The initial value problem method for time-dependent harmonic oscillator

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The initial value problem method is formulated to calculate the propagator for time- dependent harmonic oscillators. The method is based on finding the initial position operator from Heisenberg equations. The investigated models in this paper are the damped harmonic oscillator, the harmonic oscillator with strongly pulsating mass, and the harmonic oscillator with mass growing with time. The comparison of the initial value problem method with Feynman path integral and Schwinger method is also described.

Keywords: The initial value problem method; propagator; time-dependent harmonic oscillators.

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1. Introduction

The propagators have application in many areas of physics such as quantum statistical mechanics, condensed matter physics, polymer physics, and economics [1]. In non-relativistic quantum mechanics, the propagator describe the transition probability amplitude for a particle to travel from initial space-time configuration to final space-time configuration. The most popular methods to calculate the propagator are the Feynman path integral [2] and the Schwinger method [3-6]. However, both methods have some mathematical difficulties. The aim of this paper is to present the simple method called the initial value problem method to calculate the non-relativistic propagator.

The initial value problem method begins with the assumption that the propagator for the quadratic potentials can be written as [2]

$$k(x,t;x_0,0) = \phi(t)\exp\left[\frac{i}{\hbar}A(x,t;x_0,0)\right], \tag{1}$$

where the pre-exponential factor $\phi(t)$ is the pure function of time and $A(x,t;x_0,0)$ is the two-point characteristic function.

The main idea of the initial value problem method consists in the following steps.

(1) The first step is solving the Heisenberg equations for $\hat{x}(t)$ and $\hat{p}(t)$,

$$i\hbar\frac{d\hat{x}(t)}{dt} = [\hat{x}(t),\hat{H}], \quad i\hbar\frac{d\hat{p}(t)}{dt} = [\hat{p}(t),\hat{H}], \tag{2}$$

and writing the solution for $\hat{x}(0)$ only in terms of the operators $\hat{x}(t)$ and $\hat{p}(t)$.

(2) Next, we substitute the propagator in Eq. (1) into an eigenvalue equation of

$$\hat{x}(0)k(x,t;x_0,0) = x_0 k(x,t;x_0,0). \tag{3}$$

(3) Solving the differential equation for $A(x,t;x_0,0)$, we obtain the two-point characteristic function.

(4) The final step is finding $\phi(t)$ by substituting the obtained propagator form step (3) into the Schrödinger equation

$$i\hbar\frac{\partial k(x,t;x_0,0)}{\partial t} = \hat{H}(t)k(x,t;x_0,0). \tag{4}$$

The problem that use to demonstrate the application of the initial value problem method is the time-dependent harmonic oscillator described by the Hamiltonian [7]

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2}m(t)\omega^2 x^2, \tag{5}$$

where $m(t)$ is the time-dependent mass. The time-dependent mass $m(t)$ of this paper can be divided to the three system.

The first system is the damped harmonic oscillator or the Caldirola-Kanai oscillator, [7-9] which the time-dependent mass can be written as

$$m(t) = me^{rt}, \tag{6}$$

where r is the damping constant coefficient.

The second system is the harmonic oscillator with strongly pulsating mass, [7,10] which the time-dependent mass can be described by

$$m(t) = m\cos^2 vt, \tag{7}$$

where v is the frequency of mass.

The third system is the harmonic oscillator with mass growing with time, [4,11] which the time-dependent mass has the law as

$$m(t) = m(1+\alpha t)^2, \tag{8}$$

where α is a constant parameter.

In Sec. 2, the propagator for a damped harmonic oscillator is derived. The calculation of the propagator for a harmonic oscillator with strongly pulsating mass is shown in Sec. 3. In Sec. 4, the evaluation of the propagator for a harmonic oscillator with mass growing with time is illustrated. Finally, the conclusion and discussion are described in Sec. 5.

2. The initial value problem method for a damped harminic oscillator

This section is the calculation of the propagator for a damped harmonic oscillator or the Caldirola-Kanai oscillator [7-9] described by the Hamiltonian operator

$$\hat{H}(t) = e^{-rt}\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 e^{rt}\hat{x}^2. \tag{9}$$

By applying the Heisenberg equation in Eq. (2) to the Hamiltonian operator in Eq. (9), the position operator $\hat{x}(t)$ can be written as

$$\hat{x}(t) = e^{-rt}\cos\Omega t\hat{x}(0) + \frac{re^{-rt}}{2\Omega}\sin\Omega t\hat{x}(0) + \frac{e^{-rt}}{m\Omega}\sin\Omega t\hat{p}(0), \tag{10}$$

where $\hat{x}(0)$ and $\hat{p}(0)$ are the position and momentum operators respectively at $t = 0$, and $\Omega^2 = \omega^2 - (r^2/4)$

The momentum operator $\hat{p}(0) = me^r t\dot{\hat{x}}(t)$ can be written by using Eq. (10) as

$$\hat{p}(t) = -m\Omega e^{\frac{rt}{2}}\sin\Omega t\hat{x}(0) - \frac{mr^2}{4\Omega}e^{\frac{rt}{2}}\sin\Omega t\hat{x}(0) + e^{\frac{rt}{2}}\cos\Omega t\hat{p}(0) - \frac{r}{2\Omega}e^{\frac{rt}{2}}\sin\Omega t\hat{p}(0). \tag{11}$$

The next step is expressing $\hat{x}(0)$ only in terms of $\hat{x}(t)$ and $\hat{p}(t)$ by eliminating $\hat{p}(0)$ from Eq. (11) with the using of Eq. (10) to obtain

$$\hat{x}(t) = e^{\frac{rt}{2}}\cos\Omega t\hat{x}(t) - \frac{re^{\frac{rt}{2}}}{2\Omega}\sin\Omega t\hat{x}(t) - \frac{\sin\Omega t}{m\Omega}e^{-\frac{rt}{2}}\hat{p}(t). \tag{12}$$

The eigenvalue equation for the propagator in Eq. (3) can be written in coordinate representation as

$$\left(xe^{\frac{rt}{2}}\cos\Omega t - x\frac{r}{2\Omega}e^{\frac{rt}{2}}\sin\Omega t + \frac{i\hbar\sin\Omega te^{-\frac{rt}{2}}}{m\Omega}\times\frac{\partial}{\partial x}\right)K(x,t;x_0,0) = x_0K(x,t;x_0,0). \tag{13}$$

Substituting propagator assumed in Eq. (1) into Eq. (13), the result is

$$\left(e^{\frac{rt}{2}}\cos\Omega t - \frac{r}{2\Omega}e^{\frac{rt}{2}}\sin\Omega t\right)x - \frac{\sin\Omega te^{-\frac{rt}{2}}}{m\Omega}\frac{\partial A(x,t;x_0,0)}{\partial x} = x_0. \tag{14}$$

Solving Eq. (14) to find $A(x,t;x_0,0)$, the result is

$$A(x,t;x_0,0) = \frac{1}{2}m\Omega\cot\Omega te^{rt}x^2 - \frac{1}{4}mre^{rt}x^2 - m\Omega\csc\Omega te^{\frac{rt}{2}}xx_0. \tag{15}$$

So, the propagator can be written as

$$K(x,t;x_0,0) = \phi(t)\exp\left[\frac{i}{2\hbar}\left(m\Omega\cot\Omega te^{rt}x^2 - \frac{1}{2}mre^{rt}x^2 - 2m\Omega\csc\Omega te^{\frac{rt}{2}}xx_0\right)\right]. \tag{16}$$

The next step is substituting the propagator in Eq. (16) into the Schrodinger equation for a damped harmonic oscillator

$$i\hbar\frac{\partial K(x,t;x_0,0)}{\partial t} = -e^{-rt}\frac{\hbar^2}{2m}\frac{\partial^2 K(x,t;x_0,0)}{\partial x^2} + \frac{1}{2}m\omega^2x^2e^{rt}k(x,t;x_0,0), \tag{17}$$

to get the differential equation for $\phi(t)$ as

$$i\hbar\left(\frac{\partial\phi(t)}{\partial t}\right) = \frac{1}{2}m\Omega^2\csc^2\Omega tx_0^2\phi(t) - \frac{i\hbar\Omega}{2}\cot\Omega t\phi(t) + \frac{i\hbar r}{4}\phi(t). \tag{18}$$

The next step is solving Eq. (18) to obtain the pre-exponential function $\phi(t)$ as

$$\phi(t) = \frac{Ce^{\frac{rt}{4}}}{\sqrt{\sin\Omega t}}\exp\left(\frac{im\Omega}{2\hbar}\cot t\Omega tx_0^2\right), \tag{19}$$

where C is a constant. Substituting Eq. (19) into Eq. (16), the propagator becomes

$$K(x,t;x_0,0) = C\left(\frac{e^{\frac{rt}{2}}}{\sin\Omega t}\right)^{\frac{1}{2}}\exp\left\{\frac{im\Omega}{2\hbar}\times\left[(e^{rt}x^2 + x_0^2)\cot\Omega t - \frac{2xx_0}{\sin\Omega t}e^{\frac{rt}{2}}\right]\right\}\times\exp\left\{-\frac{imr}{4\hbar}e^{rt}x^2\right\}. \tag{20}$$

After applying the initial condition

$$\lim_{t\to 0^+}K(x,t;x_0,0) = \delta(x-x_0),$$

the constant C is

$$C = \sqrt{\frac{m\Omega}{2\pi i\hbar}}. \tag{21}$$

So, the propagator for a damped harmonic oscillator can be written as

$$K(x,t;x_0,0) = \left(\frac{m\Omega e^{\frac{rt}{2}}}{2\pi i\hbar\sin\Omega t}\right)^{\frac{1}{2}}$$
$$\times \exp\left\{\frac{im\Omega}{2\hbar\sin\Omega t}\Big[(e^{rt}x^2+x_0^2)\cos\Omega t - 2xx_0e^{\frac{rt}{2}}\Big]\right\},$$
$$\times \exp\left\{-\frac{imr}{4\hbar}e^{rt}x^2\right\}, \qquad (22)$$

which agree with the result of S. Pepore and *et. al.* [7] calculated by Feynman path integral.

3. The initial value problem method for a harminic oscillator with strongly pulsating mass

The Hamiltonian operator for a harmonic oscillator with strongly pulsating mass can be written as [7]

$$\hat{H}(t) = \frac{\hat{p}^2}{2m\cos^2 vt} + \frac{1}{2}\cos^2 vt\omega^2\hat{x}^2. \qquad (23)$$

By solving Heisenberg equation, the position operator $\hat{x}(t)$ becomes

$$\hat{x}(t) = \sec vt\cos\Omega t\hat{x}(0) + \frac{\sec vt}{m\Omega}\sin\Omega t\hat{p}(0). \qquad (24)$$

The momentum operator $\hat{p}(t) = m\cos^2 vt\dot{\hat{x}}(t)$ can be expressed as

$$\hat{p}(t) = -m\Omega\cos vt\sin\Omega t\hat{x}(0) + mv\sin vt\cos\Omega t\hat{x}(0)$$
$$+\cos vt\cos\Omega t\hat{p}(0) + \frac{v}{\Omega}\sin vt\sin\Omega t\hat{p}(0). \qquad (25)$$

By eliminating $\hat{p}(0)$ from Eq. (25), the initial position operator $\hat{x}(0)$ is

$$\hat{x}(0) = \cos vt\cos\Omega t\hat{x}(t)$$
$$+\frac{v}{\Omega}\sin vt\sin\Omega t\hat{x}(t) - \frac{\sec vt}{m\Omega}\sin\Omega t\hat{p}(t). \qquad (26)$$

So, we can write the eigenvalue equation for the propagator as

$$\Big(\big(\cos vt\cos\Omega t + \frac{v}{\Omega}\sin vt\sin\Omega t\big)x$$
$$+\frac{i\hbar}{m\Omega}\sec vt\sin\Omega t\frac{\partial}{\partial x}\Big)K(x,t;x_0,0)$$
$$= x_0K(x,t;x_0,0). \qquad (27)$$

Substituting the propagator in Eq. (1) into Eq. (27), the differential equation for $A(x,t;x_0,0)$ can be written as

$$\big(\cos vt\cos\Omega t + \frac{v}{\Omega}\sin vt\sin\Omega t\big)x - \frac{\sec vt}{m\Omega}\sin\Omega t$$
$$\times\frac{\partial A(x,t;x_0,0)}{\partial x} = x_0. \qquad (28)$$

Solving Eq. (28) to find $A(x,t;x_0,0)$, it can be shown that

$$A(x,t;x_0,0) = \frac{1}{2}m\Omega\cos^2 vt\cot\Omega tx^2$$
$$+\frac{1}{2}mv\sin vt\cos vtx^2$$
$$-m\Omega\cos vt\csc\Omega txx_0. \qquad (29)$$

So, the propagator becomes

$$K(x,t;x_0,0) = \phi(t)\exp\Bigg[\frac{i}{2\hbar}\Big(\big(m\Omega\cos^2 vt\cot\Omega t$$
$$+mv\sin vt\cos vt\big)x^2$$
$$-2m\Omega\cos vt\csc\Omega txx_0\Big)\Bigg]. \qquad (30)$$

Substituting the propagator in Eq. (30) into the Schrodinger equation for a harmonic oscillator with strongly pulsating mass

$$i\hbar\frac{\partial K(x,t;x_0,0)}{\partial t} = -\sec^2 vt\frac{\hbar^2}{2m}\frac{\partial^2K(x,t;x_0,0)}{\partial x^2}$$
$$+\frac{1}{2}m\cos^2 vt\omega^2x^2K(x,t;x_0,0), \qquad (31)$$

it can be shown that

$$i\hbar\left(\frac{\partial\phi(t)}{\partial t}\right) = \frac{1}{2}m\Omega^2\csc^2\Omega tx_0^2\phi(t)$$
$$-\frac{i\hbar}{2}(\Omega\cot\Omega t + v\tan vt)\phi(t). \qquad (32)$$

After solving Eq. (32), the pre-exponential function $\phi(t)$ can be written as

$$\phi(t) = C\sqrt{\frac{\cos vt}{\sin\Omega t}}\exp\left[\frac{im\Omega}{2\hbar}\cot\Omega tx_0^2\right], \qquad (33)$$

where C is a constant. Combining Eq. (33) with Eq. (30), the propagator can be written as

$$K(x,t;x_0,0) = C\sqrt{\frac{\cos vt}{\sin\Omega t}}$$
$$\times\exp\Big[\frac{i}{2\hbar}(m\Omega\cot\Omega t(\cos^2 vtx^2 + x_0^2)$$
$$-2m\Omega\cos vt\csc\Omega txx_0)\Big]$$
$$\times\exp\Big[\frac{imv}{2\hbar}\sin vt\cos vtx^2\Big]. \qquad (34)$$

To find the constant C, we apply the initial condition of the propagator

$$\lim_{t\to 0^+} K(x,t;x_0,0) = \delta(x-x_0). \tag{35}$$

The constant C becomes

$$C = \sqrt{\frac{m\Omega}{2\pi i\hbar}}. \tag{36}$$

So, the propagator for a harmonic oscillator with strongly pulsating mass is

$$\begin{aligned} K(x,t;x_0,0) &= \sqrt{\frac{m\Omega\cos vt}{2\pi i\hbar\sin\Omega t}}\exp\Big[\frac{i}{2\hbar} \\ &\times (m\Omega\cot\Omega t(\cos^2 vtx^2 + x_0^2) \\ &- 2m\Omega\cos vt\csc\Omega txx_0)\Big] \\ &\times \exp\Big[\frac{imv}{2\hbar}\sin vt\cos vtx^2\Big], \end{aligned} \tag{37}$$

which is the same result as deriving by Feynman path integral [7].

4. The initial value problem method for a harminic oscillator with mass growing with time

The last system to investigate the initial value problem method is the harmonic oscillator with mass growing with time expressed by the Hamiltonian operator

$$\hat{H}(t) = \frac{\hat{p}^2}{2m(1+\alpha t)^2} + \frac{1}{2}m(1+\alpha t)^2\omega^2\hat{x}^2. \tag{38}$$

By solving the Heisenberg equation, the operator $\hat{x}(t)$ can be written as

$$\begin{aligned} \hat{x}(t) &= \frac{\cos\omega t}{(1+\alpha t)}\hat{x}(0) + \frac{a\sin\omega t}{\omega(1+\alpha t)}\hat{x}(0) \\ &+ \frac{\sin\omega t}{m\omega(1+\alpha t)}\hat{p}(0). \end{aligned} \tag{39}$$

The momentum operator $\hat{p}(t) = m(1+\alpha t)^2\dot{\hat{x}}(t)$ can be expressed as

$$\begin{aligned} \hat{p}(t) &= m\alpha\cos\omega t\hat{x}(0) - m\omega(1+\alpha t)\sin\omega t\hat{x}(0) \\ &+ m\alpha(1+\alpha t)\cos\omega t\hat{x}(0) - \frac{m\alpha^2}{\omega}\sin\omega t\hat{x}(0) \\ &+ (1+\alpha t)\cos\omega t\hat{p}(0) - \frac{\alpha}{\omega}\sin\omega t\hat{p}(0). \end{aligned} \tag{40}$$

Eliminating $\hat{p}(t)$ from Eq. (40) by using Eq. (39), the initial position operator is

$$\begin{aligned} \hat{x}(0) &= \Big[(1+\alpha t)\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\Big]\hat{x}(t) \\ &- \frac{\sin\omega t}{m\omega(1+\alpha t)}\hat{p}(t). \end{aligned} \tag{41}$$

The eigenvalue equation for the propagator in Eq. (3) can be shown that

$$\begin{aligned} &\left(x\Big[(1+\alpha t)\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\Big] + \frac{i\hbar\sin\omega t}{m\omega(1+\alpha t)}\frac{\partial}{\partial x}\right) \\ &\times K(x,t;x_0,0) = x_0 K(x,t;x_0,0). \end{aligned} \tag{42}$$

The next step is substituting the propagator in Eq. (1) into Eq. (42) to obtain

$$\begin{aligned} &x\Big[(1+\alpha t)\cos\omega t - \frac{\alpha}{\omega}\sin\omega t\Big] - \frac{\sin\omega t}{m\omega(1+\alpha t)} \\ &\times \frac{\partial A(x,t;x_0,0)}{\partial x} = x_0. \end{aligned} \tag{43}$$

By solving Eq. (43), the answer is

$$\begin{aligned} A(x,t;x_0,0) &= \frac{1}{2}m(1+\alpha t)[\omega(1+\alpha t)\cot\omega t - \alpha]x^2 \\ &- m\omega\csc\omega t(1+\alpha t)xx_0. \end{aligned} \tag{44}$$

So, the propagator becomes

$$\begin{aligned} K(x,t;x_0,0) &= \phi(t)\exp\Big[\frac{i}{2\hbar}(m(1+\alpha t)x^2 \\ &\times [\omega(1+\alpha t)\cot\omega t - \alpha] \\ &- 2m\omega\csc\omega t(1+\alpha t)xx_0)\Big]. \end{aligned} \tag{45}$$

The next step is substituting the propagator in Eq. (45) into the Schrodinger equation for a harmonic oscillator with mass growing with time

$$\begin{aligned} i\hbar\frac{\partial K(x,t;x_0,0)}{\partial t} &= -\frac{\hbar^2}{2m(1+\alpha t)}\frac{\partial^2 K(x,t;x_0,0)}{\partial x^2} \\ &+ \frac{1}{2}m(1+\alpha t)^2\omega^2x^2K(x,t;x_0,0), \end{aligned} \tag{46}$$

to obtain the differential equation for $\phi(t)$ as

$$\begin{aligned} i\hbar\frac{\partial\phi(t)}{\partial t} &= \frac{1}{2}m\omega^2\csc^2\omega tx_0^2\phi(t) \\ &- \frac{i\hbar}{2(1+\alpha t)}(\omega(1+\alpha t)\cot\omega t - \alpha)\phi(t). \end{aligned} \tag{47}$$

The next step is solving Eq. (47) for $\phi(t)$ as

$$\phi(t) = C\sqrt{\frac{1+\alpha t}{\sin\omega t}}\exp\left(\frac{im\omega}{2\hbar}\cot\omega tx_0^2\right), \tag{48}$$

where C is a constant. Substituting Eq. (48) into Eq. (45), the result is

$$\begin{aligned} K(x,t;x_0,0) &= C\sqrt{\frac{1+\alpha t}{\sin\omega t}}\exp\Big[\frac{i}{2\hbar}([m\omega(1+\alpha t)^2 \\ &\times \cot\omega t - m\alpha(1+\alpha t)]x^2 + m\omega\cot\omega tx_0^2 \\ &- 2m\omega\csc wt(1+\alpha t)xx_0)\Big]. \end{aligned} \tag{49}$$

The last step is finding the constant C by using

$$\lim_{t\to 0^+} K(x,t;x_0,0) = \delta(x-x_0), \tag{50}$$

to get

$$C = \sqrt{\frac{m\omega}{2\pi i\hbar}}. \tag{51}$$

Finally, the propagator for a harmonic oscillator with mass growing with time can be written as

$$K(x,t;x_0,0) = \left[\frac{m\omega(1+\alpha t)}{2\pi i\hbar\sin\omega t}\right]^{\frac{1}{2}} \exp\Big[\frac{i}{2\hbar} \times \big([m\omega(1+\alpha t)^2\cot\omega t - m\alpha(1+\alpha t)]x^2 + m\omega\cot\omega t x_0^2 - 2m\omega\csc\omega t(1+\alpha t)xx_0\big)\Big], \tag{52}$$

which agree with the calculation of S. Pepore and B. Sukbot by Schwinger method and Feynman path integral [4].

5. Conclisions

We have successfully calculated the exactly propagator for three systems of time-dependent harmonics oscillators. The method in this paper is simple. It requires only solving the Heisenberg equation for $\hat{x}(t)$. This method reduce the solving second order differential equation of Schrodinger equation to solve the first order differential equation of the pre-exponential factor $\phi(t)$.

The initial value problem method in this paper have some similarity with the Schwinger method [3-6] in solving Heisenberg equation but the Schwinger methods requires the knowledge of commutator algebra for $[\hat{x}(0), \hat{x}(t)]$. The calculation of propagator for Feynman path integral have some mathematical difficulties in deriving the classical action and in time-slicing process [2].

We can conclude here that the initial value problem method may be the new techniques to calculate the non-relativistic propagator for the quadratic potentials.

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