Representation of canonical transformations in quantum mechanics

G.F. Torres del Castillo
Departamento de Física Matemática, Instituto de Ciencias,
Universidad Autónoma de Puebla, 72570 Puebla, Pue., México.

H. Bello Martínez, R.J. Mejía Sánchez, and J.M. Zárate Paz
Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla,
Apartado Postal 1152 72001 Puebla, Pue., México.

Recibido el 24 de febrero de 2009; aceptado el 20 de marzo de 2009

The transformation of the wave functions induced by a given canonical transformation in the classical phase space, $(q^i, p_i) \to (Q^i, P_i)$, is considered. In the examples presented here, the kernel of the integral transform turns out to be essentially $\exp(\mathrm{i}\Lambda/\hbar)$, where $\Lambda(q^i, Q^i)$ is defined by $P_i \mathrm{d}Q^i = p_i \mathrm{d}q^i + \mathrm{d}\Lambda$. In the case of the time evolution, which is a canonical transformation, the kernel of the transform is the propagator, and is obtained directly by making use of the solution to the classical equations of motion.

Keywords: Wave functions; coordinate representation; canonical transformations; propagators.

Se considera la transformación de las funciones de onda inducida por una transformación canónica dada en el espacio fase clásico, $(q^i, p_i) \to (Q^i, P_i)$. En los ejemplos presentados aquí, el núcleo de la transformada integral resulta ser esencialmente $\exp(\mathrm{i}\Lambda/\hbar)$, donde $\Lambda(q^i, Q^i)$ está definida por $P_i \mathrm{d}Q^i = p_i \mathrm{d}q^i + \mathrm{d}\Lambda$. En el caso de la evolución temporal, la cual es una transformación canónica, el núcleo de la transformada es el propagador, y se obtiene directamente haciendo uso de la solución de las ecuaciones de movimiento clásicas.

Descriptores: Funciones de onda; representación de coordenadas; transformaciones canónicas; propagadores.

PACS: 03.65.Ca, 45.20.Jj

1. Introduction

In the Hamiltonian formulation of classical mechanics it is convenient to make use of the canonical transformations, which can mix the coordinates of the configuration space with their conjugate momenta. A canonical transformation preserves the form of the Hamilton equations and can simplify the form of the Hamiltonian.

For example, the transformation

$$x = u + v, \qquad y = \frac{c}{eB}(p_u - p_v),$$
$$p_x = \frac{1}{2}(p_u + p_v), \qquad p_y = \frac{eB}{2c}(v - u), \tag{1}$$

where e, B, and c are constants, is canonical [1] (see below) and when applied to the Hamiltonian of a particle of mass m and electric charge e moving on the xy-plane subjected to a magnetic field B, perpendicular to this plane,

$$H = \frac{1}{2m}\left[\left(p_x + \frac{eB}{2c}y\right)^2 + \left(p_y - \frac{eB}{2c}x\right)^2\right], \tag{2}$$

yields the Hamiltonian

$$H = \frac{{p_u}^2}{2m} + \frac{m\omega^2}{2}u^2 \tag{3}$$

if $\omega = eB/mc$, which corresponds to a one-dimensional harmonic oscillator. One can convince oneself that these expressions also make sense if the phase space coordinates are substituted by operators in the framework of quantum mechanics and, therefore, the transformation (1) can be employed to relate two quantum-mechanical systems corresponding to the Hamiltonians (2) and (3). However, if we have the wave functions for the stationary states of the second Hamiltonian, we cannot simply replace the coordinates u, v in the wave functions according to Eqs. (1), since the resulting expressions would depend on the coordinates x, y and their conjugate momenta.

The relationship between wave functions is known from the elementary formalism of quantum mechanics, namely

$$\phi(Q^1, \ldots, Q^n) = \int \langle Q^1, \ldots, Q^n | q^1, \ldots, q^n\rangle \times \psi(q^1, \ldots, q^n)\, \mathrm{d}q^1 \cdots \mathrm{d}q^n, \tag{4}$$

where $|q^1, \ldots, q^n\rangle$ is a normalized common eigenket of the *operators* $\widehat{q}^1, \ldots, \widehat{q}^n$, with eigenvalues $q^1, \ldots, q^n$, respectively, that is

$$\widehat{q}^i |q^1, \ldots, q^n\rangle = q^i |q^1, \ldots, q^n\rangle, \quad (i = 1, \ldots, n), \tag{5}$$

and

$$\langle q'^1, \ldots, q'^n | q^1, \ldots, q^n\rangle = \delta(q'^1 - q^1) \cdots \delta(q'^n - q^n), \tag{6}$$

with a similar definition for $|Q^1, \ldots, Q^n\rangle$. The eigenvalue equations (5) are equivalent to the *partial differential equations*

$$\widehat{q}^i\left(Q^j, \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial Q^j}\right)\langle Q^1, \ldots, Q^n | q^1, \ldots, q^n\rangle = q^i \langle Q^1, \ldots, Q^n | q^1, \ldots, q^n\rangle, \quad (i = 1, \ldots, n), \tag{7}$$

for the functions $\langle Q^1, \ldots, Q^n | q^1, \ldots, q^n \rangle$, assuming that

$$\langle Q^1, \ldots, Q^n | \widehat{P}_i | q^1, \ldots, q^n \rangle = \frac{\hbar}{\mathrm{i}} \frac{\partial}{\partial Q^i} \langle Q^1, \ldots, Q^n | q^1, \ldots, q^n \rangle,$$

and the solution to these equations yields the required kernel of the transform (4). (In the Hamiltonians (2) and (3), x, y, u, and v are considered as Cartesian coordinates.) In the case where the $\widehat{q}^i$ depend only on the Q^i (*e.g.*, when the q^i are Cartesian coordinates and the Q^i are spherical coordinates) Eqs. (7) are not differential equations and their solutions are proportional to products of Dirac delta functions. (See the examples below.)

As we shall show below, in some cases, the kernel of the transform (4) can be expressed in terms of the generating function of the canonical transformation $(q^i, p_i) \to (Q^i, P_i)$. We recall that a coordinate transformation

$$Q^i = Q^i(q^1, \ldots, q^n, p_1, \ldots, p_n),$$
$$P_i = P_i(q^1, \ldots, q^n, p_1, \ldots, p_n), \tag{8}$$

is canonical if and only if there exists (at least locally) a function Λ such that [2–5]

$$P_i \mathrm{d}Q^i = p_i \mathrm{d}q^i + \mathrm{d}\Lambda \tag{9}$$

(with summation over repeated indices) and we shall assume that Λ is expressed as a function of q^i and Q^i: $\Lambda = \Lambda(q^i, Q^i)$. (Note that Λ can be equal to zero as in the case of a point transformation, $Q^i = Q^i(q^1, \ldots, q^n)$, with $p_i = P_j\, \partial Q^j / \partial q^i$.)

If the set $\{q^1, \ldots, q^n, Q^1, \ldots, Q^n\}$ is independent, which amounts to the condition that the Jacobian determinant

$$\left| \frac{\partial(Q^1, \ldots, Q^n)}{\partial(p_1, \ldots, p_n)} \right| \tag{10}$$

be different from zero, then Λ has a unique expression as a function of $q^1, \ldots, q^n, Q^1, \ldots, Q^n$, and Eq. (9) is equivalent to

$$P_i = \frac{\partial \Lambda}{\partial Q^i}, \qquad p_i = -\frac{\partial \Lambda}{\partial q^i}. \tag{11}$$

On the other hand, when the determinant (10) is equal to zero, the expression of Λ as a function of $q^1, \ldots, q^n, Q^1, \ldots, Q^n$ *is not unique* and the relations (11) do not hold (nor make sense). (See the examples below.)

The coordinate transformation

$$Q^i = p_i, \qquad P_i = -q^i \tag{12}$$

is canonical. In fact, $P_i \mathrm{d}Q^i = p_i \mathrm{d}q^i + \mathrm{d}(-p_i q^i)$; that is, comparing with Eq. (9),

$$\Lambda = -q^i Q^i. \tag{13}$$

The coordinate transformation in the phase space (12) gives rise to the relation between operators $\widehat{Q}^i = \widehat{p}_i$, $\widehat{P}_i = -\widehat{q}^i$, and, as is well known, in this case,

$$\langle Q^1, \ldots, Q^n | q^1, \ldots, q^n \rangle = \langle p_1, \ldots, p_n | q^1, \ldots, q^n \rangle$$
$$= (2\pi\hbar)^{-n/2} \exp(-\mathrm{i} p_i q^i / \hbar) \tag{14}$$

[see Eq. (7)]. Substitution of Eq. (14) into Eq. (4) yields the well-known relation between the wave function in the configuration space and that in the momentum space

$$\phi(p_1, \ldots, p_n) = (2\pi\hbar)^{-n/2} \int \exp(-\mathrm{i} p_i q^i / \hbar) \times \psi(q^1, \ldots, q^n)\, \mathrm{d}q^1 \cdots \mathrm{d}q^n.$$

As pointed out in Ref. 6, taking into account Eq. (13), the function (14) can be expressed as

$$\langle Q^1, \ldots, Q^n | q^1, \ldots, q^n \rangle = N \exp(\mathrm{i}\Lambda/\hbar), \tag{15}$$

where N is a normalization factor (in this case, a normalization constant).

The aim of this paper is to show that Eq. (15) applies for many other canonical transformations, including all the linear transformations [see Eqs. (1) and (12)], even if the Jacobian (10) is equal to zero, in which case the right-hand side of Eq. (15) must include Dirac delta functions, corresponding to the relations between the variables $q^1, \ldots, q^n, Q^1, \ldots, Q^n$.

If $Q^i(t)$, $P_i(t)$ are the solutions of the (classical) equations of motion, the transformation $(q^i, p_i) \to (Q^i, P_i)$, with $q^i \equiv Q^i(0)$, $p_i \equiv P_i(0)$, is canonical and the kernel of the integral transform (4), $\langle Q^1, \ldots, Q^n | q^1, \ldots, q^n \rangle$, is the propagator (see, *e.g.*, Refs. 7 and 8). Therefore, in those cases where Eq. (15) holds, one readily obtains the corresponding propagator.

The approach followed in this paper differs from that followed in Ref. 6 in several ways; in Ref. 6 only systems with one degree of freedom are considered, looking for the relation between eigenfunctions of Hamiltonians related by a canonical transformation, instead of considering the more basic problem of relating the eigenfunctions of the coordinate operators. In fact, in our treatment, a Hamiltonian need not be specified. Furthermore, the possibility that the variables $q^1, \ldots, q^n, Q^1, \ldots, Q^n$ are not independent is not even considered in Ref. 6, assuming that Eqs. (11) are always applicable.

In Sec. 2 we give some explicit examples for which the ansatz (15) gives the right expression and in Sec. 3 we prove that in the case of any linear canonical transformation Eq. (15) gives the kernel of (4).

2. Examples

In this section we consider several examples of canonical transformations for which the kernel $\langle Q^1, \ldots, Q^n | q^1, \ldots, q^n \rangle$ can be readily obtained starting from Eq. (15).

2.1. A canonical transformation for systems with one degree of freedom

We begin by considering the simple example

$$Q = -p, \qquad P = q + ap^2, \tag{16}$$

where a is a constant. This transformation is canonical since $P\mathrm{d}Q = p\mathrm{d}q + \mathrm{d}(-pq - ap^3/3)$ [*cf.* Eq. (9)] and the variables q, Q are independent. In this example $\Lambda = qQ + aQ^3/3$ and from Eq. (15) we assume that

$$\langle Q|q\rangle = N \exp\left[\frac{\mathrm{i}}{\hbar}\left(qQ + \frac{a}{3}Q^3\right)\right]. \tag{17}$$

Equations (16) can be translated into relations between the corresponding operators (that is, $\widehat{Q} = -\widehat{p}$, $\widehat{P} = \widehat{q} + a\widehat{p}^2$) and one can readily verify that (17) is an eigenfunction of $\widehat{q}$, with eigenvalue q [see Eq. (5)], and that the complex conjugate of (17) is an eigenfunction of $\widehat{Q}$, with eigenvalue Q, if N is a constant, which can be chosen as $(2\pi\hbar)^{-1/2}$.

Note that, as pointed out above, it is not necessary to specify a Hamiltonian.

2.2. Two degrees of freedom

A second example is provided by the canonical transformation already mentioned in the Introduction, from

$$(q^1, q^2, p_1, p_2) \equiv (u, v, p_u, p_v)$$

to

$$(Q^1, Q^2, P_1, P_2) \equiv (x, y, p_x, p_y),$$

given by [1]

$$x = u + v, \qquad y = \frac{c}{eB}(p_u - p_v),$$
$$p_x = \frac{1}{2}(p_u + p_v), \qquad p_y = \frac{eB}{2c}(v - u), \tag{18}$$

where e, B, and c are constants. We have

$$p_x\mathrm{d}x + p_y\mathrm{d}y = p_u\mathrm{d}u + p_v\mathrm{d}v + \mathrm{d}\left[\tfrac{1}{2}(v-u)(p_u - p_v)\right],$$

that is,

$$\Lambda = \frac{1}{2}(v-u)(p_u - p_v) = \frac{eB}{2c}y(v-u). \tag{19}$$

From the first equation in (18) we see that the coordinates q^1, q^2, Q^1, Q^2 are not independent and therefore the function (19) can be expressed in infinitely many different ways in terms of them [*e.g.*, $\Lambda = (eB/2c)y(v-u) = (eB/2c)y(2v-x) = (eB/2c)y(x-2u)$].

Assuming that relations identical to Eq. (18) hold for the corresponding operators (in this case there are no ordering ambiguities), Eqs. (7) take the explicit forms

$$\left(\frac{1}{2}x - \frac{\hbar c}{\mathrm{i}eB}\frac{\partial}{\partial y}\right)\langle x, y|u, v\rangle = u\langle x, y|u, v\rangle,$$
$$\left(\frac{1}{2}x + \frac{\hbar c}{\mathrm{i}eB}\frac{\partial}{\partial y}\right)\langle x, y|u, v\rangle = v\langle x, y|u, v\rangle. \tag{20}$$

(Note that we are treating here u and v as parameters, independent of x and y.) By combining these equations we find

$$(x - u - v)\langle x, y|u, v\rangle = 0,$$

which implies that $\langle x, y|u, v\rangle$ must be proportional to $\delta(x - u - v)$. Hence, taking into account Eqs. (15) and (19), we propose

$$\langle x, y|u, v\rangle = N\delta(x - u - v)\exp\left[\frac{\mathrm{i}eB}{2\hbar c}y(v-u)\right], \tag{21}$$

where N is a normalization factor. The presence of the delta function in Eq. (21) implies that all the expressions of Λ in terms of x, y, u, v, give an equivalent result. Taking

$$N = \left(\frac{eB}{4\pi\hbar c}\right)^{1/2},$$

the normalization condition (6) is satisfied. It can be readily verified that the expression (21) satisfies Eqs. (20).

As pointed out above, under the canonical transformation (18), the Hamiltonian

$$H = \frac{{p_u}^2}{2m} + \frac{m\omega^2}{2}u^2 \tag{22}$$

is transformed into

$$H = \frac{1}{2m}\left[\left(p_x + \frac{eB}{2c}y\right)^2 + \left(p_y - \frac{eB}{2c}x\right)^2\right], \tag{23}$$

if $\omega = eB/mc$. Hence, the function (21) allows us to relate the eigenfunctions of the Hamiltonian of a one-dimensional harmonic oscillator (22) with those of the Hamiltonian of a particle in a uniform magnetic field (23). In fact, a direct, lengthy, computation shows that the expression

$$\phi(x,y) = N\int_{-\infty}^{\infty}\psi(u, x-u)\exp\left[\frac{\mathrm{i}eB}{2\hbar c}y(x-2u)\right]\mathrm{d}u,$$

obtained by substituting Eq. (21) into Eq. (4), is an eigenfunction of the Hamiltonian (23) with eigenvalue E, if $\psi(u, v)$ is an eigenfunction with eigenvalue E of the Hamiltonian (22).

2.3. A one-parameter group of transformations

In the foregoing examples, the canonical transformations have been essentially discrete transformations. Now we consider an example of a one-parameter group of canonical transformations.

The coordinate transformations in the phase space

$$Q^1 = q^1\cos\theta + \frac{p_2}{m\omega}\sin\theta, \quad Q^2 = q^2\cos\theta + \frac{p_1}{m\omega}\sin\theta,$$
$$P_1 = p_1\cos\theta - m\omega q^2\sin\theta, \quad P_2 = p_2\cos\theta - m\omega q^1\sin\theta, \tag{24}$$

where m and ω are nonzero constants, form a one-parameter group of canonical transformations (parameterized by θ). It can be readily verified that Eq. (9) is satisfied by

$$\Lambda = \frac{m\omega}{\sin\theta}\left[(Q^1Q^2 + q^1q^2)\cos\theta - Q^1q^2 - Q^2q^1\right]. \quad (25)$$

Then, a straightforward computation shows that

$$\langle Q^1, Q^2|q^1, q^2\rangle = N \exp\left\{\frac{\mathrm{i}m\omega}{\hbar\sin\theta} \times \left[(Q^1Q^2 + q^1q^2)\cos\theta - Q^1q^2 - Q^2q^1\right]\right\} \quad (26)$$

satisfies Eq. (7), with N independent of Q^i and q^i, assuming that Eqs. (24) hold for the operators $\widehat{q}^i$, $\widehat{p}_i$, $\widehat{Q}^i$, $\widehat{P}_i$. In order to satisfy the normalization condition (6), up to a phase factor, the normalization constant N appearing in the last equation must be

$$N = \frac{m\omega}{2\pi\hbar\sin\theta}. \quad (27)$$

By contrast with the examples of the preceding subsection, where a constant phase in the normalization factor N can be chosen in an arbitrary way, in the present case it is natural to impose the condition

$$\lim_{\theta\to 0}\langle Q^1, Q^2|q^1, q^2\rangle = \delta(Q^1 - q^1)\delta(Q^2 - q^2)$$

and one finds that this condition is satisfied with N given by Eq. (27).

The canonical transformations (24) leave invariant the Hamiltonian of a two-dimensional isotropic harmonic oscillator

$$H = \frac{1}{2m}({p_1}^2 + {p_2}^2) + \frac{m\omega^2}{2}[(q^1)^2 + (q^2)^2]. \quad (28)$$

Hence, the integral transform (4) with the kernel (26) maps an eigenfunction of the Hamiltonian (28) into another eigenfunction of H with the same eigenvalue.

2.4. Propagators

As pointed out above, the time evolution is a canonical transformation and, therefore, making use of Eq. (15) we have a straightforward procedure to find the corresponding propagators.

2.4.1. Free particle

As a first example we consider the simple case of a free particle with (q^1, q^2, q^3) being Cartesian coordinates. If (Q^i, P_i) denote the values of the q^i and their conjugate momenta a time t later, we have

$$\mathbf{Q} = \mathbf{q} + \frac{\mathbf{p}}{m}t, \qquad \mathbf{P} = \mathbf{p}. \quad (29)$$

For a fixed value of t, this transformation is indeed canonical since

$$\mathbf{P}\cdot \mathrm{d}\mathbf{Q} = \mathbf{p}\cdot\mathrm{d}\mathbf{q} + \mathrm{d}\left(\frac{\mathbf{p}^2}{2m}t\right),$$

with the function Λ being given by

$$\Lambda = \frac{m}{2t}(\mathbf{Q} - \mathbf{q})^2.$$

Substituting this expression into Eq. (15), from the normalization condition (6) one finds that

$$|N| = \left(\frac{m}{2\pi\hbar t}\right)^{3/2}.$$

As in the preceding example, we impose the condition

$$\lim_{t\to 0}\langle\mathbf{Q}|\mathbf{q}\rangle = \delta(\mathbf{Q} - \mathbf{q}).$$

Then, the final expression with the appropriate phase factor is

$$\langle\mathbf{Q}|\mathbf{q}\rangle = \left(\frac{m}{2\pi\mathrm{i}\hbar t}\right)^{3/2}\exp\left[\frac{\mathrm{i}m}{2\hbar t}(\mathbf{Q} - \mathbf{q})^2\right]$$

(*cf.* Refs. 7 and 8).

2.4.2. One-dimensional harmonic oscillator

The expressions

$$Q = q\cos\omega t + \frac{p}{m\omega}\sin\omega t,$$
$$P = p\cos\omega t - m\omega q\sin\omega t, \quad (30)$$

are the solution to the equations of motion of a one-dimensional harmonic oscillator with initial conditions (q, p), and represent a canonical transformation satisfying Eq. (9) with

$$\Lambda = \frac{m\omega}{2\sin\omega t}\left[(Q^2 + q^2)\cos\omega t - 2Qq\right], \quad (31)$$

considering t as a parameter [*cf.* Ref. 8, Eq. (6.37)]. Thus, from Eq. (15) we obtain

$$\langle Q|q\rangle = N\exp\left\{\frac{\mathrm{i}m\omega}{2\hbar\sin\omega t}\left[(Q^2+q^2)\cos\omega t - 2Qq\right]\right\} \quad (32)$$

and one readily verifies that Eqs. (6) and (7) are satisfied with

$$|N| = \left(\frac{m\omega}{2\pi\hbar\sin\omega t}\right)^{1/2}.$$

The final expression with the appropriate phase factor is then

$$\langle Q|q\rangle = \sqrt{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin\omega t}} \times \exp\left\{\frac{\mathrm{i}m\omega}{2\hbar\sin\omega t}\left[(Q^2 + q^2)\cos\omega t - 2Qq\right]\right\}.$$

2.4.3. *Particle subjected to a constant force*

In the case of a particle in one dimension under the action of a constant force F, the solution to the equations of motion with initial conditions (q, p) is given by

$$Q = q + \frac{t}{m}p + \frac{Ft^2}{2m}, \qquad P = p + Ft. \tag{33}$$

These formulas represent a canonical transformation with

$$\Lambda = \frac{m}{2t}(Q-q)^2 + \frac{Ft}{2}(Q+q) - \frac{3}{8}\frac{F^2t^3}{m} \tag{34}$$

[*cf.* Ref. 8, Eq. (6.40)]. As in the preceding examples, one finds that Eq. (15) holds, with N being independent of Q.

2.5. A nonlinear transformation

As a final example we shall consider the nonlinear canonical transformation [9]

$$z = x + \frac{p_x p_y}{m^2 g}, \qquad p_z = p_x,$$

$$w = y + \frac{{p_x}^2}{2m^2 g}, \qquad p_w = p_y, \tag{35}$$

where m and g are nonzero constants. Under the coordinate transformation (35), the Hamiltonian

$$H = \frac{p_w^2}{2m} + mgw, \tag{36}$$

corresponding to a particle in one dimension in a uniform gravitational field, is transformed into

$$H = \frac{p_x^2 + p_y^2}{2m} + mgy, \tag{37}$$

which corresponds to a particle in two dimensions in a uniform gravitational field. Since

$$p_x \mathrm{d}x + p_y \mathrm{d}y = p_z \mathrm{d}z + p_w \mathrm{d}w + \mathrm{d}\left(-\frac{{p_x}^2 p_y}{m^2 g}\right),$$

the transformation is indeed canonical. Taking

$$(q^1, q^2, p_1, p_2) \equiv (z, w, p_z, p_w)$$

and

$$(Q^1, Q^2, P_1, P_2) \equiv (x, y, p_x, p_y),$$

we look for a kernel of the form (15), with

$$\Lambda = (x - z)\sqrt{2m^2 g(w-y)}.$$

Substituting into Eqs. (7), assuming that relations identical to Eqs. (35) hold for the corresponding operators, one finds that in this case N is not independent of the coordinates Q^i, but

$$N = \text{const.}(w-y)^{-1/2}$$

and the foregoing results allow us to relate the eigenfunctions of these Hamiltonians, reproducing the results of Ref. 9.

3. Discussion

In most of the examples given above, the relation between the old and the new coordinates is of the form

$$q^j = a^j_k Q^k + b^{jk} P_k + c^j, \tag{38}$$

where the coefficients a^j_k, b^{jk}, and c^j are independent of the phase space coordinates [see Eqs. (18), (24), (29), (30), and (33)]. Since no ordering ambiguities arise, we can assume that an identical relation to Eq. (38) holds for the corresponding operators. Then, in those cases where the Jacobian (10) is different from zero, making use of Eqs. (11), one readily finds that Eq. (15) satisfies Eqs. (7). In fact, assuming that N does not depend on the Q^i we have

$$\begin{aligned}\widehat{q}^j N\exp(\mathrm{i}\Lambda/\hbar) &= \left(a^j_k Q^k + b^{jk}\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial Q^k} + c^j\right) N\exp(\mathrm{i}\Lambda/\hbar)\\ &= \left(a^j_k Q^k + b^{jk}P_k + c^j\right) N\exp(\mathrm{i}\Lambda/\hbar)\\ &= q^j N\exp(\mathrm{i}\Lambda/\hbar).\end{aligned} \tag{39}$$

When the Jacobian (10) is equal to zero, that is, when the set $\{q^1, \ldots, q^n, Q^1, \ldots, Q^n\}$ is functionally dependent, it is possible to express N of the Q^i as functions of the q^j and the remaining $n - N$ Q^k only (at least locally), as in the first equation in (18), say $Q^i = F^i(q^1, \ldots, q^n, Q^{N+1}, \ldots, Q^n)$, for $i = 1, \ldots, N$. Then, $\langle Q^1, \ldots, Q^n | q^1, \ldots, q^n\rangle$ must be proportional to $\delta(Q^1 - F^1)\delta(Q^2 - F^2)\cdots\delta(Q^N - F^N)$, and we have

$$\begin{aligned}&\langle Q^1, \ldots, Q^n | q^1, \ldots, q^n\rangle\\ &= N\delta(Q^1 - F^1)\delta(Q^2 - F^2)\cdots\delta(Q^N - F^N)\exp(\mathrm{i}\Lambda/\hbar),\end{aligned}$$

with N being independent of the Q^i. (It can even happen that N is equal to n, for instance, when one is replacing Cartesian coordinates by spherical coordinates, in which case $\langle Q^1, \ldots, Q^n | q^1, \ldots, q^n\rangle$ is just a product of n delta functions.)

It may be remarked that in the procedure followed above to find the propagators we have not started from the path integral formalism, although the functions Λ obtained from Eq. (9) are equivalent to the time integral of the Lagrangian along the classical trajectory (see, *e.g.*, Ref. [4]), only that the approach employed here is much simpler and, as we have shown, it is also applicable in the case of discrete canonical transformations.

In the case of nonlinear canonical transformations, the normalization factor N appearing in Eq. (15) may depend on the coordinates (see Sec. 2.1 and 2.5), but the ansatz (15) can still be used as a starting point to find the exact expression of the kernel of Eq. (4), considering N as a function to be determined, as in Sec. 2.5.

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