Unfolding ray trace for plates and wedges

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We present a new procedure to obtain analytic expressions for tracing rays in plates and wedges used edgewise applying the unfolding method for ray tracing, to be able to analyze the behavior of these optical components for transporting light in optical systems. The plates and wedges are useful in transforming the shape and characteristics of light beams, and also for changing the focal ratio or aperture number of these beams. The analytic expressions derived are simple to use in any application.

Keywords: Optical design; ray trace; plates; wedges.

Presentamos un nuevo procedimiento para obtener expresiones analíticas para trazar rayos en placas y cuñas usadas de canto aplicando el método de desdoblaminto para trazar rayos, para analizar el comportamiento de las placas y cuñas para transportar luz en sistemas ópticos. Las placas y las cuñas son útiles en transformar la forma y las característica de los haces de luz. Y también para cambiar la razón focal o número de abertura de esos haces. Las expresiones analítcas derivadas son sencillas de usar en cualquier aplicación.

Descriptores: Diseño óptico; trazo de rayos; placas; cuñas.

PACS: 42.15.Dp; 42.15.Eq; 42.15.-i

1. Introduction

Plates and wedges are generally used facewise as windows, beam splitters, to displace images and beams, and for changing the polarization as well as the direction of rays and focal distances in optical systems, and also, as second and multiple reflection mirrors and some other applications [1,2]. The wedges are normally used as thin prisms in optical systems. For some applications it is necessary to use the plates and wedges edgewise to maximize the transfer of radiation and create a controlled distribution of illumination in optical instruments by changing the form of the entrance image and the beam shape. For example, in slit spectroscopy the entrance image is in general circular; therefore to be able to collect all the light it is necessary to change that form into a rectangular one to pass all the light through the slit of the spectrometer. Also, plates and wedges are used in non-imaging instruments, such as light pipes [3-6]. A glass plate is a smooth, homogeneous, flat, relatively, thin rigid body of uniform thickness. A glass wedge is a piece of a tapered glass plate. To analyze systems composed of plates and wedges used edgewise, it is necessary to have useful analytic expressions; therefore in this paper we develop the procedure to follow up the trajectory of a single straight ray from the entrance on the first surface, through the internal material, up to the exit surface, by unfolding the rays. Usually, given the entrance point of the ray, the successive reflections are calculated by finding the point of reflection where the ray intersects the surfaces, the normals at those points, and the angles using the reflection law. At the end, up to the exit surface, a set of equations are obtained for the point of intersection with the last surface and the exit direction of the rays. One can visualize this processes for a ray in a plate as in Fig. 1. In this procedure all the numerical calculations must be very precise, because in a sequential process the errors propagate very easily. On the other hand, the plates and wedges are amenable for carrying out the ray tracing using the technique of unfolding the rays. The objective of the technique developed here is to simulate the real behavior of the ray trajectory by a single straight ray that is equivalent to the original one; to our knowledge, this is the first time that this type of treatment has been done. This is accomplished by means of virtual rotations of the plates or wedges around the back of one exterior surface of the plate or wedge with respect to the direction of propagation for straightening or unfolding the rays. The unfolding ray trace simplifies the calculations and gives the opportunity to visualize the behavior of the system under study. Another method to study the rays inside plates and wedges, for example for analyzing the transfer of radiation in plates and wedges, is the Tunnel diagram [7,8]. For changing the shape and size of the input images in spectroscopy, in order to maintain the intrinsic resolution and throughput of spectrometers [9], there are different instruments called image slicers [10]. In general these slicers are composed of mirrors and use multiple reflections to change the circular images into rectangular ones [11]. The light lost in those devices is great, due to the multiple reflections. For the case of plates and wedges the internal reflections are total; therefore, the light lost is only due to absorption in the glass of the plate or wedge, hence is very important in the study of low intensity sources as in astronomical spectroscopy. Another novel characteristic of the results that are presented here is that, with the initial data of the entrance ray, and the knowledge of the parameters of the wedge or plate, a set of formulas can be used directly to obtain the exit data of the ray and the number of reflections without any additional calculations.

In what follows, in Sec. 2 we present the unfolding method of ray tracing. Section 3 introduces the mathematical procedure using the conventional ray tracing techniques in order to have a set of equations for the exit ray; in plates Sec. 3.1, symmetric wedges Secs. 3.2, 3.3, 3.4, and 3.5 for

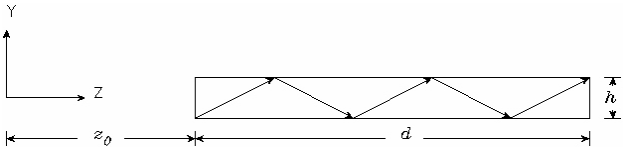

FIGURE 1. Geometry of a plate seen sideways with dimensions d in the direction of ray propagation, h in the perpendicular direction, and t in the other perpendicular direction.

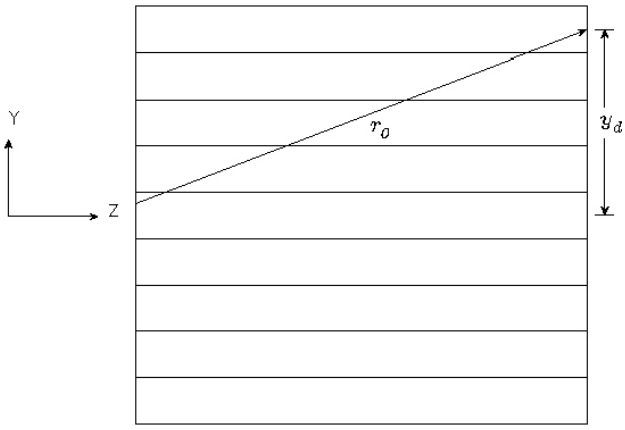

FIGURE 2. Stack of plates for unfolding the rays in our procedure, to produce a rectangular form.

different configurations; and asymmetric wedges Sec. 3.6. Some examples using the equations derived are shown in Sec. 4. Section 5 presents the conclusions.

2. Background

Any optical system that includes a number of mirrors can best be analyzed by unfolding the optical rays [12]. This is accomplished by removing the reflecting plane mirrors and replacing them with an equivalent aperture of the same size as the mirrors for straighting the optic axis. In the case of non-dispersing prisms we can substitute the prism by an equivalent plane-parallel plate that behaves in the same way as the prism. It is necessary to rotate the prism around itself with respect to the back surface of the prism to form a parallel plate made of two prisms, let's say, one prism on top of the other [13]. In case of plates and wedges, the unfolding geometry is formed by rotating the real plate (wedge) virtually around the top surface of the same plate (wedge) with respect to the direction of ray propagation to obtain a virtual plate (wedge) on top of the original one. The rotated plate (wedge) is in turn rotated now around the top of the last surface, and so on, in order to obtain a virtual stack of plates (wedges). The same procedure is used to produce the plates (wedges) below the original plate (wedge). Each plate (wedge) is turned around with respect to the one before and the same procedure used before is continued. This geometry, the stack of plates (wedges) is important for finding the correct directions and exit points in the ray trace. This procedure produces the system of virtual plates or wedges used for the mathematical analysis of our method and shown in Fig. 2 for parallel plates.

3. Mathematical Procedure

3.1. Parallel Plates

We develop the procedure first for the case of a plane-parallel plate for simplicity and because it shows all the peculiarities of the technique. In Fig. 1 we show the plate with length d in the direction (z) of ray propagation, thickness h in the (y) direction and width t in the (x) direction for one ray that travels inside the plate making internal reflections. The geometry of the unfolded ray is shown in Fig. 2. Instead of using the plate directly to trace a single ray as in Fig. 1, we use the virtual stack of plates given in Fig. 2.

A ray in three dimensions starting at the point (x_0,y_0,z_0) is represented by the following expressions:

$$\begin{aligned} x &= x_0 + \lambda l, \\ y &= y_0 + \lambda m, \\ z &= z_0 + \lambda n, \end{aligned} \tag{1}$$

where l, m and n are the direction cosines of the ray and λ is a parameter that gives the length of the ray. From Fig. 2 the ray enters the plate at the point (x_0,y_0,z_0), crosses several planes and leaves out the stack in some point on the exit surface. A plane in three dimensions is given by

$$Ax + By + Cz - D = 0, \tag{2}$$

and for the back surface $A = B = 0$; therefore, from Fig. 2 we obtain

$$z = \frac{D}{C} = d, \tag{3}$$

and from the third Eq. (1) we find

$$\lambda = \frac{d - z_0}{n}. \tag{4}$$

Knowing the height where the ray intersects the last surface $z = d$,

$$y_d = y_0 + \lambda m. \tag{5}$$

We are interested in finding out the number of reflections in order to know the exit ray direction, and this problem is solved as follows. The number of crossings that the ray makes with the surfaces gives us the direction of the ray after leaving the end of the plate. If the number is odd, the ray changes direction with respect to the entrance direction and with respect to the local normal of the exit surface. Also, this number of crossings permits us to calculate the point where the ray leaves the plate. The number of virtual surface crossings, or reflections in the original plate, is easily found to be

$$N = \left[\frac{1 + 2\frac{|y_d|}{h}}{2} \right], \tag{6}$$

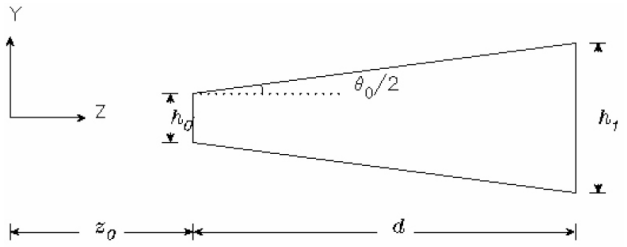

FIGURE 3. Geometry of a wedge seen sideways with dimensions d in the direction of ray propagation, h_0 in the entrance side of the wedge, h_1 in the exit side of the wedge, and t in the perpendicular direction.

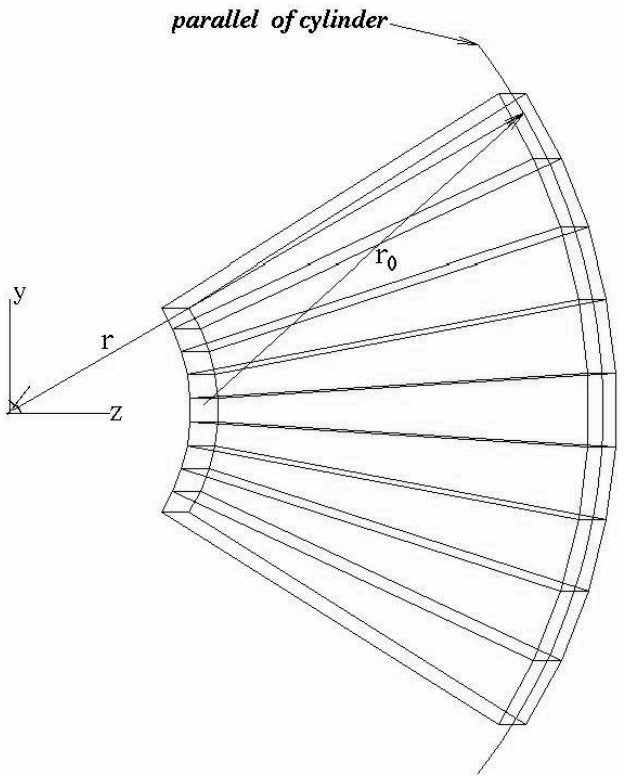

FIGURE 4. Stack of wedges for unfolding the rays in our procedure.

where the square brackets means truncation of the result. Therefore, the exit point is found using

$$y_e \;=\; (-1)^N(y_d \;\mp\; Nh), \tag{7}$$

where the plus sign is used for negative m, and the direction cosines of the exit ray are

$$l,\; (-1)^N m,\; n. \tag{8}$$

For the special cases of rays that do not have any inclination, the height is $y_e \;=\; y_d$. As a result of this section, the equations (4), (5), (6), (7) can be used sequentially, without any additional ray tracing, in order to find out information about the exit ray, crossing the plate, with the data for the incident ray and plate.

3.2. Symmetric wedges

We define the symmetric wedge by its length d in the direction of ray propagation (z), entrance height h_0, exit height h_1, and width t shown in Fig. 3. For symmetric wedges there are several cases: the first one is when the smaller face, h_0, is the entrance aperture, (see Sec. 3.3) the second one is when the entrance aperture is the other face, h_1 (see Sec. 3.4); and the last one is when the wedge is seen sideways with respect to the other two cases (see Sec. 3.4). Lastly, there exists the case of an asymmetric wedge, to be analyzed in Sec. 3.6.

3.3. First case

For the first case, the geometry of the unfolding arrangement is shown in Fig. 4. In order to know where the ray intersects the exit surface, we must initially make an approximation for that position by means of a circumscribed cylinder to the stack of wedges shown in Fig. 4. For clarity we show only the parallel of the cylinder through the point of intersection. From the geometry, we find the point of intersection of the ray $\mathbf{r}_0$ with the cylinder, and trace the position vector $\mathbf{r}$ to that point. The equation for the cylinder is given by

$$r^2 \;=\; z^2 \;+\; y^2, \tag{9}$$

where the radius of the circumscribed cylinder from Fig. 4 is

$$r = \frac{z_0 \;+\; d}{\cos\left(\frac{\theta_0}{2}\right)}, \tag{10}$$

where θ_0 is the angle of the wedge and is given by

$$\tan\left(\frac{\theta_0}{2}\right) \;=\; \frac{h_1 \;-\; h_2}{2d}, \tag{11}$$

and by substitution of Eqs. (1) into Eq. (9) we obtain

$$\lambda \;=\; \frac{(y_0m \;+\; z_0n) \;\pm\; [(y_0m \;+\; z_0n)^2 \;-\; (m^2 \;+\; n^2)(m^2 \;+\; n^2 \;-\; r^2)]^{\frac{1}{2}}}{(m^2 \;+\; n^2)} \tag{12}$$

Now the position vector of the point of intersection with the cylinder is given by

$$x = \lambda' l', \qquad y = \lambda' m', \qquad z = \lambda' n', \tag{13}$$

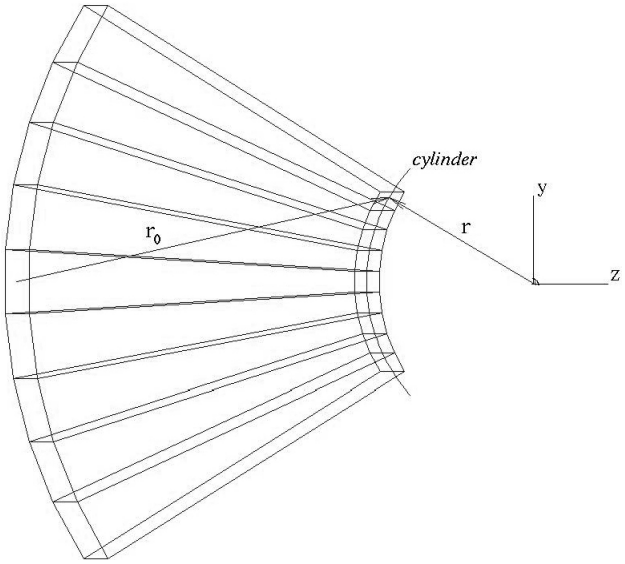

FIGURE 5. Wedge with entrance side greater than the exit side, to produce a rectangular output.

where l', m' and n' are the position vector direction cosines, with $\lambda' = r$ from Fig. 4 and therefore, from Eqs. (1) and (13), we obtain

$$l' = \frac{x_0 + \lambda l}{r}, \quad m' = \frac{y_0 + \lambda m}{r}, \quad n' = \frac{z_0 + \lambda n}{r}; \quad (14)$$

with m' we can find, from Eqs. (13), where the ray intersects the cylinder. Thus we can calculate the point on the cylinder, and in this way calculate the angle θ that the position vector makes with the optical axis. With this angle θ and with the wedge angle θ_0, one can calculate the number of crossings that the ray makes through the surfaces of the stack, or internal reflections in the original wedge, before leaving the last surface as

$$N = \left[\frac{1 \ + \ 2\frac{|\theta|}{\theta_0}}{2}\right], \quad (15)$$

where again the square bracket means truncation of the result. Now we have to rotate around the x axis the wedge the angle $\alpha = N\theta_0$ to know where the ray intersects the last plane of the given wedge. The last plane of the original wedge is given by

$$z \ = \ z_0 \ + \ d. \quad (16)$$

A rotation around the *x* axis by the angle α is represented by

$$\begin{aligned} x &= x', \\ y &= y'\cos(\alpha) \ - \ z'\sin(\alpha), \\ z &= y'\sin(\alpha) \ + \ z'\cos(\alpha). \end{aligned} \quad (17)$$

By substitution of the third of Eqs. (17) into Eq. (16) (for simplicity we neglect the primes, we obtain the rotated plane

$$By \ + \ Cz \ - \ D \ = \ 0, \quad (18)$$

with

$$B = \sin(\alpha), \qquad C = \cos(\alpha), \qquad D = z_0 \ + \ d. \quad (19)$$

Substituting Eq. (1) into Eq. (18) produces

$$\lambda \ = \ \frac{D \ - \ By_0 \ - \ Cz_0}{Bm \ + \ Cn}, \quad (20)$$

which together with Eqs. (1), can be used to find the point where the ray intersects the inclined back plane of the corresponding wedge. Having found the number of crossings and the point of intersection of the ray with some back plane, it is necessary to calculate the real exit point of the ray over the back plane of the original wedge, and for that purpose we have to fold back the virtual wedges onto the real one by the negative of the angle α. Therefore, the exit position of the ray is

$$\begin{aligned} x_x &= x, \\ y_x &= \cos(\alpha)y \ + \ \sin(\alpha)z, \\ z_x &= -\sin(\alpha)y \ + \ \cos(\alpha)z. \end{aligned} \quad (21)$$

The ray trace is accomplished using Eqs. (1), (15), (19), (20) and (21) together with the angle α. The exit direction of the ray is found by rotating by the negative of the angle α, namely

$$m_x \ = \ (-1)^N \cos(\cos^{-1}(m) \ \mp \ \alpha), \quad (22)$$

where the plus sign is applied when the initial m is negative. As can be seen, the results for a wedge are a little more cumbersome.

3.4. Second case

Now we develop the equations for the second case, a wedge with the entrance surface greater than the exit surface, shown in Fig. 5. We follow the same procedure as in the last case. Substituting Eqs. (1) into Eq. (9) for the circumscribed cylinder produces

$$\lambda \ = \ \frac{-(y_0m \ + \ z_0n) \ \pm \ [(y_0m \ + \ z_0n)^2 \ - \ (m^2 \ + \ n^2)(y_0^2 \ + \ z_0^2 \ - \ r^2)]^{\frac{1}{2}}}{(m^2 \ + n^2)}, \quad (23)$$

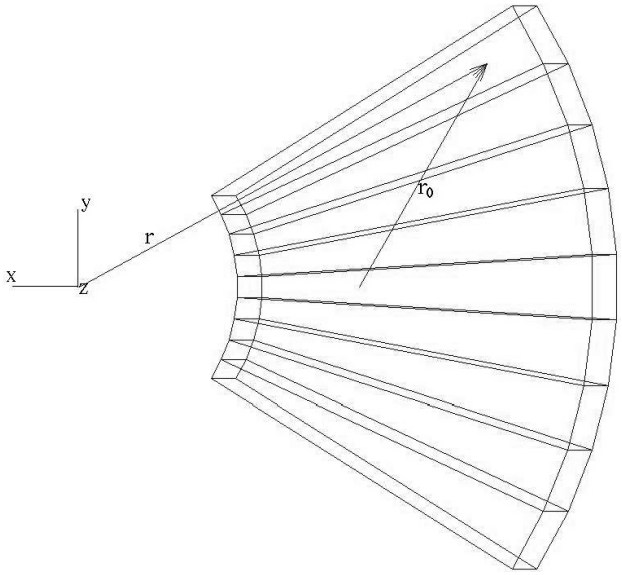

FIGURE 6. Transversal wedge, to produce a trapezoidal output.

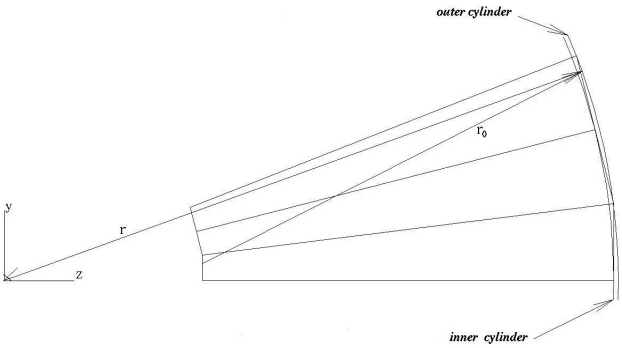

FIGURE 7. Asymmetric wedges.

where the radius of the cylinder is now given by

$$r = \frac{z_0 - d}{\cos(\frac{\theta_0}{2})}. \tag{24}$$

As in the other case $\lambda' = r$; therefore, Eqs. (14) apply together with Eqs. (13) to find the exit point of the ray. The exit direction is given by Eq. (22). Therefore, these two cases are equal except that one has to change the signs of Eqs. (23) and (24) to obtain Eqs. (12) and (10), respectively, from the first case.

3.5. Third case

If one is interested in forming an image with a trapezoidal shape, a wedge seen perpendicular to the cases discussed above as is shown in Fig. 6. This case is similar to the formulation for a plane-parallel plate given above. The ray intersects the back surface and one can find this point as in the case of a plate. The difference with the case of a plate is that the position vector to that point is found in order to calculate the direction cosines and the angle θ as in the case of the wedges. With the angle found above and the angle of the wedge, we can find the number of crossings of the ray through the wedges as was done before for the two previous cases.

3.6. Asymmetric wedges

Another case of interest is a non-symmetrical wedge that sometimes is easier to construct because it has only one inclined plane. The stack of virtual wedges is shown in Fig. 7. The analysis is similar to those carried out above for the symmetric wedge. Then we follow the same procedure: a ray is traced through the stack, producing several crossings of the surfaces, and arrives and intersects with an inscribed cylinder (inner cylinder) and then a circumscribed one (outer cylinder) although for clarity we show only the parallels through the intersection point. This is different from the previous cases for the symmetrical wedges. In this way one can ensure that the number of crossings is the correct one. From that we can say over which surface the ray will exit the wedge. We follow the same procedure as for the symmetric wedge but now with two cylinders. Eq. (9) is used for the two cylinders with radii

$$r_c = \frac{z_0 + d}{\cos(\theta_0)}, \tag{25}$$

and

$$r_i = z_0 + d, \tag{26}$$

for the circumscribed and inscribed cylinders, respectively. A ray is traced with an equation similar to Eqs. (1) and the intersection with the cylinders is found with

$$\lambda_1 = \frac{(y_0 m_1 + z_0 n_1) \pm [(y_0 m_1 + z_0 n_1)^2 - (m_1^2 + n_1^2)(m_1^2 + n_1^2 - r_c^2)]^{\frac{1}{2}}}{(m_1^2 + n_1^2)}, \tag{27}$$

and

$$\lambda_2 = \frac{(y_0 m_2 + z_0 n_2) \pm [(y_0 m_2 + z_0 n_2)^2 - (m_2^2 + n_2^2)(m_2^2 + n_2^2 - r_i^2)]^{\frac{1}{2}}}{(m_2^2 + n_2^2)}. \tag{28}$$

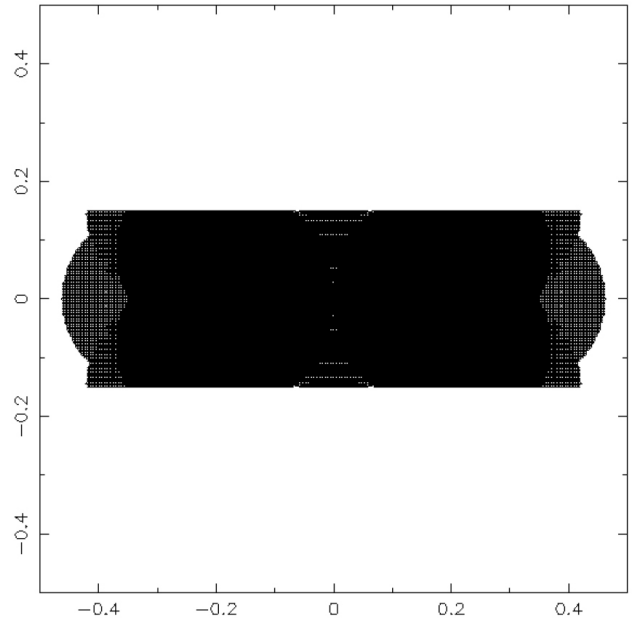

FIGURE 8. The distribution of points on the exit surface of a plate for the distribution of rays given in the text in mm.

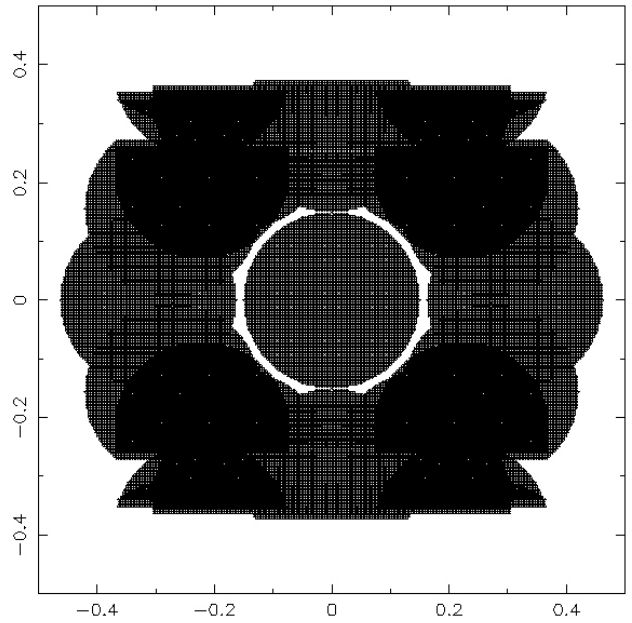

FIGURE 9. The distribution of points on the exit surface of a wedge for the distribution of rays given in the text in mm.

The position vector to these points is found with an equation similar to Eqs. (13), and therefore the direction cosines form an equation similar to Eqs. (14) to obtain the angles of inclination of the position vectors to those points. Thus one can find the number of crossings that the ray makes with the surfaces of the wedges. There are two cases, one for m positive that is given by

$$N = \left[\frac{|\theta|}{\theta_0}\right], \tag{29}$$

where θ is the angle inclination of the position vector to the circumscribed cylinder. If the value is an odd number, one takes this last value, and if the value is an even number, one uses for θ the angle for the inscribed cylinder to evaluate Eq. (29). The other case is when m is negative; then the number of crossings is given by

$$N = \left[1 + \frac{|\theta|}{\theta_0}\right], \tag{30}$$

where we now evaluate Eq. (30) with θ for the inscribed cylinder. If the value is even, we have to evaluate Eq. (30) with θ for the circumscribed cylinder, otherwise we use the value of the inscribed one. Now we follow the same procedure used for the symmetrical wedges, that is, to find the point of intersection of the ray with the rotated wedge and then rotate the wedge back to find the real exit point. Having found the exit point and direction of any ray, we can translate the wedge in the negative y-direction by $h/2$ to center the asymmetric wedge in the optical axis; that is the other difference with respect to the case of the symmetric wedge.

4. Examples

Here we show the unfolding ray trace for a plane-parallel plate with dimensions $h = 0.3$ mm, $d =$ mm and $t = 5$ mm, illuminated by a beam of 17 rays per point. Sixteen rays are distributed in a cone of aperture number 8 and one in the z direction. The points form a circle of radius 0.15 mm and are distributed uniformly. As the number of points and angles are finite, the results show the distribution of points on the exit surface as in Fig. 8. If one uses more points, the unfolding of the image is not seen, and we will see only a black image with round side edges. Also, we show in Fig. 9 the result of the application for a symmetric wedge with dimensions $h_0 = 0.3$ mm, $h_1 = 0.7625$ mm and $d = 5$ mm with an angle $\theta_0 = 3.58°$ for the same distribution of rays as before. Here again one can appreciate the distribution of the finite number of points and angles. These results show how the rays are reflected inside the plates and wedges.

5. Conclusions

A set of equations had been derived for the coordinates and directions of an exit ray, crossing a plate or wedge, just from the knowledge of some data for an incident ray and the dimensions of the plate or wedge. As a matter of fact, the resulting expressions are simple and easy to use; in the cases, for example, of spectroscopy they become very useful.

Acknowledgements

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