

Issues of duality in abelian gauge theory and in linearized gravity

J.A. Nieto^{a,b} and E.A. León^b

^a*Facultad de Ciencias Físico-Matemáticas de la Universidad Autónoma de Sinaloa,
80010, Culiacán Sinaloa, México,
e-mail: nieto@uas.uasnet.mx*

^b*Departamento de Investigación en Física de la Universidad de Sonora,
83000, Hermosillo Sonora, México,
e-mail: ealeon@posgrado.cifus.uson.mx*

Recibido el 5 de noviembre de 2008; aceptado el 4 de agosto de 2009

We start by describing two of the main proposals for duality in abelian gauge theories, namely F (ield strength)-duality approach and the S -duality formalism. We then discuss how F -duality and S -duality can be applied to the case of linearized gravity. By emphasizing the similarities and differences between these two types of dualities we explore the possibility of combining them in just one duality formalism.

Keywords: S -duality; linearized gravity; abelian gauge theory.

Comenzamos describiendo dos propuestas principales de dualidad en teorías abelianas de norma, a saber el enfoque de dualidad-(del campo) F y el formalismo de dualidad- S . De ahí, discutimos como la dualidad- F y la dualidad- S pueden implementarse para el caso de gravedad linealizada. Haciendo énfasis en las similitudes y diferencias entre estos dos tipos de dualidad, exploramos la posibilidad de combinarlos en un sólo formalismo de dualidad.

Descriptores: Dualidad- S ; gravedad linealizada; teoría abeliana de norma.

PACS: 04.60.-m, 11.25.Tq, 11.15.-q, 11.30.Ly

1. Introduction

Duality in linearized gravity [1] has been a topic of considerable interest [2-29]. There are at least two physical reasons for this increasing interest of the topic. The first possibility arises from the hope of determining the strong coupling limit for linearized gravity (see Refs. 1 and 2) via the analogue of the S -duality concept [30] in gauge field theories. In fact, just as in a dual gauge theory the coupling exchange $g^2 \rightarrow 1/g^2$ describes a basic dual symmetry, one may expect a dual gravitational theory with either one of the exchanges $l_p^2 \rightarrow 1/l_p^2$ [2] or $\Lambda \rightarrow 1/\Lambda$ [1,26], where l_p is the Planck length and Λ is the cosmological constant.

The second motivation comes from the idea of implementing a dual symmetry of the linearized gravitational field equations at the level of the corresponding action [5]. Such a dual symmetry is the gravitational analogue of the corresponding electromagnetic dual symmetry provided by the electric and magnetic field strengths. In this case, the Riemann tensor and its dual play the role of the electric and magnetic fields strengths, respectively. This dual gravitational approach has its origins in the old observation [31] that in the case of electromagnetism, this kind of dual symmetry can be implemented at the level of the action if the infinitesimal transformations are applied canonically to the gauge field rather than to the corresponding field strength.

From the above comments we observe that while in the S -duality approach [30] emphasis is placed on the coupling exchange, in the case of the canonical approach attention is focused on the dual transformation of the field strength. Both generalized approaches have, however, a common ori-

gin, namely the dual symmetry of the Maxwell equations discovered by Dirac himself [32-33]. Since linearized gravity can be understood as an abelian gauge theory [26] one becomes motivated to see whether there is a kind of dual theory for gravity in which both coupling and field strength dual exchanges are equally important. In order to find such a dual gravitational theory we first need to analyze carefully the differences between the F -duality (field strength duality) and S -duality in an abelian gauge field theory. For this purpose in Secs. 2 and 4 we briefly discuss the F -duality approach of Refs. 31 and 5, respectively. In Secs. 3 and 5, we briefly review the S -duality theory for abelian gauge fields proposed in Ref. 30 and the S -duality theory for linearized gravity described in Ref. 1, respectively. With these reviews at hand in Secs. 6 and 7, we propose a unified duality theory for abelian gauge field theory and linearized gravity, respectively. Finally, in Sec. 8 we make some final remarks.

2. F -duality for an abelian gauge field theory

In this section, we summarize the main duality ideas of the approach proposed in Ref. 31. Consider the field strength $F^{\mu\nu} = -F^{\nu\mu}$ and its dual

$$*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \quad (1)$$

where $\varepsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric Levi-Civita density in a Minkowski spacetime. The source-free Maxwell equations are

$$\partial_\nu F^{\mu\nu} = 0 \quad (2)$$

and

$$\partial_\nu^* F^{\mu\nu} = 0. \tag{3}$$

It is straightforward to see that these field equations are invariant under the transformation

$$\delta F^{\mu\nu} = \beta^* F^{\mu\nu} \tag{4}$$

and

$$\delta^* F^{\mu\nu} = -\beta F^{\mu\nu}, \tag{5}$$

where β is an arbitrary constant. Here we used the fact that $**F^{\mu\nu} = -F^{\mu\nu}$.

Since

$$F^{\mu\nu} \delta F_{\mu\nu} = \beta F^{\mu\nu*} F_{\mu\nu}, \tag{6}$$

the action

$$S_I = \frac{1}{2} \int d^4x F^{\mu\nu} F_{\mu\nu} \tag{7}$$

is not invariant under (4) unless we write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{8}$$

which means solving (3). The authors of Ref. 31 pointed out that this contradictory invariance can be solved if one considers consistent canonical variations of the potential δA_μ instead of variations of the field strength $\delta F_{\mu\nu}$. With the idea of emphasizing the invariance of the action (7) at the level of the field strength $F_{\mu\nu}$ according to (4), we shall refer to this approach as *F*-duality formalism.

3. *S*-duality for an abelian gauge field theory

Here, we shall briefly review the *S*-duality formalism for an abelian gauge theory (see Ref. 30). Our starting point is the action

$$S_{II} = \frac{1}{2g^2} \int d^4x F^{\mu\nu} F_{\mu\nu} + \frac{\theta}{2} \int d^4x F^{\mu\nu*} F_{\mu\nu}. \tag{9}$$

Here, it is assumed that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The θ -term is topological and, of course, classically it can be dropped from (9). This implies that in this case (9) can be reduced to the action (7). However, if our goal is to quantize the theory described by (9), it becomes necessary to keep the θ -term. Observe that in contrast to the formalism of Sec. 2, in this approach emphasis is placed on the role played by of the constants g^2 and θ .

Now, by introducing the (anti) self-dual field strengths

$$\pm F^{\alpha\beta} = \left(\frac{1}{2}\right)^\pm N_{\tau\lambda}^{\alpha\beta} F^{\tau\lambda}, \tag{10}$$

where

$$\pm N_{\tau\lambda}^{\alpha\beta} = \frac{1}{2} (\delta_{\tau\lambda}^{\alpha\beta} \mp i \varepsilon^{\alpha\beta}{}_{\tau\lambda}), \tag{11}$$

with $\delta_{\tau\lambda}^{\alpha\beta} = \delta_\tau^\alpha \delta_\lambda^\beta - \delta_\tau^\beta \delta_\lambda^\alpha$ denoting a generalized delta, one can prove that the action (9) can be written as

$$S_{III} = \frac{1}{2} (\tau^+) \int d^4x^+ F^{\mu\nu+} F_{\mu\nu} + \frac{1}{2} (\tau^-) \int d^4x^- F^{\mu\nu-} F_{\mu\nu}, \tag{12}$$

where τ^+ and τ^- are two different constant parameters given by

$$\tau^+ = \frac{1}{g^2} + i\theta \tag{13}$$

and

$$\tau^- = \frac{1}{g^2} - i\theta. \tag{14}$$

The fact that the parameters τ^+ and τ^- are complex means that, in addition to the field strength duality transformation,

$$\delta^\pm F^{\alpha\beta} = \pm i \beta^\pm F^{\alpha\beta}, \tag{15}$$

one can in principle implement, for $a, b, c, d \in Z$, the more general duality transformation

$$\tau' = \frac{a + c\tau}{b + d\tau}. \tag{16}$$

Observe that (16) generalizes the coupling duality transformation

$$g^2 \rightarrow \frac{1}{g^2}. \tag{17}$$

In fact, it is known that the modular group described by (16) can be generated by the elements $T : \tau \rightarrow \tau + 1$ and $S : \tau \rightarrow -1/\tau$ (see Sec. 1.4.3 of Ref. 34). So, if the vacuum angle θ vanishes, the *S*-symmetry yields precisely the transformation (17).

The next step is to write a meaningful action which may allow us to transfer information from the action (9) to its associated dual action. First, one considers the generalized field strength

$$H^{\mu\nu} = F^{\mu\nu} - G^{\mu\nu}, \tag{18}$$

where $G^{\mu\nu}$ is an auxiliary two-form. Secondly, one introduces the dual field strength $W_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, where V_μ is a one-form vector gauge field. The generalized action is then written as [30]

$$S_{IV} = \frac{1}{2} (\tau^+) \int d^4x^+ H^{\mu\nu+} H_{\mu\nu} + \frac{1}{2} (\tau^-) \int d^4x^- H^{\mu\nu-} H_{\mu\nu} + \int d^4x^+ W^{\mu\nu+} G_{\mu\nu} - \int d^4x^- W^{\mu\nu-} G_{\mu\nu}. \tag{19}$$

This action is invariant under the transformations

$$\delta A = B, \quad \delta G = dB, \tag{20}$$

where B is any one-form. If we eliminate V from (19), we see that $dG = 0$ and therefore we can set $G = 0$. Hence from (18) we see that $H^{\mu\nu} = F^{\mu\nu}$ and consequently the action (19) is reduced to (12). On the other hand the gauge

invariance (20) allows us to set $A = 0$ and therefore the action (19) becomes

$$S_{IV} = \frac{1}{2}(\tau^+) \int d^4x^+ G^{\mu\nu+} G_{\mu\nu} + \frac{1}{2}(\tau^-) \int d^4x^- G^{\mu\nu-} G_{\mu\nu} + \int d^4x^+ W^{\mu\nu+} G_{\mu\nu} - \int d^4x^- W^{\mu\nu-} G_{\mu\nu}. \quad (21)$$

Finally, after eliminating $\pm G$ one finds that (21) leads to

$$S_V = \frac{1}{2} \left(-\frac{1}{\tau^+}\right) \int d^4x^+ W^{\mu\nu+} W_{\mu\nu} + \frac{1}{2} \left(-\frac{1}{\tau^-}\right) \int d^4x^- W^{\mu\nu-} W_{\mu\nu}, \quad (22)$$

which is the dual action. We observe that the coupling constant τ transforms as $-1/\tau$. Actually, when quantum topological effects are considered, the τ transformation can be extended to the more general duality transformation given in (16) (see Ref. 30).

4. F-duality for linearized gravity

The Riemann tensor for linearized gravity is given by

$$R_{\mu\nu\alpha\beta} = \frac{1}{2} (\partial_\mu \partial_\beta h_{\nu\alpha} - \partial_\mu \partial_\alpha h_{\nu\beta} - \partial_\nu \partial_\beta h_{\mu\alpha} + \partial_\nu \partial_\alpha h_{\mu\beta}). \quad (23)$$

Here, the object $h_{\mu\nu} = h_{\nu\mu}$ can be understood as a small deviation from the full metric $g_{\mu\nu}$, namely

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (24)$$

where

$$(\eta_{\mu\nu}) = \text{diag}(-1, 1, 1, 1) \quad (25)$$

is the Minkowski flat metric. The vacuum Einstein equations are

$$R_{\nu\beta} = 0, \quad (26)$$

where $R_{\nu\beta} = \eta^{\mu\alpha} R_{\mu\nu\alpha\beta}$ is the linearized Ricci tensor.

Let us now introduce the dual tensor

$${}^*R_{\mu\nu\alpha\beta} = \frac{1}{2} \varepsilon_{\mu\nu\sigma\rho} R_{\alpha\beta}^{\sigma\rho}. \quad (27)$$

We observe that due to the Bianchi identity

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0,$$

we have that ${}^*R_{\nu\beta} = \eta^{\mu\alpha} {}^*R_{\mu\nu\alpha\beta}$ satisfies the dual field equation

$${}^*R_{\nu\beta} = 0 \quad (28)$$

or

$$\frac{1}{2} \varepsilon_{\mu\nu\sigma\rho} \eta^{\mu\alpha} R_{\alpha\beta}^{\sigma\rho} = 0. \quad (29)$$

It is not difficult to see that both field equations (26) and (28) are invariant under the infinitesimal rotations

$$\delta R_{\mu\nu\alpha\beta} = \beta^* R_{\mu\nu\alpha\beta} \quad (30)$$

and

$$\delta^* R_{\mu\nu\alpha\beta} = -\beta R_{\mu\nu\alpha\beta}, \quad (31)$$

where β is again a constant. Comparing the development of Sec. 2 with the present section, we observe that these transformations are completely analogous to the expressions (4) and (5). Thus, it is expected that the Pauli-Fierz action

$$S_{VI} = 4 \int d^4x (\partial^\alpha h^{\mu\nu} \partial_\alpha h_{\mu\nu} - 2\partial_\mu h^{\mu\nu} \partial_\alpha h_\nu^\alpha + 2\partial^\mu h \partial_\nu h_{\mu\nu} - \partial^\alpha h \partial_\alpha h), \quad (32)$$

where $h = h_\alpha^\alpha$ is not invariant under (30) and (31) unless we describe an infinitesimal canonical transformation in terms of the potential $\delta h_{\mu\nu}$ instead of the field strengths $R_{\mu\nu\alpha\beta}$ and ${}^*R_{\mu\nu\alpha\beta}$. Actually, the $SO(2)$ rotations are achieved by means of two superpotentials; one is associated with $h_{\mu\nu}$ and the other with its canonical conjugate momenta (see Ref. 5 for details).

5. S-duality for linearized gravity

Let us start by observing that the curvature Riemann tensor $R_{\mu\nu\alpha\beta}$ for linearized gravity, given in (23), can be written as

$$R_{\mu\nu\alpha\beta} = \partial_\mu A_{\nu\alpha\beta} - \partial_\nu A_{\mu\alpha\beta}, \quad (33)$$

where

$$A_{\mu\alpha\beta} = \frac{1}{2} (\partial_\beta h_{\mu\alpha} - \partial_\alpha h_{\mu\beta}). \quad (34)$$

The expression (33) immediately suggests that $R_{\mu\nu\alpha\beta}$ can be seen as an abelian field strength with $A_{\mu\alpha\beta} = -A_{\mu\beta\alpha}$ as the gauge potential. In fact, as mentioned in Refs. 1 and 26, this interpretation is reinforced by noticing that $R_{\mu\nu\alpha\beta}$ is invariant under the gauge transformation

$$\delta A_{\mu\alpha\beta} = \partial_\mu \lambda_{\alpha\beta}, \quad (35)$$

where $\lambda_{\alpha\beta} = -\lambda_{\beta\alpha}$ is an arbitrary two-form. Now, it is not difficult to prove that, up to surface term, the action (32) can be written as [1]

$$S_{VII} = \frac{1}{2} \int d^4x \varepsilon^{\mu\nu\alpha\beta} \Omega_{\mu\nu}^{\tau\lambda} R_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho}. \quad (36)$$

Here, $\Omega_{\mu\nu}^{\alpha\beta}$ is given by

$$\Omega_{\mu\nu}^{\alpha\beta} = \delta_\mu^\alpha h_\nu^\beta - \delta_\mu^\beta h_\nu^\alpha - \delta_\nu^\alpha h_\mu^\beta + \delta_\nu^\beta h_\mu^\alpha. \quad (37)$$

Suppose we add to the action (36) the topological term

$$S_T = \frac{1}{4} \int d^4x \varepsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{\tau\lambda} R_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho} \quad (38)$$

and the cosmological constant term

$$S_C = \frac{1}{4} \int d^4x \varepsilon^{\mu\nu\alpha\beta} \Omega^{\tau\lambda} Q_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho}. \quad (39)$$

What we obtain is the generalized action [1];

$$S_{VIII} = \frac{1}{4} \int d^4x \varepsilon^{\mu\nu\alpha\beta} Q_{\mu\nu}^{\tau\lambda} Q_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho}, \quad (40)$$

where $Q_{\mu\nu}^{\alpha\beta}$ is defined by

$$Q_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} + \Omega_{\mu\nu}^{\alpha\beta}. \quad (41)$$

Moreover, it is not difficult to prove that the action (40) is reduced to (see Ref. 1 for details)

$$\begin{aligned} S_{VIII} &= \frac{1}{4} \int d^4x \varepsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{\tau\lambda} R_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho} \\ &+ 8 \int d^4x h^{\mu\nu} (R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R) \\ &- 8 \int d^4x (h^2 - h^{\mu\nu} h_{\mu\nu}). \end{aligned} \quad (42)$$

We recognize in the second and third terms of (42) the Pauli-Fierz action for linearized gravity with a cosmological constant, while the first term is a total derivative (Euler topological invariant or Gauss-Bonnet term). Note that the usual cosmological factor Λ in the third term can be derived simply by changing $\Omega \rightarrow a^2 \Omega$, where a is a constant, and rescaling the total action $S_{VII} \rightarrow \frac{1}{4} \Lambda^{-1} S_{VII}$, with $\Lambda = a^2$.

In order to develop an S -dual linearized gravitational action we generalize the action (40) as follows;

$$\begin{aligned} S_{IX} &= \frac{1}{2} (\lambda^+) \int d^4x \varepsilon^{\mu\nu\alpha\beta} Q_{\mu\nu}^{\tau\lambda} Q_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho} \\ &+ \frac{1}{2} (\lambda^-) \int d^4x \varepsilon^{\mu\nu\alpha\beta} Q_{\mu\nu}^{\tau\lambda} Q_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho}, \end{aligned} \quad (43)$$

where λ^+ and λ^- are two different constant parameters (playing the analogue role of the parameters τ^+ and τ^- in the Maxwell case) and ${}^\pm Q_{\mu\nu}^{\alpha\beta}$ is given by

$${}^\pm Q_{\mu\nu}^{\alpha\beta} = (\frac{1}{2})^\pm N_{\tau\lambda}^{\alpha\beta} Q_{\mu\nu}^{\tau\lambda}, \quad (44)$$

where

$${}^\pm N_{\tau\lambda}^{\alpha\beta} = \frac{1}{2} (\delta_{\tau\lambda}^{\alpha\beta} \mp i \varepsilon^{\alpha\beta}{}_{\tau\lambda}). \quad (45)$$

It turns out that ${}^+Q_{\mu\nu}^{\alpha\beta}$ is self-dual, while ${}^-Q_{\mu\nu}^{\alpha\beta}$ is anti self-dual curvature tensors. Therefore, the action (43) describes self-dual and anti-self-dual linearized gravity.

Following the steps of section 3 let us introduce a two-form G and use it for defining

$$H_{\mu\nu}^{\alpha\beta} \equiv Q_{\mu\nu}^{\alpha\beta} - G_{\mu\nu}^{\alpha\beta}. \quad (46)$$

We assume that $G_{\mu\nu}^{\alpha\beta}$ satisfies the same indices symmetry properties as $R_{\mu\nu}^{\alpha\beta}$, namely

$$\begin{aligned} G_{\mu\nu\alpha\beta} &= -G_{\mu\nu\beta\alpha} = -G_{\nu\mu\alpha\beta} = G_{\alpha\beta\mu\nu}, \\ G_{\mu\nu\alpha\beta} + G_{\mu\beta\nu\alpha} + G_{\mu\alpha\beta\nu} &= 0. \end{aligned} \quad (47)$$

Now, let us consider the extended action

$$\begin{aligned} S_X &= \frac{1}{2} (\lambda^+) \int d^4x \varepsilon^{\mu\nu\alpha\beta} H_{\mu\nu}^{\tau\lambda} H_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho} \\ &+ \frac{1}{2} (\lambda^-) \int d^4x \varepsilon^{\mu\nu\alpha\beta} H_{\mu\nu}^{\tau\lambda} H_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho} \\ &+ \int d^4x \varepsilon^{\mu\nu\tau\lambda} W_{\mu\nu}^{\alpha\beta} G_{\tau\lambda}^{\sigma\rho} \varepsilon_{\alpha\beta\sigma\rho} \\ &- \int d^4x \varepsilon^{\mu\nu\tau\lambda} W_{\mu\nu}^{\alpha\beta} G_{\tau\lambda}^{\sigma\rho} \varepsilon_{\alpha\beta\sigma\rho}, \end{aligned} \quad (48)$$

where $W_{\mu\nu\alpha\beta} = \partial_\mu V_{\nu\alpha\beta} - \partial_\nu V_{\mu\alpha\beta}$ is the dual field strength satisfying the Dirac quantization law

$$\int W \in 2\pi\mathbf{Z}. \quad (49)$$

It is not difficult to see that, beyond the gauge invariance $A \rightarrow A - d\lambda$, $G \rightarrow G$, the partition function

$$Z = \int d^+ G d^- G dA dh dV e^{-S_X} \quad (50)$$

is invariant under

$$A \rightarrow A + B \quad \text{and} \quad G \rightarrow G + dB, \quad (51)$$

where $B_{\mu\alpha\beta} = -B_{\mu\beta\alpha}$ is an arbitrary tensor.

Starting from (48) one can proceed in two different ways. For the first possibility, we note that the path integral that involves V is

$$\begin{aligned} \int DV \exp \left(\int d^4x \varepsilon^{\mu\nu\tau\lambda} W_{\mu\nu}^{\alpha\beta} G_{\tau\lambda}^{\sigma\rho} \varepsilon_{\alpha\beta\sigma\rho} \right. \\ \left. - \int d^4x \varepsilon^{\mu\nu\tau\lambda} W_{\mu\nu}^{\alpha\beta} G_{\tau\lambda}^{\sigma\rho} \varepsilon_{\alpha\beta\sigma\rho} \right). \end{aligned} \quad (52)$$

Integrating over the dual connection V , we get a delta function setting $dG = 0$. Thus, using the gauge invariance (51), we may gauge G to zero, reducing (48) to the original action (43). Therefore, the actions (48) and (43) are, in fact, classically equivalent.

For the second possibility, we note that the gauge invariance (51) enables us to fix a gauge with $A = 0$. (It is important to note that, at this stage, we are considering $A_{\mu\alpha\beta}$ and $h_{\mu\nu}$ as independent fields.) The action (48) is then reduced to

$$\begin{aligned} S_X &= \frac{1}{2} (\lambda^+) \int d^4x \varepsilon^{\mu\nu\alpha\beta} P_{\mu\nu}^{\tau\lambda} P_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho} \\ &+ \frac{1}{2} (\lambda^-) \int d^4x \varepsilon^{\mu\nu\alpha\beta} P_{\mu\nu}^{\tau\lambda} P_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho} \\ &+ \int d^4x \varepsilon^{\mu\nu\tau\lambda} W_{\mu\nu}^{\alpha\beta} G_{\tau\lambda}^{\sigma\rho} \varepsilon_{\alpha\beta\sigma\rho} \\ &- \int d^4x \varepsilon^{\mu\nu\tau\lambda} W_{\mu\nu}^{\alpha\beta} G_{\tau\lambda}^{\sigma\rho} \varepsilon_{\alpha\beta\sigma\rho}, \end{aligned} \quad (53)$$

where

$$P_{\mu\nu}^{\tau\lambda} \equiv \Omega_{\mu\nu}^{\tau\lambda} - G_{\mu\nu}^{\tau\lambda}. \tag{54}$$

By eliminating $G_{\mu\nu}^{\tau\lambda}$ in (53), we get the dual action

$$S_{XI} = \frac{1}{2} \left(-\frac{1}{\lambda^+} \right) \int d^4x \varepsilon^{\mu\nu\alpha\beta} \Xi_{\mu\nu}^{\tau\lambda} + \Xi_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho} + \frac{1}{2} \left(-\frac{1}{\lambda^-} \right) \int d^4x \varepsilon^{\mu\nu\alpha\beta} \Xi_{\mu\nu}^{\tau\lambda} - \Xi_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho}, \tag{55}$$

Here, $\Xi_{\mu\nu}^{\tau\lambda}$ means

$$\Xi_{\mu\nu}^{\alpha\beta} = W_{\mu\nu}^{\alpha\beta} + \Omega_{\mu\nu}^{\alpha\beta}. \tag{56}$$

Observe that the complex parameter λ has been exchanged by $-1/\lambda$ as expected.

6. A relation between F -duality and S -duality for an abelian gauge field

One of our main goals is to establish, in section 7, a possible link between the F -duality and the S -duality for linearized gravity. But we shall first investigate a possible connection between F -duality and S -duality in the context of an abelian gauge field theory.

As we mentioned in Sec. 2, the Maxwell action (7) is not invariant under the infinitesimal transformations (4) and (5) in spite of the fact that field Eqs. (2) and (3) are. This problem can be overcome if one solves (3) in terms of the relation

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{57}$$

and considers canonical variations of the potential δA_μ instead of variations of the field strength $\delta F_{\mu\nu}$. In turn, in order to maintain duality invariance at the level of the corresponding canonical action, this forces us to introduce what is called superpotential (see Refs. 5 and 28 for details). However, in this case we are already using the field equations (3) which, in principle, cannot be obtained from the original action (7). This means that the action (7) needs to be properly modified in such a way that the field equations (3) are a consequence of an extended action. The procedure is well known: one introduces an auxiliary vector field Lagrange multiplier V^μ and writes the new action as

$$S = \frac{1}{2} \int d^4x F^{\mu\nu} F_{\mu\nu} + \int d^4x \varepsilon^{\mu\nu\alpha\beta} V_\mu \partial_\nu F_{\alpha\beta}. \tag{58}$$

Here, of course we are not assuming the form (57) for $F_{\mu\nu}$, otherwise the second term in (58) is identically zero. In fact, starting with (58) one can proceed in two different ways. In the first case, varying V_μ one obtains the field equation (3) which has the solution (57). Substituting (57) into the second term of (58) one sees that the action (7) is recovered. In the second case, it is first convenient to make an integration by parts obtaining (up to the surface term)

$$S = \frac{1}{2} \int d^4x F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \int d^4x \varepsilon^{\mu\nu\alpha\beta} W_{\mu\nu} F_{\alpha\beta}, \tag{59}$$

where $W_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ and then solving for $F_{\mu\nu}$. In this way, we obtain the relation

$$F^{\mu\nu} = -\frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} W_{\alpha\beta} = -^*W^{\mu\nu}, \tag{60}$$

which can be used to get the dual action

$$S = \frac{1}{2} \int d^4x W^{\mu\nu} W_{\mu\nu}. \tag{61}$$

Observe that if one assumes (57), then the second term in (59) is identically zero. An important change in this procedure arises if one assumes a nontrivial topology. In this case, the solution (57) to (3) is no longer true. But the correct expression is

$$F_{\mu\nu} \rightarrow H_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - G_{\mu\nu}, \tag{62}$$

where the two-form G is a “string” field associated with a nontrivial topology, so that $dG = 0$. This phenomenon can be emphasized if instead of starting with the action (59) one considers the action

$$S = \frac{1}{2} \int d^4x \{ H^{\mu\nu} H_{\mu\nu} + \varepsilon^{\mu\nu\alpha\beta} W_{\mu\nu} H_{\alpha\beta} \}, \tag{63}$$

with

$$H_{\mu\nu} = F_{\mu\nu} - G_{\mu\nu}. \tag{64}$$

Note that by assuming the relation (62) the action (63) is reduced to

$$S = \frac{1}{2} \int d^4x \{ H^{\mu\nu} H_{\mu\nu} - \varepsilon^{\mu\nu\alpha\beta} W_{\mu\nu} G_{\alpha\beta} \}. \tag{65}$$

This development leads to the conclusion that, rather than looking for the invariance of the action (7) under the infinitesimal transformation (4), one should consider invariance of the action (63) or (65) under such transformations. But one may recognize that the action (65) has exactly the same form as the expression (19) (see Sec. 3) which was considered in the context of S -duality approach. The main difference between (65) and (19) is that in (19) one considers ${}^\pm H^{\mu\nu}$, ${}^\pm W_{\mu\nu}$ and ${}^\pm G_{\alpha\beta}$ rather than $H^{\mu\nu}$, $W_{\mu\nu}$ and $G_{\alpha\beta}$ as in (65). Further the parameters ${}^\pm \tau$ are considered in (19), while in (65) this is not the case. This means that (65) can be considered as a particular case of (19). And in this context one should expect that invariance of (19) leads to a reduced invariance of (65). Indeed, the transformation (20), namely $\delta A = B$, $\delta G = dB$, where B is any one-form, also leaves the action (65) invariant. It is interesting to note that the infinitesimal transformation (4) can be considered as a particular case of (20) as soon as one also assumes the transformation $\delta G = \beta^* F$ for the “string” field G . One of our conclusions is that in order to implement the transformation (4) at the level of the action of Maxwell theory, one needs to introduce an auxiliary field G and consider (63) or (65) as starting point rather than (7).

Let us use the notation $D = dB$. From (64) we then observe that

$$\delta H_{\mu\nu} = \delta F_{\mu\nu} - \delta G_{\mu\nu} = D_{\mu\nu} - D_{\mu\nu}, \quad (66)$$

which is of course identically equal to zero. But writing $\delta H_{\mu\nu}$ as in (66) suggests we should consider (4) $\delta F_{\mu\nu} = \beta^* F_{\mu\nu}$ as a particular case with $D_{\mu\nu} = \beta^* F_{\mu\nu}$ and $\delta G_{\mu\nu} = \beta^* F_{\mu\nu}$. In fact, this possibility seems to have passed unnoticed before, in the context of S -duality formalism. Perhaps this is because the invariance of (66) was written in terms of δA_μ rather than in terms of $\delta F_{\mu\nu}$. It is true that δA_μ implies $\delta F_{\mu\nu}$ but the converse is not in general true; unless one considers nonlocal formalism in the sense $\delta A = d^{-1}D$, which in the case of the variation $\delta F_{\mu\nu} = \beta^* F_{\mu\nu}$ means $\delta A = B = \beta d^{-1}F$. It is tempting to assume that from the canonical point of view this is equivalent to introducing what is called superpotential [5, 31]. In other words, our conjecture is that the "string" field G and the superpotential are closely related [35].

7. Duality and S duality in linearized gravity

An application of the prescription of the previous section to the case of linearized gravity is straightforward. From (46) one sees that $H_{\mu\nu}^{\alpha\beta} = Q_{\mu\nu}^{\alpha\beta} - G_{\mu\nu}^{\alpha\beta}$ remains invariant under the transformations

$$\delta Q_{\mu\nu}^{\alpha\beta} = D_{\mu\nu}^{\alpha\beta}, \quad \delta G_{\mu\nu}^{\alpha\beta} = D_{\mu\nu}^{\alpha\beta}. \quad (67)$$

Here, $D_{\mu\nu}^{\alpha\beta}$ is an arbitrary two-form with the property $D = dB$, where B is any "one-form". This implies that the action (48) is invariant under (67).

As a particular case of (67) one writes

$$\delta Q_{\mu\nu}^{\alpha\beta} = \beta^* Q_{\mu\nu}^{\alpha\beta}. \quad (68)$$

This corresponds to considering $D_{\mu\nu}^{\alpha\beta} = \beta^* Q_{\mu\nu}^{\alpha\beta}$. The expression (68) refers of course to infinitesimal rotations and therefore we have found a mechanism to make the extended action (48) invariant under such rotations. Again, one can try to relate (68) to the gauge field $A_{\nu\alpha\beta}$ according to (33), but this would imply a nonlocal variation $\delta A = \beta d^{-1}Q_{\mu\nu}^{\alpha\beta}$. It is intriguing that with this procedure we do not even need to consider the perturbation $h_{\mu\alpha}$ as in the canonical method of Ref. 5. However, one should expect that if the action (48) is written in a canonical form, a link would have to be found between what is called a superpotential in Ref. 5 and the auxiliary field $G_{\mu\nu}^{\alpha\beta}$.

8. Discussion and final comments

In this work we have shown that the F -duality is indeed contained in the S -duality formalism as proposed in Ref. 30. One of the advantages of this identification is that it is not necessary to rely on canonical formalism in order to implement duality invariance at the level of the action. In a sense,

S -duality provides the route that it is necessary to follow in the case of the F -duality program. In fact, S -duality establishes that duality can be achieved at the level of the action by adding a θ term to the Maxwell action and by introducing an auxiliary two-form G . It turns out that this is also true for linearized gravity, as we have pointed out in Sec. 7.

These results also suggest we consider the coupling parameter τ in the F -duality formalism. This is because the partition function $Z(\tau)$ in the S -duality approach has the property

$$Z(\tau) = Z\left(-\frac{1}{\tau}\right) \quad \text{or} \quad Z(\lambda) = Z\left(-\frac{1}{\lambda}\right),$$

as can be deduced from our discussion of Secs. 3 and 5, respectively. In fact, writing symbolically

$$Z(\tau) = \int \exp(iS_{IV}), \quad (69)$$

where S_{IV} is given in (19), for the case of Maxwell theory and

$$Z(\lambda) = \int \exp(iS_X), \quad (70)$$

where S_X is given in (48), for the case of linearized gravity, from the results of Sec. 3 we may establish that (69) has the two limits

$$\int \exp(iS_{III}) \leftarrow \int \exp(iS_{IV}) \rightarrow \int \exp(iS_V), \quad (71)$$

(where S_{III} and S_V are given by (12) and (22), respectively), while from the discussion of Sec. 5 we may establish that (70) gives

$$\int \exp(iS_{IX}) \leftarrow \int \exp(iS_X) \rightarrow \int \exp(iS_{XI}), \quad (72)$$

(where S_{IX} and S_{XI} are given by (43) and (55), respectively). Therefore, one finds that (71) and (72) imply the symmetries $Z(\tau) = Z(-1/\tau)$ and $Z(\lambda) = Z(-1/\lambda)$, respectively.

It has been shown [30] that $Z(\tau)$ also contains the symmetry $Z(\tau) = Z(\tau + 1)$ thereby showing that $Z(\tau)$ is symmetric under the full group $SL(2, Z)$. So, it may appear interesting to see whether F -duality formalism may also be connected with the transformation $\tau \rightarrow \tau + 1$. In what follows we shall outline this possibility.

First we note that, if we consider the infinitesimal transformations (4) and (5), we find that the self-dual (antiself-dual) field strength transforms as

$$\delta^\pm F^{\alpha\beta} = \pm i\beta^\pm F^{\alpha\beta}. \quad (73)$$

Therefore, we discover that the action (12) transforms as

$$\delta S_{III} = i\beta \left\{ (\tau^+) \int d^4x^+ F^{\mu\nu+} F_{\mu\nu} - (\tau^-) \int d^4x^- F^{\mu\nu-} F_{\mu\nu} \right\}. \quad (74)$$

In this case we have left the parameters τ^+ and τ^- unchanged. However, we can obtain similar result if we leave the field strength $F^{\alpha\beta}$ unchanged and we require the parameters τ^+ and τ^- transform as follows:

$$\tau'^+ = \tau^+ + i\beta\tau^+, \quad \tau'^- = \tau^- - i\beta\tau^-. \quad (75)$$

An interesting possibility arises if one considers the particular cases $\beta = \frac{1}{\tau^+}$ or $\beta = \frac{1}{\tau^-}$, leading in any case to the result

$$\tau'^+ = \tau^+ + i, \quad \tau'^- = \tau^- - i, \quad (76)$$

which is similar to the expected form $\tau \rightarrow \tau + 1$.

The result (74) means that the action (12) is not invariant under (73) or (75). However, if one considers the transformations (76), this is not necessarily true for the associated partition function $Z = Z(\tau^\pm)$, namely

$$Z(\tau^\pm) = \int \exp(iS_{III}).$$

In fact the reason for this is that, using (76), one discovers that the expression (74) becomes

$$\delta S_{III} = i \left\{ \int d^4x^+ F^{\mu\nu+} F_{\mu\nu} - \int d^4x^- F^{\mu\nu-} F_{\mu\nu} \right\}, \quad (77)$$

which can be reduced to the θ term

$$\delta S_{III} = \theta \int d^4x F^{\mu\nu*} F_{\mu\nu}. \quad (78)$$

Since from (13) we have $\tau = 1/g^2 + i\theta$, one obtains $\delta\tau = i\delta\theta$ and therefore the prescription (76) implies $\delta\theta = 1$, which means

$$\theta \rightarrow \theta + 1. \quad (79)$$

So, by assuming the smallest possible value for

$$\int d^4x F^{\mu\nu*} F_{\mu\nu},$$

one may recognize that the term $\exp(\delta S_{III})$ leaves the partition function $Z = Z(\tau^\pm)$ invariant.

In Refs. 36 to 38 it is also discussed a kind of F -duality from the point of view of field equations rather than actions. For new directions of research, it may be interesting to establish the precise relations of such references with our formalism.

Finally, in Refs. 30 and 39 it is explained that the action (12) is invariant mod $2\pi n$, not only under the change $\tau \rightarrow \tau + 1$ when M is a spin manifold, but also under the change $\tau \rightarrow \tau + 2$ for any closed four manifold M . It may be interesting for further research to explore what this means in both scenarios, Maxwell theory and linearized gravity.

Acknowledgments

J.A. Nieto would like thank to L. Ruiz and J. Silvas for helpful comments. This work was partially supported by grants PIFI 3.2 and 3.3.

-
1. J.A. Nieto, *Phys. Lett. A* **262** (1999) 274.
 2. C.M. Hull, *Nucl. Phys. B* **583** (2000) 237.
 3. U. Ellwanger, (Gravitational S -duality realized on NUT-Schwarzschild and NUT-de Sitter metrics).
 4. H. Casini, R. Montemayor, and L.F. Urrutia, *Phys. Rev. D* **68** (2003) 065011.
 5. M. Henneaux and C. Teitelboim *Phys. Rev. D* **71**, (2005) 024018.
 6. A. C. Petkou, "Holography, duality and higher-spin theories" (Presented at Workshop on Higher Spin Gauge Theories, Brussels, Belgium, 12-14 May 2004).
 7. U. Ellwanger, "Vanishing cosmological constant via gravitational S -duality", (hep-th/0410265).
 8. S. Deser and D. Seminara, *Phys. Lett. B* **607** (2005) 317.
 9. K.M. Ajith, E. Harikumar, and M. Sivakumar, *Class. Quant. Grav.* **22** (2005) 5385.
 10. V.C. de Andrade, A.L. Barbosa, and J.G. Pereira, *Int. J. Mod. Phys. D* **14** (2005) 1635.
 11. S. Deser and D. Seminara, *Phys. Rev. D* **71** (2005) 081502.
 12. H. Nicolai, "Gravitational billiards, dualities and hidden symmetries", ed. A. Ashtekar (Invited contribution to the volume '100 Years of Relativity Spacetime Structure: Einstein and Beyond').
 13. A. Kleinschmidt and H. Nicolai, *Class. Quant. Grav.* **22** (2005) 4457.
 14. B. Julia, J. Levie, and S. Ray, *JHEP* **0511** (2005) 025.
 15. B.L. Julia, "Electric-magnetic duality beyond four dimensions and in general relativity", (Presented at 23rd International Conference of Differential Geometric Methods in Theoretical Physics, Tianjin, China, 20-26 Aug 2005. Published in *Tianjin 2005, Differential geometry and physics* 266).
 16. C.W. Bunster, S. Cnockaert, M. Henneaux, and R. Portugues, *Phys. Rev. D* **73** (2006) 105014.
 17. S. Cnockaert, "Higher spin gauge field theories: Aspects of dualities and interactions", (Ph.D. Thesis).
 18. A.J. Nurmagambetov, *SIGMA* **2** (2006) 020.
 19. U. Ellwanger, *Class. Quant. Grav.* **24** (2007) 785.
 20. S. de Haro and P. Gao, *Phys. Rev. D* **76** (2007) 106008.
 21. R.G. Leigh and A.C. Petkou, *JHEP* **0711** (2007) 079.
 22. M.A. Vasiliev, *Nucl. Phys. B* **793**, 469 (2008).
 23. M. Henneaux, D. Persson, and P. Spindel "Spacelike Singularities and Hidden Symmetries of Gravity", (arXiv:0710.1818).

24. A.J. Nurmagambetov, *SIGMA* **4** (2008) 022.
25. E.A. Bergshoeff, M. de Roo, S.F. Kerstan, A. Kleinschmidt, and F. Riccioni, "Dual Gravity and Matter", (arXiv:0803.1963).
26. J.A. Nieto, *Mod. Phys. Lett. A* **20** (2005) 135.
27. F. Riccioni and P. West, *JHEP* **0904**, (2009) 051.
28. G. Barnich and C. Troessaert, *JHEP* **0901** (2009) 030.
29. I. Bakas, *Class. Quant. Grav.* **26** (2009) 065013.
30. E. Witten, *Selecta Math.* **1** (1995) 383.
31. S. Deser and C. Teitelboim, *Phys. Rev. D* **13** (1976) 1592.
32. P.A.M. Dirac, *Phys. Rev.* **74** (1948) 817.
33. P.A.M. Dirac, *Int. J. Theor. Phys.* **17** (1978) 235.
34. J.M. Figueroa-O'Farrill, "Electromagnetic duality for children" (1998); <http://www.maths.ed.ac.uk/~7Ejmf/Teaching/Lectures/EDC.pdf>
35. J.A. Nieto, "Connection between the superpotential and G "string" field in linearized gravity", (work in progress, ECFM-UAS,2009).
36. C.M. Hull, *JHEP* **0109** 027 (2001).
37. P. de Medeiros and C.M. Hull, *Commun. Math. Phys.* **235** (2003) 255.
38. P. de Medeiros and C.M. Hull, *JHEP* **0305** (2003) 019.
39. S. Gukov and E. Witten, *Gauge Theory, Ramification, And The Geometric Langlands Program*, (hep-th/0612073).