa, the exterior becoming interior at the moment of close approach called the encounter. Colombo [4], estimates the amplitude of the librations of the moons in a rotating frame. The *Cassini* imaging observations at Saturn in 2006 resulted in new data [5] and analysis to fix the parameters for the dynamics of the Saturn's moons and numerically calculate their orbits.

The main theoretical approach has been made in the context of the restricted problems in the plane. Dermott and Murray [6, 7], gave the description of the coorbital motion of Janus and Epimetheus based on a combination of numerical integration and perturbation theory, and extrapolate some results to the case when the third mass is not negligible.

Yoder *et al.* [8, 9], derived a simple analytic approximation to the motion of all the Trojans and applied it to the Janus and Epimetheus motion. Llibre and Ollé [10] proved the existence of stable horseshoe orbits for the value of the mass parameter of Saturn and Janus in the context of the circular restricted three body problem.

Waldvogel & Spirig [11–13], studied the problem of Saturn's coorbital satellites by means of a singular perturbation approach, that is, the motion is initially described by an outer and an inner approximation valid when the satellites are far apart or close together, respectively; the complete description of the motion requires the matching of both approximations.

Corresponding to the Hill's problem, Petit & Henon [14, 15] dealt with satellite encounters in the framework of Hill's problem from the analytical and numerical points of view.

Cors and Hall [16] approach this problem analytically by introducing small parameters into the usual general three body problem in the plane, truncating high order terms and deriving dynamic information from the resulting equations.

A new perspective is undertaken in this paper with the use of new coordinates but regarding the Three-Body Problem without the simplifications of the restricted models. The use of new coordinates was recommended by the extreme simplification that the integrable Euler and Lagrange cases attained in these coordinates; and the relation of these two integrable cases with the actual dynamics of Janus and Epimetheus recognized by most of the cited authors. Our main purpose is to find in this paper new analytical approximations for the motion of Janus and Epimetheus in these variables, at least not

close to their encounter where we study the motion doing numerical integration as many of the previous authors working in this problem.

2. The Piña-Jiménez coordinates

To study the Janus-Epimetheus dynamics we consider the general Three Body Problem, interacting through gravitational forces in a plane.

The study of this dynamics is made in the coordinate system of Piña and Jiménez [17–19] that is further simplified by the previous hypothesis of plane two-dimensional motion.

The masses of the three bodies m_1, m_2 and m_3 are different, and ordered by the inequalities $m_1 \gg m_2 > m_3$.

Since the motion takes place in a plane, it is sufficient to take into account only one rotation angle ψ in order to transform from the inertial referential, to the frame of principal inertia axes, instead of three Euler angles.

In addition to this angle three other coordinates were introduced, named σ, R_1, R_2 , where σ is an angle, and R_1 and R_2 are two distances closely related to the two independent inertia moments

$$I_1 = \mu R_1^2, \tag{1}$$

and

$$I_2 = \mu R_2^2, \tag{2}$$

and μ is the mass

$$\mu = \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}}, \tag{3}$$

which will be used frequently in this paper.

The cartesian inertial coordinates, with the center of gravity at rest, written in terms of the new coordinates are

$$\begin{aligned} \mathbf{r}_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix} &= a_j \begin{pmatrix} R_2 \cos \sigma \cos \psi + R_1 \sin \sigma \sin \psi \\ R_2 \cos \sigma \sin \psi - R_1 \sin \sigma \cos \psi \end{pmatrix} \\ &+ b_j \begin{pmatrix} R_2 \sin \sigma \cos \psi - R_1 \cos \sigma \sin \psi \\ R_2 \sin \sigma \sin \psi + R_1 \cos \sigma \cos \psi \end{pmatrix}, \end{aligned} \tag{4}$$

where the a_j and b_j are constants forming two constant, linearly independent vectors \mathbf{a} and \mathbf{b} , in the mass space, orthogonal to the vector $\mathbf{m} = (m_1, m_2, m_3)$:

$$a_1 m_1 + a_2 m_2 + a_3 m_3 = b_1 m_1 + b_2 m_2 + b_3 m_3 = 0. \tag{5}$$

We introduce the following notation for the matrix

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{6}$$

In order to complete the definition of vectors \mathbf{a} and \mathbf{b} we assume

$$\mathbf{b M a}^T = 0, \quad \mathbf{b M}^2 \mathbf{a}^T = 0, \tag{7}$$

which fixes the two directions of \mathbf{a} and \mathbf{b} in the plane orthogonal to \mathbf{m} , and we assume the normalizations

$$\mathbf{a M a}^T = \mathbf{b M b}^T = \mu, \tag{8}$$

that define the vectors \mathbf{a} and \mathbf{b} with no dimensions.

These two vectors are easily computed from the previous orthogonality conditions. One has

$$\mathbf{a} = \mu y_a \left(\frac{1}{m_1 - x_a}, \frac{1}{m_2 - x_a}, \frac{1}{m_3 - x_a} \right). \tag{9}$$

and

$$\mathbf{b} = \mu y_b \left(\frac{1}{m_1 - x_b}, \frac{1}{m_2 - x_b}, \frac{1}{m_3 - x_b} \right), \tag{10}$$

where y_a and y_b are normalization factors, and x_a and x_b are the roots of the quadratic equation

$$\frac{x^2}{\mu^2} - 2\frac{x}{\mu}\alpha + 3 = 0, \tag{11}$$

where

$$\alpha = \mu \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \right). \tag{12}$$

The two solutions to that quadratic equation are

$$x_a = \mu(\alpha + \sqrt{\alpha^2 - 3}) \tag{13}$$

and

$$x_b = \mu(\alpha - \sqrt{\alpha^2 - 3}). \tag{14}$$

These quantities are defined in this form only for different masses. In that case we have the inequalities

$$m_1 > x_a > m_2 > x_b > m_3, \tag{15}$$

that imply

$$a_1 > 0, a_2 < 0, a_3 < 0; b_1 > 0, b_2 > 0, b_3 < 0. \tag{16}$$

We define the angles $\sigma_1, \sigma_2, \sigma_3$ by means of

$$a_j = k_j \cos \sigma_j, \quad b_j = k_j \sin \sigma_j \tag{17}$$

where k_j are positive constants. The angles σ_j obey the inequalities

$$0 < \sigma_1 < \pi/2 < \sigma_2 < \pi < \sigma_3 < 3\pi/2. \tag{18}$$

To express the potential energy (G is the gravitational constant)

$$V = -\frac{Gm_2 m_3}{p} - \frac{Gm_3 m_1}{q} - \frac{Gm_1 m_2}{r}, \tag{19}$$

we need the relationship among the distances between particles p, q, r , and the new coordinates.

$$\begin{pmatrix} p^2 \\ q^2 \\ r^2 \end{pmatrix} = \mathbf{Be} \begin{pmatrix} R_1^2 + R_2^2 \\ (R_1^2 - R_2^2) \cos 2\sigma \\ (R_1^2 - R_2^2) \sin 2\sigma \end{pmatrix}, \quad (20)$$

where \mathbf{Be} is the constant matrix, depending only on the masses,

$$\mathbf{Be} = \frac{1}{\mu^2} \begin{pmatrix} m_1^2 \frac{a_1^2 + b_1^2}{2} & m_1^2 \frac{a_1^2 - b_1^2}{2} & m_1^2 a_1 b_1 \\ m_2^2 \frac{a_2^2 + b_2^2}{2} & m_2^2 \frac{a_2^2 - b_2^2}{2} & m_2^2 a_2 b_2 \\ m_3^2 \frac{a_3^2 + b_3^2}{2} & m_3^2 \frac{a_3^2 - b_3^2}{2} & m_3^2 a_3 b_3 \end{pmatrix}. \quad (21)$$

It has been convenient to introduce polar coordinates for R_1 and R_2 namely

$$R_1 = R \cos \theta, \quad R_2 = R \sin \theta. \quad (22)$$

Writing these quantities in terms of the coordinates and the angles σ_j one has

$$\begin{aligned} p^2 &= \frac{R^2 \sin(\sigma_3 - \sigma_2)}{2 \sin(\sigma_1 - \sigma_3) \sin(\sigma_2 - \sigma_1)} \\ &\quad \times [1 + \cos(2\theta) \cos(2\sigma_1 - 2\sigma)], \\ q^2 &= \frac{R^2 \sin(\sigma_1 - \sigma_3)}{2 \sin(\sigma_2 - \sigma_1) \sin(\sigma_3 - \sigma_2)} \\ &\quad \times [1 + \cos(2\theta) \cos(2\sigma_2 - 2\sigma)], \\ r^2 &= \frac{R^2 \sin(\sigma_2 - \sigma_1)}{2 \sin(\sigma_3 - \sigma_2) \sin(\sigma_1 - \sigma_3)} \\ &\quad \times [1 + \cos(2\theta) \cos(2\sigma_3 - 2\sigma)]. \end{aligned} \quad (23)$$

The binary collisions are defined in our coordinates for constant values of the coordinates σ and θ . The value of the θ angle at collision implies that one of the principal moments of inertia becomes zero. These collisions are defined by the equations

$$\cos(2\theta) \cos(2\sigma_i - 2\sigma) = -1$$

when $\theta = \pi/2 \pmod{\pi}$ and $\sigma = \sigma_i \pmod{\pi}$, σ_i being one of the previous defined constants, or when $\theta = 0 \pmod{\pi}$ and $\sigma = \sigma_i + \pi/2 \pmod{\pi}$. We call these σ_i angles the collision angles.

We compute also the kinetic energy as a function of the new coordinates, it becomes

$$K = \frac{\mu}{2} \left[\dot{R}^2 + R^2(\dot{\theta}^2 + \dot{\sigma}^2 - 2 \sin(2\theta)\dot{\sigma}\dot{\psi} + \dot{\psi}^2) \right]. \quad (24)$$

3. Equations of motion

The equations of motion follow from the Lagrange equations derived from the Lagrangian $K - V$ as presented in any standard text of Mechanics [20].

We note ψ is a cyclic variable that is not present in the kinetic energy nor in the potential energy. Therefore

$$P_\psi = \frac{\partial K}{\partial \dot{\psi}} = \mu R^2 (\dot{\psi} - \dot{\sigma} \sin(2\theta)) \quad (25)$$

is a constant parameter.

The equations of motion for the other three coordinates are

$$\begin{aligned} \frac{d}{dt} \mu \dot{R} - \mu R [\dot{\theta}^2 + \dot{\psi}^2 + \dot{\sigma}^2 - 2\dot{\sigma}\dot{\psi} \sin(2\theta)] \\ + \frac{V^*(\sigma, \theta)}{R^2} = 0, \end{aligned} \quad (26)$$

$$\frac{d}{dt} \mu R^2 \dot{\theta} + 2\mu R^2 \dot{\sigma}\dot{\psi} \cos(2\theta) - \frac{1}{R} \frac{\partial V^*}{\partial \theta} = 0, \quad (27)$$

$$\frac{d}{dt} [\mu R^2 \dot{\sigma} - \mu R^2 \dot{\psi} \sin(2\theta)] - \frac{1}{R} \frac{\partial V^*}{\partial \sigma} = 0, \quad (28)$$

where $V^*(\sigma, \theta)$ is the function defined by $V = -V^*/R$.

The equations of motion have other constant of motion, the total energy namely:

$$\begin{aligned} E = K + V = \frac{\mu}{2} \\ \times \left[\dot{R}^2 + R^2(\dot{\theta}^2 + \dot{\sigma}^2 - 2 \sin(2\theta)\dot{\sigma}\dot{\psi} + \dot{\psi}^2) \right] - \frac{V^*}{R}. \end{aligned} \quad (29)$$

4. The Euler and Lagrange cases of motion of the three-body problem

It is convenient to combine the two well known constants of motion of the Three-Body Problem in the following form, by substitution of the angular momentum P_ψ in the energy constant to obtain

$$\begin{aligned} E = \frac{\mu}{2} \left[\dot{R}^2 + R^2 (\dot{\theta}^2 + \dot{\sigma}^2 \cos^2(2\theta)) \right] \\ + \frac{P_\psi^2}{2\mu R^2} - \frac{V^*(\sigma, \theta)}{R}. \end{aligned} \quad (30)$$

The Euler and Lagrange cases are obtained when the coordinates σ and θ are constants of motion, then the previous two equations (25) and (30) become identical to the equations of motion for the elliptic two body relative motion in terms of the radius R and the true anomaly ψ , namely

$$P_\psi = \mu R^2 \dot{\psi}, \quad (31)$$

and

$$E = \frac{\mu}{2} \dot{R}^2 + \frac{P_\psi^2}{2\mu R^2} - \frac{V^*(\sigma, \theta)}{R}. \quad (32)$$

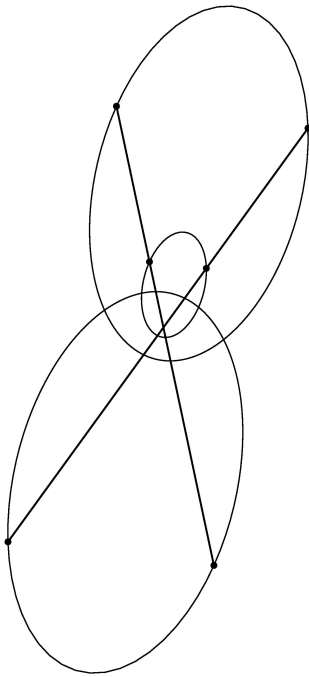


FIGURE 1. Euler case. Typical elliptic trajectories and two simultaneous positions on a straight line. The center of mass is at the intersection of the two straight lines.

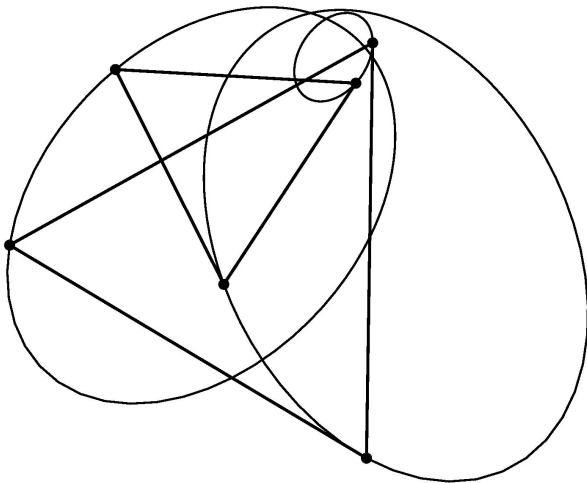


FIGURE 2. Lagrange case. Typical elliptic trajectories and two simultaneous positions on an equilateral triangle. The center of mass is at the common focus.

In particular the orbit for these coordinates is the ellipse

$$\frac{P_\psi}{R} = \frac{\mu V^*}{P_\psi} + \sqrt{2E\mu + \left(\frac{\mu V^*}{P_\psi}\right)^2} \cos(\psi - \psi_0), \quad (33)$$

where ψ_0 is a constant of integration.

The equations of motion for the two coordinates σ and θ give for the Lagrange case the constant values

$$\sigma = 0, \quad \cos(2\theta) = -\frac{\sqrt{\alpha^2 - 3}}{\alpha}. \quad (34)$$

For the Euler case the coordinate θ is chosen equal to 0 or $\pi/2$ and then σ is a constant that results from the equation

$$\begin{pmatrix} q^3 r^3 & p^3 r^3 & q^3 p^3 \end{pmatrix} C \begin{pmatrix} p^2 \\ q^2 \\ r^2 \end{pmatrix} = 0, \quad (35)$$

where C is the matrix

$$C = \begin{pmatrix} m_2 - m_3 & -(m_2 + m_3) & m_2 + m_3 \\ m_3 + m_1 & m_3 - m_1 & -(m_3 + m_1) \\ -(m_1 + m_2) & m_1 + m_2 & m_1 - m_2 \end{pmatrix}. \quad (36)$$

The equation (35), obtained for the Euler case, is interesting not only because it is one for the three Euler cases, but because it gives also the θ and σ coordinates for the Lagrange cases, for the collision angles, and for fulfill the condition for equal moments of inertia.

5. Saturn-Janus-Epimetheus dynamics

Our study of the Saturn-Janus-Epimetheus dynamics rest on numerical integrations of the equations of motion in our coordinates. The first integration produced an orbit that was very close to a circular orbit in the polar R and ψ coordinates with major perturbations in the neighborhood of the encounter which occurs every four years. The following analysis was developed in parallel to complete the numerical output.

Based on numerical integration of the Three-Body Problem in these coordinates we [5] look the Janus Epimetheus motion around Saturn as an adiabatic motion perturbation of the Euler and the Lagrange cases. The fast motion is an elliptic motion for the R and ψ coordinates as it is in the Euler and Lagrange cases. The slow motion occurs for the coordinates θ and σ , that were constants in the Euler and Lagrange cases and now are assumed to vary slowly.

This hypothesis is related to the observed two time scales in the actual motion of these two Saturn's moons. One is a fast revolution of the moons around Saturn with almost the same period of 2/3 of a day. The second is the slow relative approach of the two moons every 4 years, producing an interchange of relative positions and velocities.

The two particles have almost the same radius of 151450 km with a difference of ± 50 km. We write

$$\frac{q}{r} = 1 + \delta, \quad (37)$$

with $\delta = \pm 50/151450 \doteq \pm 1/3000$. These two values for δ remain after an encounter that happens each 4 years, when δ changes of sign between these two values. Forgetting the encounter for a while, most of the time the two particles move in the combination of the fast elliptic motion of the coordinates R and ψ , and the slow motion of the other two coordinates that we study now.

Since the masses of Janus and Epimetheus are smaller than the mass of Saturn by a factor of 10^{-9} , we have some

simplifications (for example $\sigma_3 \doteq \sigma_2 + \pi/2$), hence the ratio of the distances q and r is given by

$$\frac{q}{r} = \sqrt{\frac{m_2}{m_3}} \sqrt{\frac{1 - \cos(2\sigma_3 - 2\sigma) \cos(2\theta)}{1 + \cos(2\sigma_3 - 2\sigma) \cos(2\theta)}}. \quad (38)$$

From these two equations it follows to first order in δ

$$\cos(2\sigma_3 - 2\sigma) \cos(2\theta) = \frac{m_2 - m_3}{m_2 + m_3} - 4\delta \frac{m_2 m_3}{(m_2 + m_3)^2}. \quad (39)$$

The slow motion occurs very near this curve in the σ/θ plane represented in Fig.3, that was drawn for $\delta = 0$. This curve crosses the constant values corresponding to binary collision, one Euler case, and two Lagrange cases. The curve matches the condition for equal distance to Saturn of the two moons. The actual motion happens on two parallel curves very near the previous one, forming a horseshoe with the two ends pointing to the binary collision point.

The slow motion is an adiabatic motion [21] of the two coordinates θ and σ . We assume the adiabatic hypothesis which implies the conservation of the action J of the elliptic motion, that is equal to [22]

$$J = \frac{\pi V^*}{\sqrt{-\frac{E}{2\mu}}}. \quad (40)$$

We repeat the computation of the invariance of this action as was made by Becker [23] for the pendulum with a slow motion of the length; but here for the case of the elliptic motion. The differential of this action, assuming a slow motion of the parameters V^* and E is

$$dJ = \frac{\pi}{\sqrt{-\frac{E}{2\mu}}} \left(dV^* - \frac{V^*}{2E} dE \right). \quad (41)$$

The Hamiltonian of the elliptic movement is written as

$$\mathcal{H} = K - \frac{V^*}{R}. \quad (42)$$

We change V^* by the action of a force F

$$F = -\frac{\partial \mathcal{H}}{\partial V^*} = \frac{1}{R} = -\frac{V}{V^*}. \quad (43)$$

The last expression was written to take the time average force in one period of the elliptic motion. According to the Virial theorem the time average in a period of the potential energy is equal to $2E$,

$$\langle F \rangle = -\frac{1}{V^*} \langle V \rangle = -\frac{2E}{V^*}. \quad (44)$$

The adiabatic work to change dV^* is

$$dE = -\langle F \rangle dV^* = \frac{2E}{V^*} dV^*. \quad (45)$$

Therefore substitution into (41) results in the action J remaining invariant during this adiabatic change.

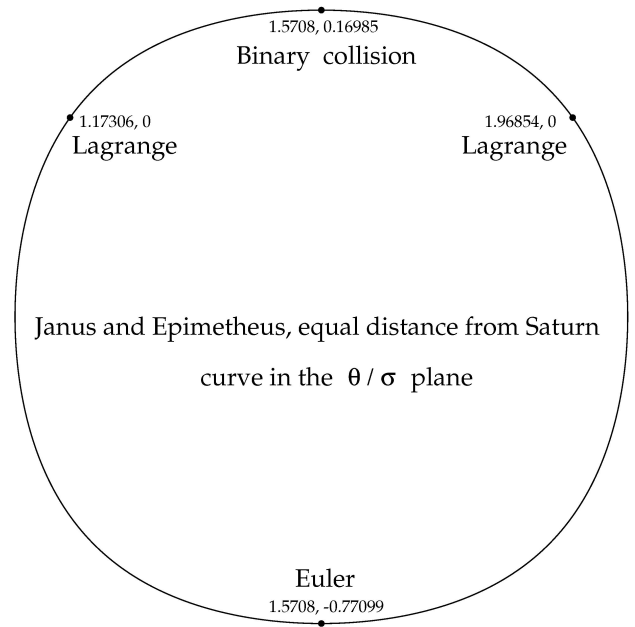


FIGURE 3. Central curve for the motion of the σ and θ coordinates of Janus and Epimetheus slow dynamics.

The adiabatic hypothesis demands the invariance of the V^* function in the slow motion of the coordinates θ and σ . We turn now to study this function of those coordinates. We begin by writing it in terms of the distances p, q, r

$$V^* = -VR = Gm_1 m_2 m_3 \left(\frac{1}{m_1 p} + \frac{1}{m_2 q} + \frac{1}{m_3 r} \right) \times \sqrt{\mu} \sqrt{\frac{p^2}{m_1} + \frac{q^2}{m_2} + \frac{r^2}{m_3}} \quad (46)$$

and we concentrate on the two factors

$$\left(\frac{1}{m_1 p} + \frac{1}{m_2 q} + \frac{1}{m_3 r} \right) \sqrt{\frac{p^2}{m_1} + \frac{q^2}{m_2} + \frac{r^2}{m_3}} \quad (47)$$

on both of them the terms divided by the mass m_1 are smaller by a factor of 10^{-9} compared with the other terms. Neglecting the smaller terms for a while we study the larger ones

$$\left(\frac{1}{m_2 q} + \frac{1}{m_3 r} \right) \sqrt{\frac{q^2}{m_2} + \frac{r^2}{m_3}}, \quad (48)$$

that is a function of $z = q/r$ namely

$$f(z) = \left(\frac{1}{m_2 z} + \frac{1}{m_3} \right) \sqrt{\frac{z^2}{m_2} + \frac{1}{m_3}}. \quad (49)$$

This function has a minimum in $z = 1$ since

$$2f(z)f'(z) = \frac{2}{m_2 m_3 z^2} \left(\frac{1}{m_2 z} + \frac{1}{m_3} \right) (z^3 - 1). \quad (50)$$

According to (37), the system moves near this minimum $z = 1 + \delta$, hence we approximate the function $f(z)$ by a Taylor expansion that to second order in δ gives

$$f(z) = \left(\frac{1}{m_2} + \frac{1}{m_3}\right)^{3/2} + \frac{3}{2\sqrt{m_2 m_3} (m_2 + m_3)^{1/2}} \delta^2. \quad (51)$$

But now we see that because δ is of the order of $1/3000$, the corrections to the first term are of the order of the neglected terms. These terms are now calculated assuming that $q = r$ since the corrections are of higher order. We consider then the factors

$$\left[1 + \frac{1}{\frac{m_1 p}{m_2 q} + \frac{m_1 p}{m_3 r}}\right] \sqrt{1 + \frac{\frac{p^2}{m_1}}{\frac{q^2}{m_2} + \frac{r^2}{m_3}}} \quad (52)$$

that are approximated by

$$1 + \frac{1}{\frac{m_1}{m_2} + \frac{m_1}{m_3}} \left(\frac{1}{w} + \frac{w^2}{2}\right), \quad (53)$$

where $w = p/q$. The function in parenthesis has a minimum at $w = 1$, ($p = q$) where it has the value 1.5. For Janus and Epimetheus w has the maximum value at $w = 2$ when the function is equal to 2.5. The function becomes important at the smaller values of w near the value $1/12$ when the function become larger than 12 at the encounter. To low order in small quantities the function V^* becomes

$$V^* = Gm_1 m_2 m_3 \sqrt{\mu} \left(\frac{1}{m_2} + \frac{1}{m_3}\right)^{1/2} \left[\frac{1}{m_2} + \frac{1}{m_3} + \frac{3}{2(m_2 + m_3)} \delta^2 + \frac{1}{m_1} \left(\frac{1}{w} + \frac{w^2}{2}\right)\right]. \quad (54)$$

We confirm the hypothesis that V^* is almost a constant by verifying the adiabatic hypothesis that makes constant the angular part of the potential energy. According to (38) if we assume V^* is a function of q/r , then it is a functional of the function (39) that is almost constant during the motion. The correction term in δ^2 is very small. The w -dependent term is the smallest except at the encounter when it grows to his highest value and simultaneously the δ goes to zero.

The leading time behavior of the σ and θ coordinates is predicted satisfactorily as follows. We write the equation (30) of energy conservation in the form

$$\begin{aligned} \left(\frac{\dot{R}}{R}\right)^2 + \dot{\theta}^2 + \dot{\sigma}^2 \cos^2(2\theta) \\ = \frac{1}{\mu^2 R^2} \left[2\mu E + \frac{P_\psi}{R} \left(\frac{2\mu V^*}{P_\psi} - \frac{P_\psi}{R}\right)\right] \end{aligned} \quad (55)$$

We take the average on the right hand side over the fast variable R making the substitution of the average R by

$$R \longrightarrow -\frac{V^*}{2E} \quad (56)$$

to obtain

$$\begin{aligned} \overline{\left(\frac{\dot{R}}{R}\right)^2} + \overline{\dot{\theta}^2} + \overline{\dot{\sigma}^2 \cos^2(2\theta)} \\ = -\frac{8E^3}{\mu V^{*2}} \left(1 + \frac{2EP_\psi^2}{\mu V^{*2}}\right), \end{aligned} \quad (57)$$

where the over line denotes the above substitution

$$\overline{X(R)} = X[-V^*/(2E)]. \quad (58)$$

We introduce the hypothesis, confirmed by numerical integration that the slow variables satisfy the equation

$$\dot{\theta}^2 + \dot{\sigma}^2 \cos^2(2\theta) = \lambda^2 \text{ (constant)} \quad (59)$$

where λ^2 is a constant of the order of the quantity on the right hand side of (57). Assuming the equations (39) and (59) hold for the variables σ and θ , they lead to the time dependence of these slow variables

$$\begin{aligned} \sin(2\theta) &= \pm \sqrt{1 - k^2} \sin(\omega' t), \\ \tan(2\sigma_3 - 2\sigma) &= \frac{\sqrt{1 - k^2}}{k} \cos(\omega' t), \end{aligned} \quad (60)$$

where ω' is

$$\omega' = 2\lambda/\sqrt{1 - k^2}, \quad (61)$$

and k is the constant on the right hand side of (39)

$$\begin{aligned} k = \cos(2\sigma_3 - 2\sigma) \cos(2\theta) &= \frac{m_2 - m_3}{m_2 + m_3} \\ &- 4\delta \frac{m_2 m_3}{(m_2 + m_3)^2}. \end{aligned} \quad (62)$$

Numerical integration agrees up to a maximum of 2% of the predicted values (60) of the θ and σ coordinates if one tunes the ω' frequency to the observed value near 4 years, that we remark it is half the value of the complete journey. The amplitudes of these variables fit very well the numerical integration values. Small discrepancies arises at the encounter.

Comparing experimental versus numerical evidence give a value for λ^2 near of the right hand side of (57)

$$\lambda^2 = -\frac{8E^3}{\mu V^{*2}} \left(1 + \frac{2EP_\psi^2}{\mu V^{*2}}\right). \quad (63)$$

The sign of the time dependent quantities in (60) has been fixed from comparison with the numerical integration. The function of σ is the same for any time in the 8 years period. Both signs are included in the $\sin 2\theta$ function because a change of sign occurs in the neighborhood of the encounter at the closest approach of Janus and Epimetheus each four years.

Upon comparing the frequency ω' predicted by (61) and (63) with the frequency ω of the elliptic motion

$$\omega = \sqrt{\frac{-8E^3}{\mu V^{*2}}}, \tag{64}$$

we obtain

$$\left(\frac{\omega'}{\omega}\right)^2 = \frac{4e^2}{1-k^2}, \tag{65}$$

which has been written in terms of the eccentricity of the elliptic motion that obeys the equation

$$1 - e^2 = \frac{-2EP_\psi^2}{\mu V^{*2}}. \tag{66}$$

For the eccentricity we have the expression

$$e = \frac{\omega'}{2\omega} \sqrt{(1-k^2)}. \tag{67}$$

that predicts a value of the eccentricity of the ellipse near $2 \cdot 10^{-4}$, that is one order of magnitude smaller than the computed best values obtained by Spitale *et al.* [5] for the eccentricities of Janus and Epimetheus. The Spitale *et al.* computation takes in account the influence of the Sun, Jupiter and many of the moons of Saturn, and deviates essentially from the point of view in this paper.

6. Perturbing the circular solution

Most of the results presented in the previous sections were suggested and confirmed on the basis of a first numerical integration of the equations of motion. However the elliptic motion associated to the fast variables R and ψ in this first integration was nearly circular with an eccentricity of 10^{-8} .

Some of the previous results are true only for this particular numerical integration. Nevertheless it is easy to find the changes one must to attain with a similar solution but with an eccentricity small, of the order of 10^{-4} .

Starting from the equations of motion (27) and (28) we substitute the ψ dependence by using the conservation of angular momentum and then multiply the first by $R^2\dot{\theta}$ and the second by $R^2\dot{\sigma}$, and add the two resulting equations to obtain

$$\frac{d}{dt} \frac{\mu}{2} R^4 [\dot{\theta}^2 + \dot{\sigma}^2 \cos^2(2\theta)] = R \frac{d}{dt} V^*. \tag{68}$$

This equation shows that assuming $V^* = \text{constant}$, to obtain a constant value of $\dot{\theta}^2 + \dot{\sigma}^2 \cos^2(2\theta)$, one should have also $R = \text{constant}$. Suppressing this last restriction but conserving $V^* = \text{constant}$, which is confirmed by a numerical integration with initial conditions driven to an eccentricity non zero, we are led to the result

$$\dot{\theta}^2 + \dot{\sigma}^2 \cos^2(2\theta) = \frac{\Lambda^2 P_\psi^2}{\mu^2 R^4}, \tag{69}$$

where Λ is a constant of integration.

Now we introduce a new independent variable τ defined by

$$dt = \frac{\mu}{P_\psi} R^2 d\tau. \tag{70}$$

Variable τ could be identified with the angle ψ if the quantity $\dot{\sigma}$ could be equal to zero.

We obtain the equation

$$\left(\frac{d\theta}{d\tau}\right)^2 + \left(\frac{d\sigma}{d\tau}\right)^2 \cos^2(2\theta) = \Lambda^2, \tag{71}$$

that is similar to the equation (59). Assuming again that expression (62) remain constant we deduce the couple of equations that are similar to the previous (60)

$$\sin(2\theta) = \pm \sqrt{1-k^2} \sin(\Omega\tau),$$

$$\tan(2\sigma_3 - 2\sigma) = \frac{\sqrt{1-k^2}}{k} \cos(\Omega\tau), \tag{72}$$

with Ω defined by

$$\Omega = 2 \frac{\Lambda}{\sqrt{1-k^2}} \tag{73}$$

That is the analogous to (61).

Next we obtain the coordinate R as a function of τ . To this end we write the equation of energy conservation in terms of the new independent variable, and using as is usual in the Kepler problem the change of variable

$$u = \frac{1}{R}, \tag{74}$$

we have

$$\begin{aligned} \left(\frac{du}{d\tau}\right)^2 = & - \left[1 + \left(\frac{d\theta}{d\tau}\right)^2 + \left(\frac{d\sigma}{d\tau}\right)^2 \cos^2(2\theta) \right] u^2 \\ & + \frac{2\mu V^*}{P_\psi^2} u + \frac{2\mu E}{P_\psi^2}. \end{aligned} \tag{75}$$

Making the substitution of equation (71) in the square bracket we obtain

$$\left(\frac{du}{d\tau}\right)^2 = - [1 + \Lambda^2] u^2 + \frac{2\mu V^*}{P_\psi^2} u + \frac{2\mu E}{P_\psi^2}. \tag{76}$$

If Λ were zero, the previous equation could be the differential equation for the orbit of the Kepler motion in terms of the true anomaly τ . Solution to this equation is similar, although the orbit now is a precessing ellipse in the polar coordinates R and τ . It is also an ellipse if the polar angle is defined as $\sqrt{1 + \Lambda^2}\tau$ as is shown next:

$$\begin{aligned} \frac{1}{R} = u = & \frac{\mu V^*}{P_\psi^2(1 + \Lambda^2)} + \sqrt{\frac{2\mu E}{P_\psi^2(1 + \Lambda^2)} + \frac{\mu^2 V^{*2}}{P_\psi^4(1 + \Lambda^2)^2}} \\ & \times \cos\left(\sqrt{1 + \Lambda^2}(\tau - \tau_0)\right) \end{aligned} \tag{77}$$

here τ_0 is an integration constant. Comparing with (33) we note the constants are modified by the transformation

$$P_\psi \longrightarrow P_\psi \sqrt{1 + \Lambda^2}. \quad (78)$$

This ellipse has been compared with a numerical integration having the qualitative expected behavior for the Janus and Epimetheus dynamics. The initial conditions were selected near the Euler collinear configuration. The numerical data fit very well these equations, in particular the mean radius and eccentricity of the ellipse are very close to both the numerical integration and to the theoretical value predicted by equation (77); the mean radius is as usual

$$a = -\frac{V^*}{2E}, \quad (79)$$

whereas the eccentricity is modified by the transformation (78)

$$e = \sqrt{1 + \frac{2EP_\psi^2(1 + \Lambda^2)}{\mu V^{*2}}}. \quad (80)$$

The cyclic variable ψ is now expressed in terms of the independent variable. From the conservation of angular momentum (25) we deduce the differential identity

$$d\psi = d\tau + \sin(2\theta)d\sigma. \quad (81)$$

Substitution in it of the functions θ and σ as functions of τ , according to (72), produces the derivative

$$\frac{d\psi}{d\tau} = 1 \pm \frac{k\Omega}{2} \frac{(1 - k^2) \sin^2(\Omega\tau)}{1 - (1 - k^2) \sin^2(\Omega\tau)} \quad (82)$$

this is trivially integrated to

$$\psi = \tau \mp \frac{k\Omega}{2} \tau \pm \frac{1}{2} \tan^{-1}(k \tan(\Omega\tau)). \quad (83)$$

In this form we have written the four coordinates in terms of the new variable τ .

The quantities ψ and τ are near the same. This is confirmed numerically by plotting and computing the previous equation. The numerical integration agrees very well with the predicted equations of this section.

This solution is based on the assumptions that the potential energy is constant and the slow variables move on the curve (39). Both conditions are nearly satisfied by the numerical solution and confirmed by the theoretical and experimental analysis.

7. Conclusions

Our coordinates seem suitable for formulate important cases of motion of the Three-Body Problem.

In the square determined by the coordinates σ and θ , the binary collisions are determined by a finite number of points of this square with coordinates determined only by the masses of the three particles. The Lagrange and Euler cases of motion are also determined as functions of the three masses by a finite number of points in this square.

The elliptic motion of Janus and Epimetheus is represented in this paper by a corresponding elliptic motion of the coordinates R and ψ corresponding to a small adiabatic perturbation of the Euler and Lagrange cases of motion, where now the coordinates σ and θ move slowly on a curve in the σ/θ square. This last motion is periodic in these two coordinates with the frequency of the encounter, and the time behavior of the slow coordinates is predicted satisfactorily.

To compute this behavior of the slow coordinates we follow the data of a numerical integration that allow to determine two approximate constants of motion for those coordinates. The assumption that in some equations R should be replaced by the constant value $-V^*/(2E)$ is consistent with the neglect of the term $(\dot{R}/R)^2$ and with a first numerical integration.

In this paper we find that the angular part of the potential energy represented by the function V^* is almost a constant. That the minimum of this function is the Lagrange case with three equal distances between particles. That the correction terms allowed by the experiment are of the order 10^{-8} . Based on this reality we deduce the behavior of the four coordinates: the R and ψ move in a precessing ellipse with an average radius determined as usual by the energy and the angular part of the potential energy. For the eccentricity we discover a perturbation of the classical terms by the slow terms. The slow dynamics has been described in these conclusions with small corrections that take into account the eccentricity up the order 10^{-3} .

The difficult point in this study is the behavior at the encounter. We fill this gap only by a numerical integration as was made by other authors [6-16]. We have noted however the change of sign of the θ derivative at the encounter and the opposite behavior of the correction terms in the angular part of the potential energy at the encounter: one term becoming important when the other term vanishes. The use of the Hill model as was developed by other authors requires also of a numerical integration and no similar approach was undertaken in this paper.

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