Entanglement and control operations in Heisenberg 3-D interactions of two qubits

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Entanglement generated by the Heisenberg model has been studied by several authors in order to understand its relation to the magnetic properties of materials, using mainly particular cases in one or two dimensions for two or more particles. In this work, the evolution of the Heisenberg model is solved for two particles including an inhomogeneous magnetic field in three dimensions, giving a detailed study of the entanglement properties derived from this interaction. Some relations between entanglement and energy or spin are verified, based on known relations for these observables. Finally, some possible quantum control operations are suggested to drive bipartite qubits with an external magnetic field, controlling their evolution into a periodical behavior. These operations become useful to preserve the system properties as well as to transfer information between two parts which can be exploited in engineering applications (*e.g.* quantum computation or quantum information).

Keywords: Heisenberg model; entanglement; quantum control.

El enmarañamiento generado por el modelo de Heisenberg ha sido estudiado por diversos autores con la finalidad de comprender su relación con las propiedades magnéticas de los materiales, usando para ello casos particulares para la interacción entre dos o más partículas en una y dos dimensiones. En este trabajo, la evolución del modelo de Heisenberg es resuelta para tres dimensiones y dos partículas, incluyendo además un campo magnético no homogéneo, dando un estudio detallado de las propiedades del enmarañamiento generado por esta interacción. Algunas relaciones del enmarañamiento con la energía o el espín son revisadas, de acuerdo a propiedades conocidas para estos observables. Finalmente, algunas operaciones de control son sugeridas para qubits bipartitas bajo la acción de campos magnéticos externos induciendo a la evolución hacia un comportamiento periódico. Estas operaciones resultan útiles para conservar las propiedades del sistema o bien para transferir información entre las dos partes que la conforman para aplicaciones útiles ingeniería cuántica (*e.g.* cómputo cuántico o información cuántica).

Descriptores: Modelo de Heisenberg; enmarañamiento; control cuántico.

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1. Introduction

Entanglement is used in quantum computing as a central aspect to improve information processing to take advantage of the special features of quantum mechanics [1, 2]. For this reason, entanglement is the subject of intense research to understand completely its complexity, properties and potential benefits [3–5]. In this last sense, entanglement control is one of the most important aspects [6, 7]; nevertheless its study will not have a complete road map until its quantification and its behavior can be understood.

Nielsen [8] was the first to report entanglement results in magnetic systems based on two spin systems using the Heisenberg model with an external magnetic field. After that, different authors have extended this research for more complex systems involving external parameters (temperature and external field strength) [9, 10], considering different Heisenberg models (XX, XY, XYZ depending on the focus given by each one in order to reproduce calculations related to lattices in one or two dimensions) [11–17].

In this paper we study bipartite systems in three dimensions to learn about entanglement behavior, control operations and information transfer processes in Heisenberg mod-

els, rather than to study lattice properties. These results are extended in a parallel work about control on entangled bipartite qubits [18].

2. Heisenberg interaction and variants

Heisenberg models are motivated mainly by the far-field strength of a magnetic dipole interaction between two particles in which the binding energy is given by:

$$E = \frac{\mu_0}{4\pi r^3}(\vec{\mathbf{m}}_1 \cdot \vec{\mathbf{m}}_2 - 3\vec{\mathbf{m}}_1 \cdot \hat{\mathbf{r}}\vec{\mathbf{m}}_2 \cdot \hat{\mathbf{r}}) \qquad (1)$$

where $\vec{\mathbf{r}}$ is the relative position vector between particles, $\hat{\mathbf{r}}$ its associated unitary vector and $\vec{\mathbf{m}}_i$ is the magnetic momentum of particle i. In addition, a model which relates magnetic momentum to spin is:

$$\vec{\mathbf{m}}_i = \mathbf{g}_i \cdot \vec{\mathbf{s}}_i \qquad (2)$$

where $\mathbf{g}_i$ is a tensor. Combining the last two expressions we obtain:

$$E = \frac{\mu_0}{4\pi r^3} \sum_{j,l=1}^{3} \left(\sum_{i=1}^{3} g_{1ij}g_{2il} - 3 \sum_{i,k=1}^{3} g_{1ij}g_{2kl}r_i r_k \right)$$

$$\times s_{1j}s_{2l} \equiv - \sum_{j,l=1}^{3} J_{jl}s_{1j}s_{2l} \tag{3}$$

Depending on the g_{kij} values, the Heisenberg-like interactions give rise to different models which have been tried by several authors in entanglement studies [12–16]. The original Heisenberg model given by J_{jl} proportional to the identity matrix (see Appendix A) is:

$$E = -J\vec{\mathbf{s}}_1 \cdot \vec{\mathbf{s}}_2 \tag{4}$$

This kind of interactions were first used in statistical physics to describe the magnetic behavior of lattices by Ising [19, 20] in statistical physics and later by Heisenberg [21] in quantum mechanics by introducing $\vec{s}_i$ proportional to Pauli matrices. Works listed before [12–16] and more recent ones show transference and control of entanglement in bipartite qubits [22] and lattices [23,24] but normally focused on describing magnetic properties of materials related to entanglement. Cai [25], on the other hand, has considered a more general model with J_{jl} diagonal in order to study the entanglement relation with local information. In a similar way, this work is focused on the Heisenberg model in three dimensions, in order to explore some entanglement properties which arise between particles including an inhomogeneous external magnetic field in the z direction:

$$H = -J\vec{\sigma}_1 \cdot \vec{\sigma}_2 + B_1\sigma_{1z} + B_2\sigma_{2z} \tag{5}$$

and to prescribe some operations that one could apply in different situations to control the entanglement.

Using an explicit form of the Pauli matrices we obtain in matrix form for (5):

$$H = \begin{pmatrix} B_+ - J & 0 & 0 & 0 \\ 0 & B_- + J & -2J & 0 \\ 0 & -2J & -B_- + J & 0 \\ 0 & 0 & 0 & -B_+ - J \end{pmatrix} \tag{6}$$

where:

$$B_+ = B_1 + B_2, B_- = B_1 - B_2, R \equiv \sqrt{B_-^2 + 4J^2} \tag{7}$$

Our Hamiltonian has the eigenvalues (Fig. 1):

$$E_1 = -J - B_+$$
$$E_2 = -J + B_+$$
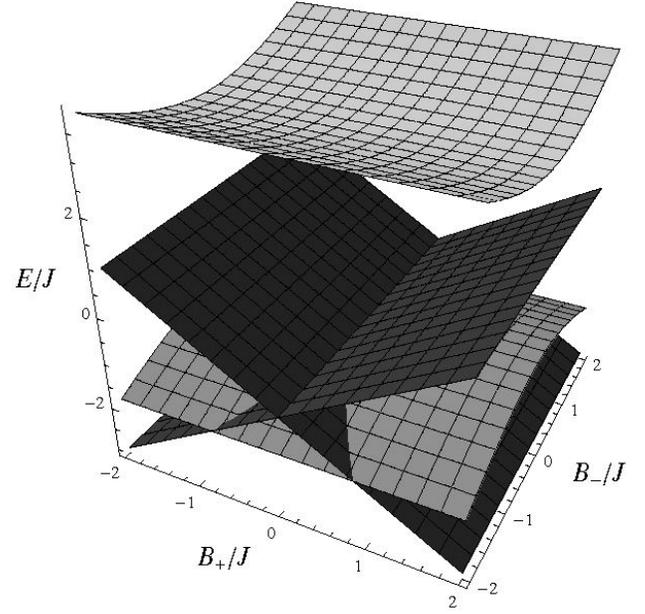$$E_3 = J - R$$
$$E_4 = J + R \tag{8}$$

FIGURE 1. Energy eigenvalues as functions of magnetic field parameters (B_-/J and B_+/J). From E_1/J (darkest) to E_4/J (lightest). Note how E_4/J is separated, suggesting a different behavior which can be verified trough some properties of the corresponding eigenstate.

Precisely by diagonalizing the Hamiltonian (6), it is clear that a more suitable selection of parameters is (to reduce some intermediate expressions in the calculations):

$$b_+ = B_+/R, \qquad b_- = B_-/R \in [-1, 1],$$
$$j = J/R \in [0, 1/2], \qquad t' = Rt \tag{9}$$

We will not drop the prime in the time variable in order to warn the reader about the actual selection of parameters and to avoid misconceptions, because nevertheless we use it [set of variables of Eq. (9)] in almost all calculations, at the end, in final expressions, we preserve the original parameters [set of variables in Eq. (7)], in order to analize some results based on physical parameters.

Thus, we can obtain the evolution operator in Dirac notation, which becomes:

$$U(t') = e^{-it'(b_+ - j)} |0_1 0_2\rangle \langle 0_1 0_2|$$
$$+ e^{-it'j}(\cos t' - ib_- \sin t') |0_1 1_2\rangle \langle 0_1 1_2|$$
$$+ i2je^{-it'j} \sin t' |0_1 1_2\rangle \langle 1_1 0_2|$$
$$+ i2je^{-it'j} \sin t' |1_1 0_2\rangle \langle 0_1 1_2|$$
$$+ e^{-it'j}(\cos t' + ib_- \sin t') |1_1 0_2\rangle \langle 1_1 0_2|$$
$$+ e^{it'(b_+ + j)} |1_1 1_2\rangle \langle 1_1 1_2| \tag{10}$$

3. Evolution and properties in the Ising model for bipartite systems

3.1. Generalities

The corresponding eigenvectors for eigenvalues (8) are:

$$|u_1\rangle = |0_1 0_2\rangle \qquad |u_2\rangle = |1_1 1_2\rangle$$

$$|u_3\rangle = \sqrt{2}j\left(\frac{|0_1 1_2\rangle}{\sqrt{1+b_-}} + \frac{|1_1 0_2\rangle}{\sqrt{1-b_-}}\right)$$

$$|u_4\rangle = \sqrt{2}j\left(\frac{|0_1 1_2\rangle}{\sqrt{1-b_-}} - \frac{|1_1 0_2\rangle}{\sqrt{1+b_-}}\right) \qquad (11)$$

so, these states are invariant under interaction (10). As a consequence, if the magnetic field is homogeneous, then $|u_3\rangle = |\beta_{01}\rangle$ and $|u_4\rangle = |\beta_{11}\rangle$ remain invariant, and $|\beta_{00}\rangle$ and $|\beta_{10}\rangle$ are invariant only in absence of magnetic field (here, $|\beta_{ij}\rangle$ are the usual notation for Bell states, see Ref. 5 on page 25).

3.2. Energy, spin and entanglement

It is well known that Heisenberg-like interactions generate entanglement [12]. Energy and spin are normally related to entanglement. The present section makes some remarks about the relation between energy and spin with entanglement. We will begin from the simplest scenario and move to the most general one.

3.2.1. Absence of magnetic field

From Section A, in the absence of magnetic field, all Bell states are invariant under evolution (10) until a global phase. In addition, one can show by direct calculation that for a separable state $|\psi\rangle$ (see Appendix B):

$$E = \langle\psi| H |\psi\rangle \in [-J, J] \qquad (12)$$

this means that the Hamiltonian operator could be used to define an entanglement witness operator [26] (this is obvious because proportionality between H and $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ in the absence of a magnetic field) since $\langle\beta_{11}|H|\beta_{11}\rangle = 3J$. Indeed, in this case $|0_1 0_2\rangle$, $|1_1 1_2\rangle$ (or alternatively, $|\beta_{00}\rangle$, $|\beta_{10}\rangle$), $|\beta_{01}\rangle$, form an invariant subspace of $\mathcal{H}^{\otimes 2}$ (all states obtained by combining these states become invariant) isolated from $|\beta_{11}\rangle$. In this case, the states are degenerated in a triplet with $E_{1,2,3} = -J$, while for the fourth state, a singlet, we have $E_4 = 3J$. In the absence of a field, non a invariant evolution is only possible by combining both subspaces.

3.2.2. Homogeneous magnetic field

If we add a homogeneous field, two of the first three degenerate eigenstates become unfolded with energies $E_1 = -J - B_+$, $E_2 = -J + B_+$ (depending on the magnetic field, $B = B_+/2$) with separable eigenstates $|0_1 0_2\rangle$

and $|1_1 1_2\rangle$ respectively. On the other hand, the remaining two states $E_3 = -J$ and $E_4 = 3J$ are unchanged and independent of the only invariant Bell states, $|\beta_{01}\rangle$ and $|\beta_{11}\rangle$.

3.2.3. Inhomogeneous magnetic field

In the general case, when the magnetic field is inhomogeneous, energy of the two unfolded separable eigenstates remains unchanged. Nevertheless, the previous maximal entangled eigenstates become just partially entangled with energies $E_3 = J - R \leq -J$, $E_4 = J + R \geq 3J$. Otherwise, one can prove than the last two eigenvectors, $|u_3\rangle$ and $|u_4\rangle$, are not separable unless $|B_-| \to \infty^i$. By direct calculation one can show that for a separable $|\psi\rangle$ (see appendix B):

$$E = \langle\psi| H |\psi\rangle \in [-M, M] \qquad (13)$$

where $M = \text{Max}\{J + |B_-|, |B_+|\}$. In this case, the Hamiltonian is no longer an entanglement witness (even in the homogeneous field case) given that $\langle\beta_{11}|H|\beta_{11}\rangle = 3J + B_- \equiv E_{11}$ and therefore E_{11} together with E_4 do not necessarily fall within the interval given in (13), depending on the magnetic field parameters, B_+ and B_-. In some sense, the complexity and the difference in intensity of the magnetic field appear to destroy the energy-entanglement relation present only in the absence of a field.

The connection between entanglement and the product of two spin operators is well known [27]. In our case, the eigenvalues of $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ are:

$$\Lambda_{1,2,3} = 1, \Lambda_4 = -3 \qquad (14)$$

with eigenvectors:

$$|v_1\rangle = |0_1 0_2\rangle \qquad |v_2\rangle = |1_1 1_2\rangle$$

$$|v_3\rangle = \frac{1}{\sqrt{2}}(|0_1 1_2\rangle + |1_1 0_2\rangle) = |\beta_{01}\rangle$$

$$|v_4\rangle = \frac{1}{\sqrt{2}}(|0_1 1_2\rangle - |1_1 0_2\rangle) = |\beta_{11}\rangle \qquad (15)$$

An important issue is that for $|\psi\rangle$ separable (see Appendix B):

$$\langle\vec{\sigma}_1 \cdot \vec{\sigma}_2\rangle = \langle\psi| \vec{\sigma}_1 \cdot \vec{\sigma}_2 |\psi\rangle \in [-1, 1] \qquad (16)$$

so the $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ operator can be used as an entanglement witness, given that:

$$\langle\beta_{01}| \vec{\sigma}_1 \cdot \vec{\sigma}_2 |\beta_{01}\rangle = -3 \qquad (17)$$

The usefulness of entanglement witnesses in bipartite systems is relative because entanglement is well understood and entanglement entropy is a robust measure to identify entangled states. Nevertheless, states considered before are important inputs for quantum computing, so relations of entanglement behavior with energy and spin should be understood.

The evolution (10) of $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ is only dependent of the field inhomogeneity:

$$
\begin{aligned}
\vec{\sigma}_1 \cdot \vec{\sigma}_2(t') = U^\dagger(t')\vec{\sigma}_1 \cdot \vec{\sigma}_2 U(t') = {}& |0_1 0_2\rangle \langle 0_1 0_2| \\
& - (1 + 8jb_- \sin^2 t') |0_1 1_2\rangle \langle 0_1 1_2| \\
& + 2(1 - 2b_-^2 \sin^2 t' + ib_- \sin 2t') |0_1 1_2\rangle \langle 1_1 0_2| \\
& + 2(1 - 2b_-^2 \sin^2 t' - ib_- \sin 2t') |1_1 0_2\rangle \langle 0_1 1_2| \\
& - (1 + 8jb_- \sin^2 t') |1_1 0_2\rangle \langle 1_1 0_2| \\
& + |1_1 1_2\rangle \langle 1_1 1_2| \qquad (18)
\end{aligned}
$$

and obviously for the energy eigenstates (11), $\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2 \rangle$ is time independent:

$$
\begin{aligned}
\langle u_1 | \vec{\sigma}_1 \cdot \vec{\sigma}_2 | u_1\rangle &= 1 \\
\langle u_2 | \vec{\sigma}_1 \cdot \vec{\sigma}_2 | u_2\rangle &= 1 \\
\langle u_3 | \vec{\sigma}_1 \cdot \vec{\sigma}_2 | u_3\rangle &= 4j - 1 \in (-1, 1] \\
\langle u_4 | \vec{\sigma}_1 \cdot \vec{\sigma}_2 | u_4\rangle &= -4j - 1 \in [-3, -1) \qquad (19)
\end{aligned}
$$

noting that the last eigenstate has values corresponding to non-separable states, verifying the fact that $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ is an entanglement witness [27].

3.3. Entanglement and separability

Using the evolution operator we can verify that while $|0_1 0_2\rangle$, $|1_1 1_2\rangle$ are invariant, the states:

$$
\begin{aligned}
U(t') |0_1 1_2\rangle = {}& e^{-ijt'} ((\cos t' - ib_- \sin t') |0_1 1_2\rangle \\
& + 2ij \sin t' |1_1 0_2\rangle) \\
U(t') |1_1 0_2\rangle = {}& e^{-ijt'} ((\cos t' + ib_- \sin t') |1_1 0_2\rangle \\
& + 2ij \sin t' |0_1 1_2\rangle) \qquad (20)
\end{aligned}
$$

have an interesting behavior. Calculating the Schmidt coefficients, we note that these states are maximally entangled when:

$$
\begin{aligned}
\cos^2 t' + b_-^2 \sin^2 t' &= 4j^2 \sin^2 t' \\
\Rightarrow b_-^2 &= -4j^2 \cos 2t' \qquad (21)
\end{aligned}
$$

From this we get (in terms of the non-normalized variables) the times when maximal entanglement is reached:

$$
t_a = \frac{1}{2R} \arccos\left(-\frac{B_-^2}{4J^2} \right) + n\frac{T}{2}
$$

$$
t_b = \frac{\pi}{R} - \frac{1}{2R} \arccos\left(-\frac{B_-^2}{4J^2} \right) + n\frac{T}{2}, n \in \mathbb{Z} \qquad (22)
$$

with $T = 2\pi/R$, the period of the process^{ii}. Note that we get maximum entanglement only if $B_-^2/4J^2 \leq 1$. The resulting

states at these times are respectively (dropping some unitary factors):

$$
U(t_{a,b}) |0_1 1_2\rangle = \frac{1}{\sqrt{2}} \Bigg(e^{i \arctan \left(\frac{\sqrt{4j^2 - b_-^2}}{b_-} \operatorname{sgn}(\tan t'_{a,b}) \right)}
$$
$$
\times |0_1 1_2\rangle + |1_1 0_2\rangle \Bigg)
$$

$$
U(t_{a,b}) |1_1 0_2\rangle = \frac{1}{\sqrt{2}} \Bigg(e^{-i \arctan \left(\frac{\sqrt{4j^2 - b_-^2}}{b_-} \operatorname{sgn}(\tan t'_{a,b}) \right)}
$$
$$
\times |1_1 0_2\rangle + |0_1 1_2\rangle \Bigg) \qquad (23)
$$

where $\operatorname{sgn}(x) = x/|x|$ if $x \neq 0$. They reach four times the maximal entanglement in each cycle, except if $|B_-/2J| = 1$. In the last case, the maximally entangled states reached are $|\beta_{01}\rangle$, $|\beta_{11}\rangle$ (depending on the sign of B_-).

Figure 2 shows the entanglement entropy versus time for $|0_1 1_2\rangle$ and $|1_1 0_2\rangle$ under this evolution (graphs are the same for both states). Note the behavior of their maximal value for different field strengths, summarizing our findings for these initial states.

Conversely, if we start with the Bell states at these times they become separable. We do not show the evolution for initial maximally entangled states $|\beta_{01}\rangle$ and $|\beta_{11}\rangle$, but the behavior is analogous. When $B_-^2/4J^2 \leq 1$ these states reach
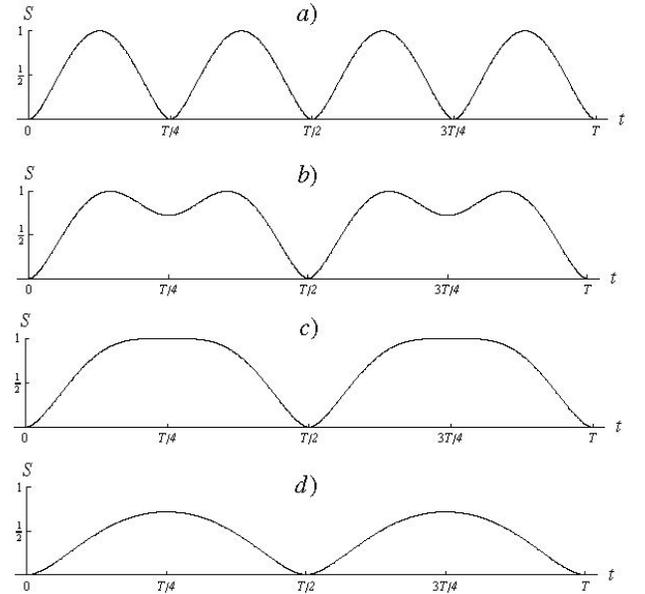

FIGURE 2. Entropy of entanglement behavior for the evolution of $|0_1 1_2\rangle$ and $|1_1 0_2\rangle$ for different values of $|B_-/J|$ in which maximally entangled states can be reached. Using $J = 1$, the periods shown are different: a) $|B_-/J| = 0 \Rightarrow T = \pi$, b) $|B_-/J| = 1 \Rightarrow T = 2\pi/\sqrt{5}$, c) $|B_-/J| = 2 \Rightarrow T = \pi/\sqrt{2}$, d) $|B_-/J| = 4 \Rightarrow T = \pi/\sqrt{5}$. For higher values, as in d), only partially entangled states are reached. Note in c), the limit case, a maximally entangled state, precisely $|\beta_{01}\rangle$ or $|\beta_{11}\rangle$. Note too that while magnetic field is more inhomogeneous the process is faster but far away of limit case $|B_-/J| = 2$, the maximum of entropy reached goes to zero.
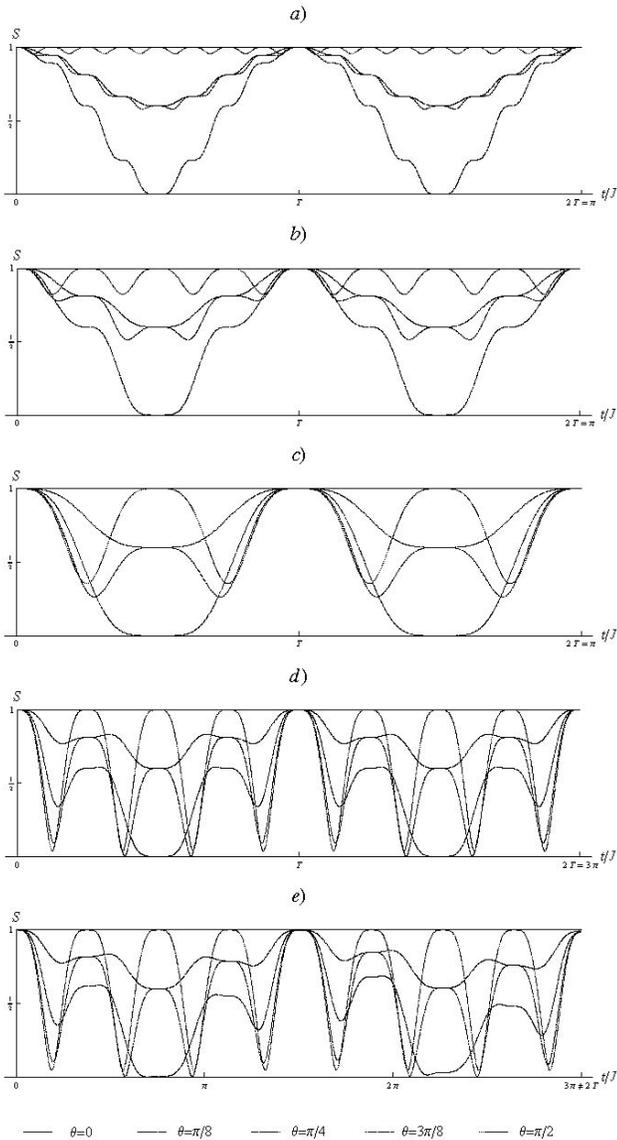
FIGURE 3. Entanglement evolution for different values of θ and $j = J/R \in [0, 1/2]$. Periodic behavior is only possible for rational j: a) 1/16, b) 1/8, c) 1/4 and d) 3/8. By increasing j we increase the number of oscillations of the entropy of entanglement, while for $\theta = \pi/4$ the maximum variation is reached in all cases. Evolution for irrational j is shown in e), with $j = 1/\sqrt{7} \approx 3/8$, showing a non-periodic behavior.

separable states, otherwise they reach partially entangled states. In the same way, they are invariant when $B_- \to 0$.

Note that the periodic behavior shown by the initial states (23) is only a partial view of the phenomenon. This behavior is inherited from Rabi oscillations which are present in this Hamiltonian, generating a swap between spin states. Since different energy eigenvalues do not necessarily have rational quotients, the evolution is not periodic in general. For example, let us take as initial state:

$$|\psi\rangle = \sin\theta \, |\beta_{01}\rangle - \cos\theta \, |\beta_{10}\rangle \qquad (24)$$

where θ is a parameter that varies from 0 to $\pi/2$. With a situable choice of θ we can either get a maximally entangled state ($\theta = 0, \pi/2$) or a partially entangled state otherwise. We find that the Schmidt coefficients become the following after time t':

$$\lambda_{1,2} = \frac{1}{2}\Big(1 \pm \big(16j^2(1-4j^2)\sin^4 t' \sin^4\theta + \sin^2 2\theta(\sin^2 2jt'$$

$$+ 4j^2 \sin^2 t' \cos 4jt' - j\sin 2t \sin 4jt')\big)^{\frac{1}{2}}\Big) \qquad (25)$$

Here, $|b_-| = \sqrt{1-4j^2}$ has been used. Figure 3 shows the entanglement evolution of this state for different values of θ and j, which exhibits the properties of the entanglement entropy in this interaction versus time, specially for separable states.

Appendix C shows a detailed calculation for the entanglement evolution properties and conditions for periodicity of separability in the general case, which are not included here because of their length. Some interesting results about separability recurrence can be obtained in the non-periodical case. In general, for $j \in \mathbb{Q}'$, there is a non-periodic behavior for that recurrence and the continuous preservation of separability recurrence coming from the nearest $j \in \mathbb{Q}$ cases is not always fulfilled (it means that prescriptions for repetitive separability coming from rational j does not remain for closest irrational j). This result is in agreement with some numerical works about separability for ground states in multiqubits systems [28] or ground state factorization [29,30] for XYZ-type Hamiltonians. These works use a more local approximation for the chain interaction than [31], which is centered on mean-field solutions. All of them agree with this work about recurrence of separable states. However, this recurrence is sensible in regard to the interaction strength.

By comparison with the results of the previous subsection, one can calculate the following using (18) for $|0_1 1_2\rangle$, $|1_1 0_2\rangle$:

$$\langle 0_1 1_2| \, \vec{\sigma}_1 \cdot \vec{\sigma}_2(t') \, |0_1 1_2\rangle = -(1 + 8jb_- \sin^2 t) \in [-3, 1]$$

$$\langle 1_1 0_2| \, \vec{\sigma}_1 \cdot \vec{\sigma}_2(t') \, |1_1 0_2\rangle = -(1 - 8jb_- \sin^2 t) \in [-3, 1] \quad (26)$$

with $8jb_- \in [-2, 2]$. In particular, $\langle \vec{\sigma}_1 \cdot \vec{\sigma}_2(t) \rangle = -3$ when $8jb_- = 2$ for the first state and $8jb_- = -2$ for the second state, both for $t = 2n + 1/2R\pi$, $n \in \mathbb{Z}$. These are precisely the times when these states become $|\beta_{11}\rangle$, so these initially separable states reach an entangled stage by the action of magnetic field. For the same reason an invariant entangled state in the absence of a magnetic field, such as $|\beta_{01}\rangle$ or $|\beta_{11}\rangle$, becomes a separable state in an inhomogeneous magnetic field.

4. Elementary procedures of control

4.1. Evolution loops

Rabi oscillation control has been studied for different applications [32, 33]. Our Hamiltonian could be understood in different ways for applications. For example, non-periodical

behavior of states (with or without parasite fields) could be driven into periodical behavior by adding an extra magnetic field. Otherwise, a magnetic field can be used to drive the information exchange of particles. Some more concrete applications are presented in Ref. 18.
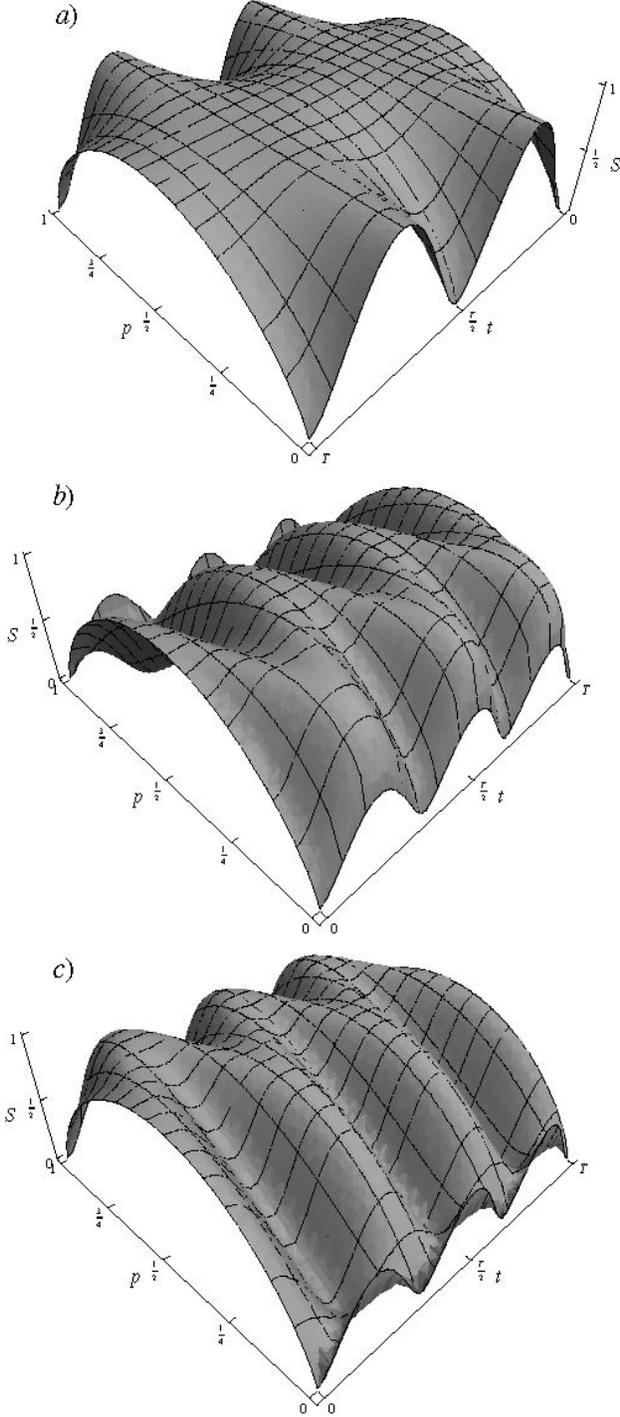

a)

b)

c)

FIGURE 4. Time evolution of entanglement under evolution loops in one period for different values of p. a) The case $m, n, s = 1$ for $|\varphi_\pm\rangle$; the case $n = 3, m = 0, s = 2$ for b) $|\varphi_+\rangle$ and c) $|\varphi_-\rangle$. Note that the period for S is smaller than T, the evolution loop period.

Evolution loops were introduced by Mielnik [34] and they were applied and extended in other directions of control

by several authors [35–37] as simple operations to pursue the specific behavior of quantum systems. Basically, note that we do not introduce here any stochastic element as considered in Refs. 6 and 7.

An evolution loop effect is a dynamical process produced by a cyclic Hamiltonian with period T, if at each time T, the evolution operator takes the form:

$$U(t = NT) = e^{-i\phi}I \qquad (27)$$

where $N \in \mathbb{Z}$, I is the identity operator and ϕ is an arbitrary global phase. Looking at the evolution operator (10), we note a first condition: T has to be a multiple integer of π/R making zero the off-diagonal terms. A second condition is needed for the diagonal terms in order to fit them into the form (27). As a result, we get for the non normalized parameters:

$$B_+ = \frac{2(s-m)J}{m+s-n},$$

$$B_- = \pm\frac{2J\sqrt{(2n-m-s)(m+s)}}{m+s-n},$$

$$T = \frac{(m+s-n)\pi}{2J} \qquad (28)$$

with $n, m, s \in \mathbb{Z}$ and $0 < n < m + s \le 2n$, from which we obtain:

$$U(T) = (-1)^n e^{-iJT}I \qquad (29)$$

It is clear that if $k \in \mathbb{Z}^+$, then all cases with: $m, s, n, T \to km, ks, kn, kT$ are physically equivalent. The fastest process, with $m + s - n = 1$, requires stronger fields in general, although $B_+ = B_- = 0$ is possible by choosing $n, m, s = 1$. Figure 4 shows the entanglement evolution for the family of states:

$$|\varphi_\pm\rangle = \sqrt{p}\,|0_1 1_2\rangle \pm \sqrt{1-p}\,|1_1 0_2\rangle\,, p \in [0, 1] \qquad (30)$$

under these evolution loops, illustrating this effect and showing that entanglement evolution can have a shorter period than the evolution loop.

4.2. Information transfer

Another control operation which could be induced is the information transfer between particles. Suppose that two particles in separate states begin to interact as in (5). Is it possible that after some time T, the particle states become exchanged? In order to get this effect, the evolution operator should have the form:

$$U(T) = e^{-i\phi}I_{1\leftrightarrow 2} \qquad (31)$$

where ϕ is some global phase and $I_{1\leftrightarrow 2}$ is the unitary exchange operator between particles 1 and 2:

$$I_{1\leftrightarrow 2} = |0_1 0_2\rangle \langle 0_1 0_2| + |0_1 1_2\rangle \langle 1_1 0_2|$$
$$+ |1_1 0_2\rangle \langle 0_1 1_2| + |1_1 1_2\rangle \langle 1_1 1_2| \qquad (32)$$
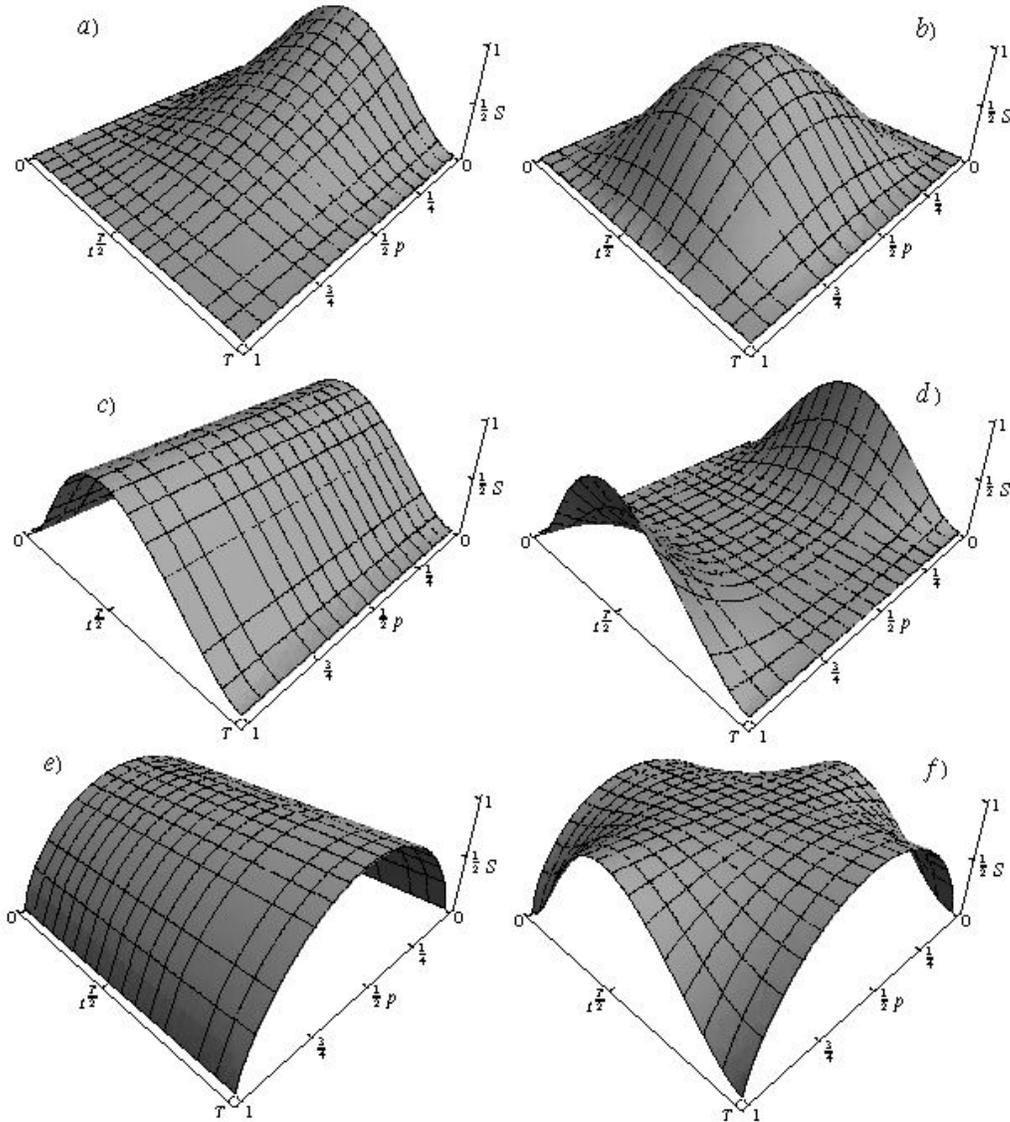
FIGURE 5. Time evolution of entanglement under information transfer depending on parameter p, using $m, n = 0$. For the initial states:
a) $|\varphi_a\rangle = (\sqrt{p}\,|0_1\rangle + \sqrt{1-p}\,|1_1\rangle) \otimes |0_2\rangle$, b) $|\varphi_b\rangle = (\sqrt{p}\,|0_1\rangle + \sqrt{1-p}\,|1_1\rangle) \otimes (\sqrt{p}\,|0_2\rangle - \sqrt{1-p}\,|1_2\rangle)$,
c) $|\varphi_c\rangle = (\sqrt{1-p}\,|0_1\rangle + \sqrt{p}\,|1_1\rangle) \otimes (\sqrt{p}\,|0_2\rangle - \sqrt{1-p}\,|1_2\rangle)$, d) $|\varphi_d\rangle = (\sqrt{1-p}\,|0_1\rangle + \sqrt{p}\,|1_1\rangle) \otimes (\sqrt{p}\,|0_2\rangle + \sqrt{1-p}\,|1_2\rangle)$,
e) $|\varphi_e\rangle = \sqrt{p}\,|0_1 0_2\rangle + \sqrt{1-p}\,|1_1 1_2\rangle$, f) $|\varphi_f\rangle = \sqrt{p}\,|0_1 1_2\rangle + \sqrt{1-p}\,|1_1 0_2\rangle$.

This effect could, for example, induce the following information transfer:

$$U(T)(\alpha\,|0_1\rangle + \beta\,|1_1\rangle) \otimes |\psi\rangle_2$$
$$= e^{i\phi}\,|\psi\rangle_1 \otimes (\alpha\,|0_2\rangle + \beta\,|1_2\rangle) \qquad (33)$$

Fitting the Ising evolution operator to (33), we obtain the following conditions for non-normalized parameters:

$$B_+ = 2B_z = \frac{8Jm}{2n+1},$$
$$B_- = 0,$$
$$T = \frac{(2n+1)\pi}{4J} \qquad (34)$$

with $n \in \mathbb{Z}^+, m \in \mathbb{Z}$ so that the evolution operator becomes:

$$U(T) = i(-1)^n e^{-iJT} I_{1 \leftrightarrow 2} \qquad (35)$$

Note that this phenomenon happens only with homogeneous fields. In addition this effect could happen without a magnetic field (choosing $m = 0$), including the fastest process, $n = 0$. In general, repeating this process two times, we get an evolution loop. This effect is similar to other results for Heisenberg chains with XX and XY Hamiltonians presented for fluxes with homonymous decomposition [38, 39].

Figure 5 exhibits the time evolution of entanglement for different initial separable states for one period as a function of parameter p. Graphs suggest that in the case of initial separable states, the process requires intermediate entangled states.

The same behavior is observed for e) and f) (initially entangled states): in the first case, entropy of entanglement remains unchanged, while in the second case, a weaker initial entanglement gives rise to higher entanglement in intermediate stages.

5. Conclusions

The interaction between a single pair of qubits is rich in complexity, especially when inhomogeneous magnetic fields are introduced. Non-periodical behavior is one of the principal aspects of complexity, so generalization to larger Heisenberg chains is not easy, specifically those related to entanglement. System properties for periodical behavior cases could be exploited by quantum control, with some useful features emerging as a result of information transfer, which can be driven by an external magnetic field (related to Ref. 25). The non-linear features of these strings may provide other noteworthy properties. Controlled generation of entanglement or separability with this kind of interaction is possible by the selection of an adequate external magnetic field, so the study of control to drive this kind of systems in order to transfer information in larger chains or rings or to create some input states for quantum information should be considered an extension of the procedures presented here.

Intermediate increasing of entanglement in an exchange information process should be studied in perspective to understand the deep meaning of this quantum phenomenon.

Acknowledgements

APPENDIX

A. Reliability of symmetric Ising-Heisenberg models

A correct selection of parameters g_{kij} shows that (4) is always possible. For example, we can select a symmetrical relative position of particles: $\vec{r} = (1/\sqrt{3})(1, 1, 1)$ together with:

$$0 = -g_{22}(g_{11} + g_{31}) - g_{12}(g_{21} + g_{31}) - (g_{11} + g_{21})g_{32}$$

$$0 = -g_{23}(g_{11} + g_{31}) - g_{13}(g_{21} + g_{31}) - (g_{11} + g_{21})g_{33}$$

$$0 = -g_{23}(g_{12} + g_{32}) - g_{13}(g_{22} + g_{32}) - (g_{12} + g_{22})g_{33}$$

$$g_{12} = -c, g_{13} = c, g_{23} = -c$$

$$g_{21} = c, g_{31} = -c, g_{32} = c \tag{A.1}$$

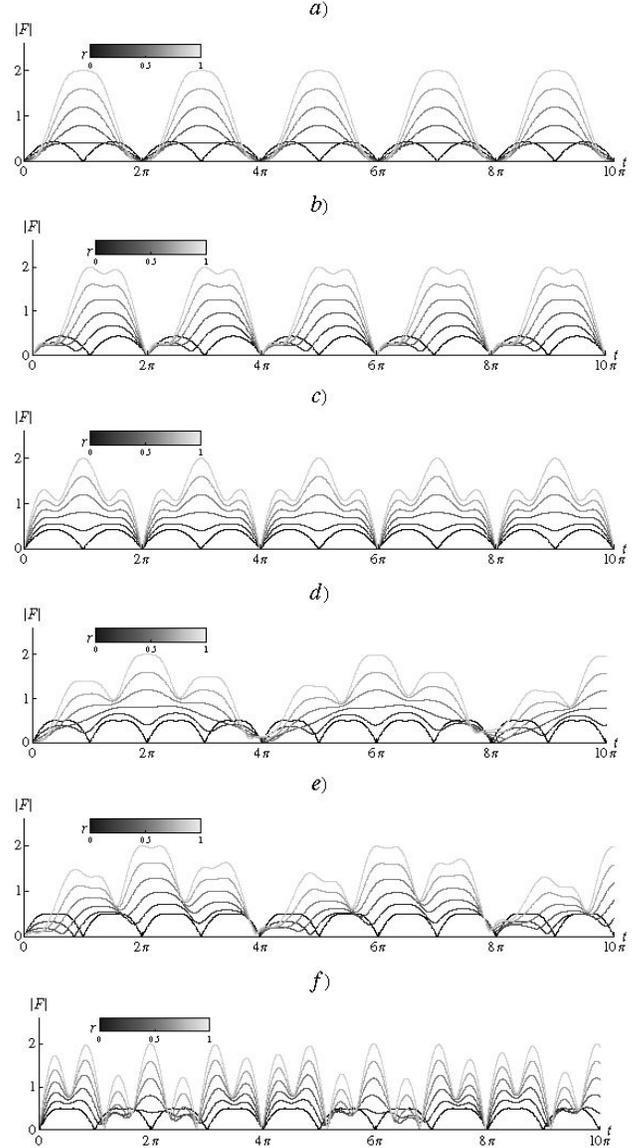with c a constant. If $c \in \mathbb{R}$ then $J > 0$, and if $c \in \mathbb{I}$ then $J < 0$.

FIGURE 6. Behavior of $|F(r, \phi, j, t')|$ for a set of values of variable r (decreasing with darkness). For rational $j = 1/4$ exhibiting 2π periodicity of the function (and therefore its roots), a) $\phi = 0$, b) $\phi = \pi/2$ and c) $\phi = \pi$; note that ϕ is related to the symmetry of the roots. For irrational $j = 1/\sqrt{7}$, d) $\phi = 0$, e) $\phi = \pi/4$ and f) $\phi = \pi$; it is remarkable that some are roots accumulated around $t' = 4\pi p$, $p \in \mathbb{Z}$ [specially for the cases $r = 0, 1$ or $\phi = 0, \pi$ in agreement with (C.12)]. For rational j this corresponds to $m = 4$, which implies $j = n/(2 \cdot 4)$. Indeed, by taking $n = 3$ we note that $j = 3/8 = 0.375$ which is close to $j = 1/\sqrt{7} \approx 0.377 \ldots$. We can see that the recurrence of values for entanglement is normally present in the Heisenberg model, even it they do not follow a strictly periodic pattern.

B. Range for mean values for an observable of bipartite separable states

If $|\psi\rangle = (a|0_1\rangle + b|1_1\rangle) \otimes (c|0_2\rangle + d|1_2\rangle)$ is an arbitrary bipartite separable state, then the problem of finding the range of the observable $\langle\psi| H |\psi\rangle$ for H in (5) reduces to optimizing the function (this is obtained by direct calculation making the convenient selection: $a \to x, b \to \sqrt{1 - x^2}$, $c \to \mp y, d \to \sqrt{1 - y^2}$, with $x, y \in [0,1]$, dividing by J in order to find the extrema):

$$f(x,y) = 2(x\sqrt{1-y^2} \pm y\sqrt{1-x^2})^2$$
$$+ b_1(2x^2 - 1) + b_2(2y^2 - 1) - 1 \qquad \text{(B.1)}$$

with $x, y \in [0,1]$, which leads to solutions for the extrema:

$$\{-1 - b_+, 1 - b_-, 1 + b_-, -1 + b_+\} \qquad \text{(B.2)}$$

where $b_+ = b_1 + b_2$ and $b_- = b_1 - b_2$, when $x = 0, 1$ and $y = 0, 1$. With this same procedure it is possible to find the extrema of $\langle\psi| \vec{\sigma}_1 \cdot \vec{\sigma}_2 |\psi\rangle$ by setting $J = 1, B_1 = B_2 = 0$.

C. Periodicity of entanglement and separability in the Heisenberg model for two qubits

Given an arbitrary initial bipartite state:

$$|\varphi(0)\rangle = \sum_{i,j=0}^{1} \alpha_{ij} |i_1 j_2\rangle \qquad \text{(C.1)}$$

(with $\sum_{i,j=0}^{1} |\alpha_{ij}|^2 = 1$) we can represent it using the state matrix:

$$A(0) = \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \qquad \text{(C.2)}$$

By applying the Heisenberg evolution operator (10) we obtain:

$$A(t) = \begin{pmatrix} \alpha_{00}(t) & \alpha_{01}(t) \\ \alpha_{10}(t) & \alpha_{11}(t) \end{pmatrix} \qquad \text{(C.3)}$$

where:

$$\alpha_{00}(t') = \alpha_{00} e^{-i(b_+ - j)t'}$$
$$\alpha_{01}(t') = e^{-ijt'}(\alpha_{01}(\cos t' - ib_- \sin t') + 2ij\alpha_{10}\sin t')$$
$$\alpha_{10}(t') = e^{-ijt'}(\alpha_{01} 2ij \sin t'$$
$$+ \alpha_{10}(\cos t' + ib_- \sin t') + \alpha_{10} 2ij \sin t')$$
$$\alpha_{11}(t') = \alpha_{11} e^{i(b_+ + j)t'} \qquad \text{(C.4)}$$

The eigenvalues of:

$$A(t)A^\dagger(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{(C.5)}$$

will be the Schmidt coefficients:

$$\lambda_{a,b} = \frac{1}{2}\left(1 \pm \sqrt{1 - 4|\Delta(t')|^4}\right) \qquad \text{(C.6)}$$

with[iii]:

$$|\Delta(t')|^2 = (ac - |b|^2)$$
$$= |\alpha_{00}(t')\alpha_{11}(t') - \alpha_{01}(t')\alpha_{10}(t')|^2 \qquad \text{(C.7)}$$

Thus, the state $|\varphi(t')\rangle$ will be separable at time t' iff $\Delta(t') = 0$. After some calculations we obtain:

$$\Delta(t') = \alpha_{00}\alpha_{11}e^{2ijt'} - \alpha_{01}\alpha_{10}e^{-2ijt'} - e^{-2ijt'}$$
$$\times \left((\alpha_{01}^2 + \alpha_{10}^2)ij\sin 2t' + (\alpha_{01}^2 - \alpha_{10}^2)2jb_-\sin^2 t'\right.$$
$$\left. - 8\alpha_{01}\alpha_{10}j^2\sin^2 t'\right) \qquad \text{(C.8)}$$

Using $\Delta \equiv \Delta(0)$, $\alpha_{01} \equiv \alpha_{10}re^{i\phi}$ one can write $(\alpha_{10} \neq 0)^{iv}$:

$$F(r, \phi, j, t') = \frac{e^{-i\phi}}{\alpha_{10}^2}(\Delta(t') - \Delta e^{2ijt'}) = \frac{e^{-i(\phi - \phi_{\Delta'})}}{\alpha_{10}^2}$$
$$\times (|\Delta(t')| - |\Delta|e^{2ijt' - i(\phi_{\Delta'} - \phi_\Delta)}) = -2r^2 ij$$
$$\times e^{i\phi - 2ijt'}\sin t'(\cos t' - ib_-(j)\sin t') + 2r$$
$$\times (i\sin 2jt' + 4e^{-2ijt'}j^2\sin^2 t) - 2ij$$
$$\times e^{-i\phi - 2ijt'}\sin t'(\cos t' + ib_-(j)\sin t') \qquad \text{(C.9)}$$

where $b_-(j)$ denotes the dependence of b_- on j.

Here ϕ_Δ, $\phi_{\Delta'}$ are the phases of $\Delta, \Delta(t')$ respectively. (C.9) shows the non-periodic behavior of separability and entanglement, at least for $j \in \mathbb{Q}'$. When the right side of this equation is zero for $t' > 0$ and separable initial state, we have separability again at time t'. This holds true for any value of $|\Delta|$, given that $F(r, \phi, j, t')$ vanishes, then: $|\Delta(t')| = |\Delta|$. In addition the relative phase $\phi_{\Delta'} - \phi_\Delta$ should be equal to $2jt'$ (plus an integer multiple of 2π). Figure 6 shows the behavior of $F(r, \phi, j, t')$.

$F(r, \phi, j, t')$ has some interesting properties. By writing $F(r, \phi, j, t') = F_r(r, \phi, j, t') + iF_i(r, \phi, j, t')$ in terms of the real and the imaginary parts of F, then, for fixed values of r and ϕ, we have that if $F(j = \chi, t' = \tau') = 0$ to first order:

$$F_r(\chi + \mathrm{d}j, \tau' + \mathrm{d}t') = \left.\frac{\partial F_r}{\partial j}\right|_{\chi,\tau'} \mathrm{d}j + \left.\frac{\partial F_r}{\partial t'}\right|_{\chi,\tau'} \mathrm{d}t'$$
$$F_i(\chi + \mathrm{d}j, \tau' + \mathrm{d}t') = \left.\frac{\partial F_i}{\partial j}\right|_{\chi,\tau'} \mathrm{d}j + \left.\frac{\partial F_i}{\partial t'}\right|_{\chi,\tau'} \mathrm{d}t' \qquad \text{(C.10)}$$

Given that it is possible to have roots of F if we slightly change the value of j by setting $F_r(\chi + \mathrm{d}j, \tau' + \mathrm{d}t') = F_i(\chi + \mathrm{d}j, \tau' + \mathrm{d}t') = 0$ in (C.10), we get:

$$\left.\frac{\partial F_r}{\partial j}\right|_{\chi,\tau'}\left.\frac{\partial F_i}{\partial t'}\right|_{\chi,\tau'} - \left.\frac{\partial F_r}{\partial t'}\right|_{\chi,\tau'}\left.\frac{\partial F_i}{\partial j}\right|_{\chi,\tau'} = 0 \qquad \text{(C.11)}$$

By taking $\tau' = m\pi, \chi = n/2m$ (the cases where we know that F vanishes for j−rational solutions), we find the following condition:

$$-4\pi n r (r^2 - 1)\sin\phi = 0 \qquad \text{(C.12)}$$

which means that F has roots for irrational values of j (see d-f in Fig. 6), and the ones near the rational values have roots, when $r = 0, 1$ or $\phi = 0, \pi$.

i. One can show that in this case the Schmidt coefficients of $|u_3\rangle$, $|u_4\rangle$ are

$$\lambda_{1,2} = \frac{1}{2}\left(1 \pm \frac{|B_-|}{R}\right) \to 1, 0.$$

Because

$$\lim_{B_- \to \infty} R(R \pm B_-) = \infty, \qquad 2J^2$$

and

$$\lim_{B_- \to -\infty} R(R \pm B_-) = 2J^2, \qquad \infty,$$

the eigenvectors become: $|0_1 1_2\rangle$, $|1_1 0_2\rangle$.

ii. Actually, $T = \pi/R$ is the period to get a maximally entangled state, but the initial state is not repeatable at this time. The period to get the same initial state is $T = 2\pi/R$. Because of that, it happens four times in one period (except for $|B_-/2J| = 1$).

iii. $|\Delta(t)|$ is itself an entanglement measure because it is monotone between 0 (separable) and $1/\sqrt{2}$ (maximally entangled state). It means that $S(|\Delta|)$ is monotone.

iv. Case $\alpha_{01} = \alpha_{10} = 0$ exhibits separability or entanglement invariance as was seen. Case $\alpha_{10} = 0, \alpha_{01} \neq 0$ is similar to this changing $r \to 1/r, \phi \to -\phi$. In addition, the study of the right side of this equation can be restricted to $r \in (0, 1)$, because cases with $r > 1$ are obtained with the transformation $\phi \to -\phi, b_- \to -b_-$.

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