Quasinormal frequencies of the Dirac field in the massless topological black hole

A. López-Ortega
Centro de Investigación en Ciencia Aplicada y Tecnología Avanzada,
Unidad Legaria, Instituto Politécnico Nacional, Calzada Legaria # 694. Colonia Irrigación,
Miguel Hidalgo México, D.F., 11500 México.

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Motivated by the recent computations of the quasinormal frequencies of higher dimensional black holes, we exactly calculate the quasinormal frequencies of the Dirac field, propagating in D-dimensional ($D \geq 4$) massless topological black hole. From the exact values of the quasinormal frequencies for the fermion and boson fields we discuss whether the recently proposed bound on the relaxation time of a perturbed thermodynamical system is satisfied in the D-dimensional massless topological black hole. Also we study the consequences of these results.

Keywords: Quasinormal modes; massless topological black hole; Dirac field; relaxation time.

Motivados por el cálculo de las frecuencias cuasinormales de agujeros negros cuyo número de dimensiones D es mayor o igual a cuatro, en el presente artículo calculamos exactamente las frecuencias cuasinormales del campo de Dirac moviéndose en el agujero negro topológico de masa cero con $D \geq 4$. Usando los valores exactos de las frecuencias cuasinormales para los fermiones y bosones, discutimos si el límite, recientemente propuesto, sobre el tiempo de relajamiento de un sistema termodinámico perturbado se satisface en el agujero negro topológico de masa cero con $D \geq 4$. Adicionalmente estudiamos algunas consecuencias de estos resultados.

Descriptores: Modos cuasinormales; agujero negro topológico de masa cero; campo de Dirac; tiempo de relajamiento.

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1. Introduction

The physical systems for which we exactly solve their equations of motion can be expected to play a significant role in several lines of research. For these physical systems we exactly calculate the physical quantities that for other systems we calculate by using approximate methods. Also in many research areas, the physical insight that is obtained by studying the exactly solvable systems can be used to infer some details about the behavior of more complex physical systems.

The quasinormal modes (QNMs) of a black hole are solutions to the equations of motion for a classical field that satisfy the appropriate radiation boundary conditions at the horizon and at the asymptotic region. The quasinormal frequencies (QNFs) of a field are valuable quantities since these are determined by a few parameters of the black hole and the field [1-3], for example, the QNFs of the Kerr-Newman black hole are determined by the mass, angular momentum, and charge of the black hole and the mode of the field. Hence if we measure the QNFs of a field, then we can infer the values of the mass, angular momentum, and charge of the Kerr-Newman black hole.

Also the QNMs allow us to study the linear stability of the black holes, because if we find QNMs whose amplitude increases in time, then the black hole may be unstable [1-3]. Recently the QNMs have found applications in several research lines. For example,

a) the AdS/CFT correspondence of string theory [2,4,5],

b) the determination of the area quantum of the black hole event horizon [6,7],

c) the expansion of functional determinants in some thermal spacetimes [8,9],

d) the expansion of the "distant past" Green functions used in self-force calculations [10].

For many relevant spacetimes their QNFs must be calculated approximately, hence we use numerical methods or perturbation methods [1-3]. Nevertheless, recently exact calculations of the QNFs for several spacetimes have been presented. Among these we enumerate the following:

a) three-dimensional static and rotating BTZ black holes [5,11-13],

b) three-dimensional charged and rotating black holes of the Einstein-Maxwell-dilaton with cosmological constant theory [14-17],

c) two-dimensional dilatonic black hole [18,19]

d) five-dimensional dilatonic black hole [18,19],

e) D-dimensional de Sitter spacetime ($D \geq 3$) [20-26],

f) BTZ black string [27],

g) Nariai spacetime [28][i]

In the following paragraphs we comment on another D-dimensional anti-de Sitter black hole for which the exact values of its QNFs have been calculated.

We notice that the AdS/CFT correspondence of string theory motivated many studies on the QNFs of anti-de Sitter black holes [2,4,5], because this correspondence proposes that the QNFs of the anti-de Sitter black holes determine the relaxation time of the dual conformal field theory [4,5]. (See

Ref. 5 for an explicit verification of this proposal in three-dimensional rotating BTZ black hole.)

Furthermore, we recall that in asymptotically anti-de Sitter spacetimes, there are solutions to the Einstein equations that represent black holes whose horizons are negative curvature Einstein manifolds [30-36]. These solutions are usually known as topological black holes, and for some of these solutions the mass parameter can assume negative or zero values [30-36].

Among these exact solutions of the Einstein equations, there is one that has attracted a lot of attention. It is the asymptotically anti-de Sitter black hole whose mass is equal to zero [30-36]. In the rest of the present paper, we shall call it the massless topological black hole (MTBH). According to Ref. 37, we can consider the MTBH to be a higher dimensional generalization of the three-dimensional static BTZ black hole, and we expect that it will play a significant role in future research.

The metric of the D-dimensional MTBH is simple and as a consequence many of its physical properties can be calculated exactly [37-41]. For example, the QNFs of the gravitational Klein Gordon, and electromagnetic perturbations were calculated exactly in Refs. 37 and 38 and Sec. 6 of Ref. 39, respectively. Also its stability against the three types of gravitational perturbations was proven in Refs. 40 and 41. For numerical and analytical computations of the QNFs for other topological black holes, see Refs. 42 to 48.

Here we exactly calculate the QNFs of the Dirac field evolving in the D-dimensional MTBH and thus we extend the results of Refs. 37 to 39. The computation of the QNFs for this fermion field is interesting because in some backgrounds the Dirac field behaves in a different way from the boson fields; for example, it is well known that in a rotating black hole, the Dirac field does not show superradiant scattering [49-52], in contrast to boson fields [53]. Also notice that the QNFs of the Dirac field allow us to discuss some additional details about the behavior of the MTBH under perturbations.

Note that in higher dimensional spacetimes, for the Dirac field we only know the QNFs reported in Refs. 19, 26, 54 to 57; thus for this fermion field, its resonances have not been studied as extensively as for other fields. Hence this paper extends our knowledge of the QNMs of the Dirac field in higher dimensional black holes.

This paper is organized as follows. In Sec. 2 we find exact solutions to the Dirac equation in D-dimensional MTBHs and using these solutions we exactly calculate the QNFs of the Dirac field. Exploiting these results we enumerate some facts about the behavior of the MTBHs under perturbations. In Sec. 3 we investigate whether the fundamental QNFs of the MTBH satisfy the bound recently proposed by Hod in Ref. 58. In Sec. 4, following Chandrasekhar [53], in MTBHs we write the Dirac equation as a pair of Schrödinger type differential equations and identify the effective potentials. Finally in Sec. 5 we discuss the results obtained.

2. QNFs of the Dirac field

The line element of a G_{D-2}-symmetric spacetime may be written as [59]

$$ds^2 = F(r)^2 dt^2 - G(r)^2 dr^2 - H(r)^2 d\Sigma^2_{D-2}, \qquad (1)$$

where $F(r)$, $G(r)$, and $H(r)$ are functions only of the coordinate r, and $\mathrm{d}\Sigma^2_{D-2}$ denotes the line element of a $(D-2)$-dimensional G_{D-2}-invariant base spacetime Σ_{D-2}, which depends only on the coordinates ϕ_i, $i = 1, 2, \ldots, D-2$.

Our aim is to calculate exactly the QNFs of the Dirac field evolving in D-dimensional MTBHs. Thus first we explicitly write the Dirac equation

$$i\nabla\!\!\!\!/\,\psi = m\psi \qquad (2)$$

in MTBHs to find its exact solutions. Note that we follow the usual conventions; thus in the formula (2) the symbol $\nabla\!\!\!\!/$ denotes the Dirac operator, m stands for the mass of the Dirac field, and ψ denotes the spinor of dimension $2^{[D/2]}$, where $[D/2]$ denotes the integer part of $D/2$ [56,60-65].

As is well known, in a D-dimensional G_{D-2}-symmetric spacetime with line element (1), the Dirac equation reduces to a pair of coupled partial differential equations in two variables (see for example Eqs. (30) of [56] and Refs. 60 to 65 for more details):

$$\partial_t\psi_2 - \frac{F}{G}\partial_r\psi_2 = \left(i\kappa\frac{F}{H} - imF\right)\psi_1,$$
$$\partial_t\psi_1 + \frac{F}{G}\partial_r\psi_1 = -\left(i\kappa\frac{F}{H} + imF\right)\psi_2, \qquad (3)$$

where κ stands for the eigenvalues of the Dirac operator on the manifold Σ_{D-2} with line element $\mathrm{d}\Sigma^2_{D-2}$, and the functions ψ_1 and ψ_2 are the components of a two-dimensional spinor ψ_{2D} which depends only on the coordinates (t, r) of the G_{D-2}-symmetric spacetime with line element (1), that is

$$\psi_{2D}(r,t) = \begin{pmatrix} \psi_1(r,t) \\ \psi_2(r,t) \end{pmatrix}. \qquad (4)$$

We point out that in Eqs. (3) and in the rest of this paper, we write the functions $\psi_1(r,t)$, $\psi_2(r,t)$, $F(r)$, $G(r)$, and $H(r)$ simply as ψ_1, ψ_2, F, G, and H, respectively. We shall use a similar convention for the functions to be defined in the rest of the present work.

The line element of the D-dimensional MTBHs is given by [30-36]

$$\mathrm{d}s^2 = \left(-1 + \frac{r^2}{L^2}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{\left(-1 + \frac{r^2}{L^2}\right)} - r^2\mathrm{d}\Sigma^2_{D-2}, \quad (5)$$

where $r \in (L, +\infty)$, L is related to the cosmological constant Λ by

$$L^2 = -\frac{(D-1)(D-2)}{2\Lambda}, \qquad (6)$$

and $\mathrm{d}\Sigma^2_{D-2}$ stands for the line element of a $(D-2)$-dimensional compact space of negative curvature

Σ_{D-2} [30-36]. Notice that the (t, r) sector of line element (5) for the MTBH is similar to that of the three-dimensional static BTZ black hole with mass $M = 1$. Taking into account this fact, it was proposed that the D-dimensional MTBH (5) is a higher dimensional generalization of the three-dimensional static BTZ black hole [37].

The QNMs of the MTBH are solutions to the equations of motion for a field that are purely ingoing near the event horizon and, since this black hole is asymptotically anti-de Sitter, we impose that at infinity the radial functions go to zero (Dirichlet's boundary condition) [37,39]. In this section, we compute the QNFs of the Dirac field propagating in D-dimensional MTBHs to find out about the behavior of this black hole under fermion perturbations and compare with its behavior under boson perturbations. We note that the results of this section are an extension of those already published in Refs. 37 to 39.

The line element of the D-dimensional MTBH (5) has the same form as the line element of the G_{D-2}-symmetric spacetime (1). Thus making the appropriate identifications, we get that the functions F, G, and H for the MTBH are equal to

$$F = \frac{1}{G} = \left(-1 + \frac{r^2}{L^2}\right)^{1/2}, \qquad H = r. \tag{7}$$

Therefore in D-dimensional MTBHs the coupled partial differential equations (3) reduce to

$$\begin{aligned}\partial_t \psi_2 - \frac{z^2-1}{L}\partial_z \psi_2 &= (z^2-1)^{1/2}\left(\frac{i\kappa}{zL} - im\right)\psi_1,\\ \partial_t \psi_1 + \frac{z^2-1}{L}\partial_z \psi_1 &= -(z^2-1)^{1/2}\left(\frac{i\kappa}{zL} + im\right)\psi_2,\end{aligned} \tag{8}$$

where $z = r/L$ and therefore $z \in (1, +\infty)$. In what follows, we write in detail the procedure used to solve exactly Eqs. (8).

Choosing for the components ψ_1 and ψ_2 a harmonic time dependence of the form

$$\begin{aligned}\psi_1(z,t) &= \bar{R}_1(z)\,\mathrm{e}^{-i\omega t},\\ \psi_2(z,t) &= R_2(z)\,\mathrm{e}^{-i\omega t},\end{aligned} \tag{9}$$

and defining $\tilde{\omega} = \omega L$, $\tilde{m} = mL$, and $K = -i\kappa$, we get that the system of partial differential equations (8) transforms into the coupled system of ordinary differential equations for the functions R_2 and $R_1 = -i\bar{R}_1$:

$$\begin{aligned}(z^2-1)\frac{\mathrm{d}R_2}{\mathrm{d}z} + i\tilde{\omega}R_2 &= (z^2-1)^{1/2}\left(\frac{iK}{z} - \tilde{m}\right)R_1,\\ (z^2-1)\frac{\mathrm{d}R_1}{\mathrm{d}z} - i\tilde{\omega}R_1 &= -(z^2-1)^{1/2}\left(\frac{iK}{z} + \tilde{m}\right)R_2.\end{aligned} \tag{10}$$

If we make the following ansatz for the functions R_1 and R_2 (see the formulas (26) of Ref. 26 for a similar ansatz for the radial functions of the Dirac field evolving in D-dimensional de Sitter spacetime):

$$\begin{aligned}R_1(z) &= (z^2-1)^{-1/4}(z+1)^{1/2}\tilde{R}_1(z),\\ R_2(z) &= (z^2-1)^{-1/4}(z-1)^{1/2}\tilde{R}_2(z),\end{aligned} \tag{11}$$

then we find that the functions $\tilde{R}_1$ and $\tilde{R}_2$ satisfy

$$\begin{aligned}(z^2-1)\frac{\mathrm{d}\tilde{R}_2}{\mathrm{d}z} + \left(i\tilde{\omega} + \tfrac{1}{2}\right)\tilde{R}_2 &= \left(\frac{iK}{z} - \tilde{m}\right)(z+1)\tilde{R}_1,\\ (z^2-1)\frac{\mathrm{d}\tilde{R}_1}{\mathrm{d}z} - \left(i\tilde{\omega} + \tfrac{1}{2}\right)\tilde{R}_1 &= -\left(\frac{iK}{z} + \tilde{m}\right)(z-1)\tilde{R}_2.\end{aligned} \tag{12}$$

Next, we define the functions f_1 and f_2 by

$$f_1(z) = \tilde{R}_1(z) + \tilde{R}_2(z), \qquad f_2(z) = \tilde{R}_1(z) - \tilde{R}_2(z), \tag{13}$$

to obtain that these functions must be solutions to the coupled system of ordinary differential equations

$$\begin{aligned}(z^2-1)\frac{\mathrm{d}f_1}{\mathrm{d}z} + \left(\tilde{m}z - \frac{iK}{z}\right)f_1 &= \left(i\tilde{\omega} + \tfrac{1}{2} + iK - \tilde{m}\right)f_2,\\ (z^2-1)\frac{\mathrm{d}f_2}{\mathrm{d}z} - \left(\tilde{m}z - \frac{iK}{z}\right)f_2 &= \left(i\tilde{\omega} + \tfrac{1}{2} - iK + \tilde{m}\right)f_1.\end{aligned} \tag{14}$$

From Eqs. (14) we obtain that functions f_1 and f_2 satisfy the decoupled ordinary differential equations

$$\begin{aligned}(z^2-1)^2\frac{\mathrm{d}^2 f_1}{\mathrm{d}z^2} + 2z(z^2-1)\frac{\mathrm{d}f_1}{\mathrm{d}z} + (z^2-1)\left(\tilde{m} + \frac{iK}{z^2}\right)f_1 - \left(\tilde{m}^2 z^2 - 2\tilde{m}iK - \frac{K^2}{z^2}\right)f_1 &= \left(\left(i\tilde{\omega} + \tfrac{1}{2}\right)^2 - (iK - \tilde{m})^2\right)f_1,\\ (z^2-1)^2\frac{\mathrm{d}^2 f_2}{\mathrm{d}z^2} + 2z(z^2-1)\frac{\mathrm{d}f_2}{\mathrm{d}z} - (z^2-1)\left(\tilde{m} + \frac{iK}{z^2}\right)f_2 - \left(\tilde{m}^2 z^2 - 2\tilde{m}iK - \frac{K^2}{z^2}\right)f_2 &= \left(\left(i\tilde{\omega} + \tfrac{1}{2}\right)^2 - (iK - \tilde{m})^2\right)f_2.\end{aligned} \tag{15}$$

To solve Eqs. (15), we make the changes of variables $x = z^2$ and $u = (x-1)/x$, and take functions f_1 and f_2 in the form

$$\begin{aligned}f_1(u) &= u^{B_1}(1-u)^{F_1}\hat{R}_1(u),\\ f_2(u) &= u^{B_2}(1-u)^{F_2}\hat{R}_2(u),\end{aligned} \tag{16}$$

where

$$B_1 = B_2 = \begin{cases} \frac{i\tilde{\omega}}{2} + \frac{1}{4}, \\[2mm] -\frac{i\tilde{\omega}}{2} - \frac{1}{4}, \end{cases} \tag{17}$$

$$F_1 = \begin{cases} \frac{1}{4} + \frac{1}{2}\sqrt{\tilde{m}^2 - \tilde{m} + \frac{1}{4}}, \\ \frac{1}{4} - \frac{1}{2}\sqrt{\tilde{m}^2 - \tilde{m} + \frac{1}{4}}, \end{cases}$$

$$F_2 = \begin{cases} \frac{1}{4} + \frac{1}{2}\sqrt{\tilde{m}^2 + \tilde{m} + \frac{1}{4}}, \\ \frac{1}{4} - \frac{1}{2}\sqrt{\tilde{m}^2 + \tilde{m} + \frac{1}{4}}, \end{cases}$$

to find that the functions $\hat{R}_1$ and $\hat{R}_2$ must be solutions of the hypergeometric differential equation [66,67]:

$$u(1-u)\frac{\mathrm{d}^2 f}{\mathrm{d}u^2} + (c - (a+b+1)u)\frac{\mathrm{d}f}{\mathrm{d}u} - abf = 0. \quad (18)$$

If the parameter c is not an integer, then the solutions to Eq. (18) are given in terms of the standard hypergeometric functions ${}_2F_1(a, b; c; u)$ [66,67].

For the functions $\hat{R}_1$ and $\hat{R}_2$, the quantities a, b, and c of Eq. (18) are equal to (a_i, b_i, and c_i correspond to the function $\hat{R}_i$, $i = 1, 2$)

$$\begin{aligned} a_1 &= B_1 + C_1 + \tfrac{1}{4} + \tfrac{1}{2}\sqrt{\tilde{m}^2 - \tilde{m} + \tfrac{1}{4}}, \\ b_1 &= B_1 - C_1 + \tfrac{3}{4} + \tfrac{1}{2}\sqrt{\tilde{m}^2 - \tilde{m} + \tfrac{1}{4}}, \\ c_1 &= 2B_1 + 1, \end{aligned}$$

$$\begin{aligned} a_2 &= B_2 + C_2 + \tfrac{1}{4} + \tfrac{1}{2}\sqrt{\tilde{m}^2 + \tilde{m} + \tfrac{1}{4}}, \\ b_2 &= B_2 - C_2 + \tfrac{3}{4} + \tfrac{1}{2}\sqrt{\tilde{m}^2 + \tilde{m} + \tfrac{1}{4}}, \\ c_2 &= 2B_2 + 1, \end{aligned} \quad (19)$$

where the quantities C_1 and C_2 take on the values

$$C_1 = \begin{cases} \frac{1}{2} + \frac{iK}{2}, \\ -\frac{iK}{2}, \end{cases} \qquad C_2 = \begin{cases} \frac{iK}{2}, \\ \frac{1}{2} - \frac{iK}{2}. \end{cases} \quad (20)$$

At this point we notice that the coordinate x lies in the range $x \in (1, +\infty)$. Hence the variable u satisfies $u \in (0, 1)$. Also the tortoise coordinate of the MTBH is [39]

$$r_* = \int \left(-1 + \frac{r^2}{L^2}\right)^{-1} \mathrm{d}r = -L\,\mathrm{arccoth}(z); \quad (21)$$

thus $r_* \in (-\infty, 0)$, $r_* \to -\infty$ near the event horizon and $r_* \to 0$ near infinity. From these definitions of the coordinates u and r_*, we get that

$$\begin{aligned} &\text{as} \quad r_* \to -\infty, \qquad u \approx \mathrm{e}^{2r_*/L}, \qquad \text{and} \\ &\text{as} \quad r_* \to 0, \qquad u \approx 1. \end{aligned} \quad (22)$$

Now we use these results to compute the QNFs of the Dirac field exactly. First let us study the function f_1. We choose the quantities C_1, B_1, and F_1 as $C_1 = 1/2 + iK/2$, $B_1 = i\tilde{\omega}/2 + 1/4$, and $F_1 = 1/4 + \sqrt{\tilde{m}^2 - \tilde{m} + 1/4}/2$. If we assume that the quantity c_1 is not an integer, then we obtain that function f_1 is equal to

$$f_1 = (1-u)^{F_1}\left\{\mathbb{D}_1 u^{i\tilde{\omega}/2+1/4}{}_2F_1(a_1, b_1; c_1; u) + \mathbb{E}_1 u^{-i\tilde{\omega}/2-1/4}{}_2F_1(a_1 - c_1 + 1, b_1 - c_1 + 1; 2 - c_1; u)\right\}, \quad (23)$$

where $\mathbb{D}_1$ and $\mathbb{E}_1$ are constants. Taking into account expressions (22), we find that near the horizon function f_1 behaves as

$$f_1 \approx \mathbb{D}_1 \mathrm{e}^{i\omega r_* + r_*/(2L)} + \mathbb{E}_1 \mathrm{e}^{-i\omega r_* - r_*/(2L)}; \quad (24)$$

thus in order to have a purely ingoing wave near the event horizon, we must impose the condition $\mathbb{D}_1 = 0$ [37-39]. Hence the function f_1 becomes

$$\begin{aligned} f_1 &= \mathbb{E}_1 u^{-i\tilde{\omega}/2-1/4}(1-u)^{1/4+\sqrt{\tilde{m}^2-\tilde{m}+1/4}/2}{}_2F_1(a_1 - c_1 + 1, b_1 - c_1 + 1; 2 - c_1; u) \\ &= \mathbb{E}_1 u^{-i\tilde{\omega}/2-1/4}(1-u)^{1/4+\sqrt{\tilde{m}^2-\tilde{m}+1/4}/2}{}_2F_1(\alpha_1, \beta_1; \gamma_1; u). \end{aligned} \quad (25)$$

We recall that if the quantity $c - a - b$ is not an integer, then the hypergeometric function ${}_2F_1(a, b; c; u)$ satisfies [66,67]:

$$\begin{aligned} {}_2F_1(a, b; c; u) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a, b; a+b+1-c; 1-u) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-u)^{c-a-b}{}_2F_1(c-a, c-b; c+1-a-b; 1-u), \end{aligned} \quad (26)$$

where $\Gamma(x)$ stands for the gamma function. Hence if the quantity $\gamma_1 - \alpha_1 - \beta_1$ is not an integer, then we write the function f_1 of the formula (25) as

$$\begin{aligned} f_1 = \mathbb{E}_1 u^{-i\tilde{\omega}/2-1/4}\Bigg[&\frac{\Gamma(\gamma_1)\Gamma(\gamma_1-\alpha_1-\beta_1)}{\Gamma(\gamma_1-\alpha_1)\Gamma(\gamma_1-\beta_1)}(1-u)^{1/4+\sqrt{\tilde{m}^2-\tilde{m}+1/4}/2}{}_2F_1(\alpha_1, \beta_1; \alpha_1+\beta_1+1-\gamma_1; 1-u) \\ &+ \frac{\Gamma(\gamma_1)\Gamma(\alpha_1+\beta_1-\gamma_1)}{\Gamma(\alpha_1)\Gamma(\beta_1)}\,(1-u)^{1/4-\sqrt{\tilde{m}^2-\tilde{m}+1/4}/2}{}_2F_1(\gamma_1-\alpha_1, \gamma_1-\beta_1; \gamma_1+1-\alpha_1-\beta_1; 1-u)\Bigg]. \end{aligned} \quad (27)$$

Due to the MTBHs being asymptotically anti-de Sitter, the QNM boundary conditions at infinity require that $f_1 \to 0$ as $u \to 1$ [37,39]. From expression (27) we note that the first term in square brackets vanishes as $u \to 1$. The second term vanishes for $1/2 > \sqrt{\tilde{m}^2 - \tilde{m} + 1/4}$, (that is, for $1 > \tilde{m}$). Thus the function f_1 goes to zero as $u \to 1$, and therefore if $1 > \tilde{m}$ the boundary condition at infinity does not impose any restriction on the frequencies, that is, there is a continuum of frequencies that satisfy the boundary condition at infinity of the QNMs. For $\tilde{m} \geq 1$, in order that $f_1 \to 0$ as $u \to 1$, we must impose the condition

$$\alpha_1 = -n_1, \quad \text{or} \quad \beta_1 = -n_1, \qquad n_1 = 0, 1, 2, \ldots \tag{28}$$

Therefore for $\tilde{m} \geq 1$, from Eqs. (28) we find that the QNFs of the function f_1 are equal to

$$\tilde{\omega}_1 = K - i\left(2n_1 + 1 + \sqrt{\tilde{m}^2 - \tilde{m} + \tfrac{1}{4}}\right), \qquad \text{or}$$

$$\tilde{\omega}_1 = -K - i\left(2n_1 + \sqrt{\tilde{m}^2 - \tilde{m} + \tfrac{1}{4}}\right), \tag{29}$$

whereas for $1 > \tilde{m}$ there is a continuum of QNFs.

To calculate the QNFs of the function f_2, we choose the quantities C_2, B_2, and F_2 as $C_2 = iK/2$, $B_2 = i\tilde{\omega}/2 + 1/4$, and $F_2 = 1/4 + \sqrt{\tilde{m}^2 + \tilde{m} + 1/4}/2$. A similar method to that used for the function f_1 allows us to find that, for all $\tilde{m}$, the QNFs of the function f_2 are ($n_2 = 0, 1, 2, \ldots$):

$$\tilde{\omega}_2 = K - i\left(2n_2 + \sqrt{\tilde{m}^2 + \tilde{m} + \tfrac{1}{4}}\right), \qquad \text{or}$$

$$\tilde{\omega}_2 = -K - i\left(2n_2 + 1 + \sqrt{\tilde{m}^2 + \tilde{m} + \tfrac{1}{4}}\right). \tag{30}$$

From formulas (13) we find that functions $\hat{R}_1$ and $\hat{R}_2$ are linear combinations of functions f_1 and f_2; therefore only the QNFs that are equal for both functions f_1 and f_2 will be QNFs of the Dirac field in D-dimensional MTBHs. Thus when $\tilde{m} < 1$, for the function f_1 we find a continuum of QNFs, but for the function f_2 we only find QNFs (30). Hence for $\tilde{m} < 1$ the QNFs of the Dirac field are equal to

$$\omega = \frac{K}{L} - \frac{i}{L}\left(2n + \frac{1}{2} + \tilde{m}\right), \qquad n = 0, 1, 2, \ldots$$

$$\omega = -\frac{K}{L} - \frac{i}{L}\left(2n + \frac{3}{2} + \tilde{m}\right). \tag{31}$$

When $\tilde{m} \geq 1$ for function f_1 we find QNFs (29), whereas for function f_2 we find QNFs (30). After some simplifications we find that for $\tilde{m} \geq 1$, the QNFs frequencies of the Dirac field are also determined by expressions (31). Thus in MTBHs formulas (31) give the QNFs of the Dirac field for any value of the mass $\tilde{m}$. In the massless limit the QNFs (31) reduce to

$$\omega = \frac{K}{L} - \frac{i}{L}\left(2n + \frac{1}{2}\right),$$

$$\omega = -\frac{K}{L} - \frac{i}{L}\left(2n + \frac{3}{2}\right). \tag{32}$$

For QNFs (31) and (32), we find that $\mathbb{I}\mathrm{m}\,(\tilde{\omega}) < 0$, hence these QNMs decay in time. Thus the D-dimensional MTBH is linearly stable against Dirac perturbations. Something similar happens for the QNFs of the electromagnetic and gravitational perturbations [37,39]. The stability of the MTBHs against the gravitational perturbations was shown in Refs. 40, and 41.

As we previously commented, in Refs. 37 to 39 the QNFs of the gravitational, electromagnetic, and minimally coupled massless Klein-Gordon perturbations were calculated. The values obtained for the QNFs of these fields are

$$\omega = \pm\frac{\xi}{L} - \frac{2i}{L}\left(n + \frac{\mathbb{A}}{4}\right), \tag{33}$$

where the quantity $\mathbb{A}$ takes on the values

$$\mathbb{A} = \begin{cases} D-1 & \text{for the vector-type gravitational} \\ & \text{and electromagnetic perturbations,} \\ |D-5|+2 & \text{for the scalar-type gravitational} \\ & \text{and electromagnetic perturbations,} \\ D+1 & \text{for the tensor-type gravitational} \\ & \text{perturbation } (D \geq 5) \text{ and minimally} \\ & \text{coupled massless Klein-Gordon field,} \end{cases} \tag{34}$$

the quantity ξ depends on the perturbation type and is related to the eigenvalues of the Laplacian on the manifold Σ_{D-2} [37-39].

For the non-minimal coupled to gravity massive Klein-Gordon field, the QNFs are equal to [38]

$$\omega = \pm\frac{\xi}{L} - \frac{i}{L}\left(2n + 1 + \sqrt{\left(\tfrac{D-1}{2}\right)^2 + m_{eff}^2 L^2}\right), \tag{35}$$

where $m_{eff}^2 = m^2 - \gamma D(D-2)/(4L^2)$, m denotes the mass of the Klein-Gordon field, and γ is the coupling constant between the scalar curvature and the Klein-Gordon field. Notice that in Ref. 38, a different time parameter was chosen to that used in the present paper. This fact implies that the QNFs (35) have an additional factor of $1/L$ to the QNFs reported in Ref. 38.

From formulas (31)–(35), we find that for the Dirac field the imaginary part of QNFs (31) and (32) does not depend on the spacetime dimension; unlike boson fields, the imaginary part of their QNFs (33) and (35) shows an explicit dependence on the spacetime dimension. Thus for the Dirac field, the decay time $\tau_d = 1/|\mathbb{I}\mathrm{m}(\omega)|$ depends on the mode number n and not on the spacetime dimension. In contrast, for the boson fields the decay time is inversely proportional to the spacetime dimension; thus for a given boson field and fixed mode number, the decay time decreases as the spacetime dimension increases. Hence in D-dimensional MTBHs the decay time for the Dirac field and the decay time for the boson fields show a different behavior when the spacetime dimension changes.

Furthermore, from formulas (33) and (34) for the massless boson fields with mode number, and for $D \geq 5$, we find that the tensor type gravitational perturbation and minimally coupled massless Klein-Gordon field decay faster than vector-type and scalar-type electromagnetic and gravitational perturbations.

From QNFs (32) and (33), we see that for $D \geq 6$, the decay time of the massless Dirac field is greater than the decay time of the massless boson fields. Thus for $D \geq 6$ the massless boson fields decay faster than massless Dirac field. Also for $D = 5$, the tensor type gravitational perturbation and minimally coupled massless Klein-Gordon field decay faster than the other massless boson fields and massless Dirac field. For $D = 4$, we find that the minimally coupled massless Klein-Gordon field decays faster than the electromagnetic, gravitational, and massless Dirac perturbations.

It is convenient to note that for the massive Klein-Gordon and Dirac fields, the imaginary part of the QNF depends on the mass of the field. Taking into account formulas (31) and (35), we find that if the mass of the Dirac and the minimally coupled Klein-Gordon fields are equal and the condition

$$mL < \left(\frac{D}{2} - 1\right)\frac{D}{2} \tag{36}$$

is satisfied, then the minimally coupled Klein-Gordon field decays faster than the Dirac field.

In MTBHs the oscillation frequencies of the boson and fermion fields do not depend on the mass of the field. For the boson fields, the oscillation frequencies are determined by the eigenvalues of the Laplace operator on the negative curvature manifold Σ_{D-2}, whereas for the Dirac field the oscillation frequencies are determined by the eigenvalues of the Dirac operator on Σ_{D-2}.

Thus for a complete determination of the QNFs (31) for the Dirac field moving in MTBHs, we need to know the eigenvalues of the Dirac operator on the base manifold Σ_{D-2} with metric $\mathrm{d}\Sigma^2_{D-2}$. We expect that the event horizon of a black hole will be a compact and orientable manifold [68]. For the MTBH, the negative curvature manifold Σ_{D-2} usually is a quotient of the form H^{D-2}/G, where G is a freely acting discrete subgroup of the isometry group for the $(D-2)$-dimensional hyperbolic space H^{D-2}. Therefore, for the QNFs (31) of the MTBH, we need to find the spectrum of the Dirac operator on a compact spin manifold of the hyperbolic type. Regarding the spectrum of the Dirac operator on hyperbolic manifolds, we know the following facts.

In contrast to the Laplace operator, the spectrum of the Dirac operator depends on the geometry of the manifold and the spin structure, which is a topological object that is necessary to define spinors [69,70]. In general, the spin structure of a spin manifold is not unique; for example the circle S^1 has two spin structures, but note that some manifolds do not admit even a spin structure, for example the complex projective plane $\mathbb{CP}^2$ [69,70].

The hyperbolic space H has an unique spin structure (due to the fact that the hyperbolic space is contractible) [71]. It is known that on the hyperbolic space for the Dirac operator, the discrete spectrum is empty and its continuous spectrum is $\mathbb{R}$ [69,71]. We note that the conventions used in Refs. 69 and 71 and the present paper are different. In the conventions that we use here, the eigenvalues of the Dirac operator on the hyperbolic space are purely imaginary as in Ref. 72 (and therefore $K \in \mathbb{R}$), whereas in Refs. 69 and 71 the eigenvalues of the Dirac operator on the hyperbolic space are real numbers.

If the manifold is compact, then general elliptic theory asserts that the spectrum of the Dirac operator is discrete [69,70]. Thus we expect that on the base manifold Σ_{D-2} of the MTBH, the eigenvalues of the Dirac operator will be discrete. Furthermore, for a D-dimensional compact manifold Σ, the eigenvalues κ of the Dirac operator satisfy the Weyl asymptotic law [69]

$$\lim_{\kappa\to\infty} \frac{N(\kappa)}{\kappa^D} = \frac{2^{[D/2]}\mathrm{vol}(\Sigma)}{(4\pi)^{D/2}\Gamma\left(\frac{D}{2}+1\right)}, \tag{37}$$

where $\mathrm{vol}(\Sigma)$ is the volume of the D-dimensional manifold Σ and $N(\kappa)$ is the number of eigenvalues whose modulus is $\leq \kappa$.

On a compact symmetric manifold with a homogeneous spin structure, the square of the Dirac operator $\not{\nabla}^2$ satisfies [70,73]

$$\not{\nabla}^2 = \Omega + \frac{\mathcal{R}}{8}, \tag{38}$$

where Ω is the Casimir operator of the isometry group and $\mathcal{R}$ is the scalar curvature of the compact symmetric manifold. Therefore, for these manifolds, the computation of the spectrum for the square of the Dirac operator can be done by algebraic methods. Also on these manifolds the spectrum of the Dirac operator is symmetric with respect to the origin, and the spectrum of the Dirac operator is determined by the spectrum of its square. Nevertheless, there are technical difficulties and the spectrum of the Dirac operator is explicitly known for a small number of manifolds [73].

As far as we know, for compact hyperbolic manifolds the spectrum of the Dirac operator is calculated exactly for the manifold $\Sigma = PSL_2(\mathbb{R})/\Gamma$, where $PSL_2(\mathbb{R})$ is the projective special linear group of $\mathbb{R}^2$ and Γ is a co-compact Fuchsian subgroup [73,74]. The complicated spectrum of the Dirac operator on $\Sigma = PSL_2(\mathbb{R})/\Gamma$ appears in Theorem 2.2.3 of Ref. 73. Notice that the case relevant to our work is when the parameter t of Theorem 2.2.3 is equal to 1, and therefore the manifold $PSL_2(\mathbb{R})/\Gamma$ has negative constant sectional curvature[ii].

We notice that for the Dirac operator, eigenvalue estimates can be found in several manifolds for which an exact calculation of the spectrum is not possible [73]. We believe that the following result is relevant for our work.

In Proposition 2 of Ref. 75, it is asserted that for a compact and oriented two-dimensional surface Σ of genus $g \neq 1$, there is an eigenvalue κ of the Dirac operator that satisfies

$$| \kappa | \leq c(g) \max\{\text{principal curvatures of } \Sigma\}, \tag{39}$$

where

$$c(g) = \begin{cases} 1 & \text{if} \quad g = 0, \\ 3 & \text{if} \quad g = 2, 3, \\ 4 & \text{if} \quad g \geq 4. \end{cases} \tag{40}$$

For $g \geq 2$, this result is pertinent for the four-dimensional MTBHs. We do not know similar estimates for the eigenvalues of the Dirac operator on higher dimensional compact hyperbolic manifolds.

From these comments it is deduced that in the mathematical literature, we do not find many calculations of the eigenvalues of the Dirac operator on compact hyperbolic manifolds, and we believe that the computation of these quantities is a challenging mathematical problem.

In Ref. 5 it was shown that the momentum space poles of the retarded correlation functions in the dual conformal field theory and the QNFs of the three-dimensional BTZ black hole are identical. Calculating whether something similar happens for the QNFs of the D-dimensional MTBH is an interesting problem.

3. Hod's bound

Taking into account quantum information theory and thermodynamic concepts, in Ref. 58 Hod found a bound on the relaxation time τ of a perturbed thermodynamic system. This bound is

$$\tau \geq \tau_{min} = \frac{\hbar}{\pi T}, \tag{41}$$

where τ_{min} stands for the minimum relaxation time and T denotes the temperature of the thermodynamic system. This bound is called "TTT bound" (time times temperature bound) by Hod in Ref. 58.

In Ref. 58 it was shown that strong self-gravity systems, such as the black holes, are appropriate systems for testing the TTT bound (41). For a black hole the TTT bound states that at least for the fundamental QNFs the following inequality is satisfied [58]:

$$\frac{\hbar \omega_I}{\pi T_H} \leq 1, \tag{42}$$

where ω_I is the absolute value of the imaginary part of the fundamental QNFs and T_H is Hawking's temperature of the black hole (see Refs. 58, 76 to 79 for more details). The fundamental QNM is the least damped mode of the black hole, and it determines its relaxation time scale [1-3].

The Hawking temperature of the MTBHs is equal to [30-36]

$$T_H = \frac{\hbar}{2\pi L}, \tag{43}$$

and from QNFs (32) of the massless Dirac field, we find

$$\frac{\hbar \omega_I}{\pi T_H} = 1 \qquad \text{and} \qquad \frac{\hbar \omega_I}{\pi T_H} = 3. \tag{44}$$

We see that the first expression in formulas (44) saturates the inequality (42), and that the second expression does not satisfy the previously mentioned inequality.

Furthermore, from QNFs (33) of the massless boson fields, we obtain that

$$\frac{\hbar \omega_I}{\pi T_H} = \mathbb{A}. \tag{45}$$

Hence, taking into account the values of quantity $\mathbb{A}$ given in formula (34), for $D \geq 4$ we find that in MTBHs the fundamental QNFs of the massless bosons do not satisfy inequality (42).

Thus we find that in D-dimensional MTBHs the fundamental QNFs of the massless boson and Dirac fields do not satisfy inequality (42). We expect that inequality (42) be satisfied in MTBHs [58] owing to the fact that Hawking's temperature of the MTBHs is of the same order of magnitude as the reciprocal of the characteristic length (L) of the spacetime. According to Hod, the TTT bound (41) is universal and we do not know the cause of its failure for the fundamental QNFs of the MTBH.

4. Effective potentials

Following the method of Chandrasekhar's book [53], we take for the Dirac field a harmonic time dependence as in formula (9) to transform Eqs. (8) into the pair of decoupled Schrödinger-type equations:

$$\frac{\mathrm{d}^2 Z_\pm}{\mathrm{d}\hat{r_*}^2} + \omega^2 Z_\pm = V_\pm Z_\pm, \tag{46}$$

where

$$\begin{aligned} Z_\pm &= \mathrm{e}^{i\theta/2} \bar{R}_1 \pm \mathrm{e}^{-i\theta/2} R_2, \\ \theta &= \arctan \frac{mz}{\hat{K}}, \\ V_\pm &= W^2 \pm \frac{\mathrm{d}W}{\mathrm{d}\hat{r_*}}, \\ W &= \frac{\sqrt{z^2-1}\left(\hat{K}^2 + (mz)^2\right)^{3/2}}{z(\hat{K}^2 + (mz)^2) + \frac{\hat{K}m}{2\omega L} z(z^2-1)}, \end{aligned} \tag{47}$$

$\hat{K} = K/L$, we define $\hat{r_*}$ by

$$\frac{\mathrm{d}\hat{r_*}}{\mathrm{d}r_*} = 1 + \frac{z^2-1}{2\omega L} \frac{m\hat{K}}{\hat{K}^2 + (mz)^2}, \tag{48}$$

and, as in Sec. 2, r_* denotes the tortoise coordinate of the D-dimensional MTBHs (see formula (21)).

From formulas (46) and (47), we see that the effective potentials $V_\pm$ are complicated functions of the different parameters. Nevertheless, in the massless limit we find that the formulas for W and $V_{\pm}$ reduce to

$$W = \hat{K}\frac{\sqrt{z^2-1}}{z} = -\hat{K}\text{sech}(r_*/L),$$
$$V_{\pm} = \frac{\hat{K}^2}{\cosh^2(r_*/L)} \pm \frac{(\hat{K}/L)\sinh(r_*/L)}{\cosh^2(r_*/L)}. \qquad (49)$$

Thus for the massless Dirac field the effective potentials (49) are of the Morse type [80]. In Ref. 39 it was shown that in D-dimensional MTBHs the effective potentials of the Schrödinger differential equations for the massless boson fields are of the Pöschl-Teller type. We note that for many of the spacetimes for which we exactly calculate their QNFs the effective potentials of Schrödinger-type equations are of the Pöschl-Teller or Morse type (see for example Table 1 in Ref. 23).

5. Discussion

For the D-dimensional MTBHs in Sec. 2, we found that the real part of the QNFs is determined by the eigenvalues of the Laplace operator (boson fields) or the Dirac operator (Dirac field) on the negative curvature manifold Σ_{D-2}. Nevertheless, to our knowledge there are not many calculations of the spectrum of the Dirac operator on compact spin manifolds of the hyperbolic type. We believe that this mathematical problem deserves detailed study. Furthermore, we notice that the imaginary part of QNFs (31) is independent of the eigenvalues of the Dirac operator on Σ_{D-2}. This fact allows us to discuss some phenomena (see Secs. 2 and 3), even if we do not know explicitly the value of the eigenvalues of the Dirac operator on the manifold Σ_{D-2}.

For the massless boson and Dirac fields, the imaginary part of the QNFs shows a different dependence on the spacetime dimension. For the boson fields, the decay time depends on the spacetime dimension, whereas for the Dirac field, it is independent of the spacetime dimension. Also we point out that for $D \geq 6$, the massless boson fields decay faster than the massless Dirac field.

In MTBHs, the QNFs of the Klein-Gordon, gravitational, electromagnetic, and Dirac perturbations have been calculated (see Refs. 37 to 39 and Sec. 2 of this paper). Nevertheless, as far as we know the QNFs of the Rarita-Schwinger field have not been computed. We believe that the calculation of the QNFs for this field is an interesting problem.

According to Hod, the TTT bound of, formula (41) is universal [58], [76-79], but we found in Sec. 3 that for the fundamental QNFs of the MTBHs the inequality (42) is not satisfied [see formulas (44) and (45)]. We believe that this puzzling result deserves detailed study.

For the D-dimensional MTBHs, from our results and those already published, we obtain that the real part of the QNFs depends on the eigenvalues of the Laplace or Dirac operators on the negative curvature manifold Σ_{D-2}. These values can be different for distinct fields, also for a fixed field these eigenvalues may depend on the mode of the field. Thus the asymptotic limit of the real part of the QNFs for the D-dimensional MTBHs depend on the physical parameters of the black hole and the field (and the mode of the field).

An interesting proposal is the so-called Hod's conjecture [6]; it states that in the semiclassical limit the area quantum of an event horizon can be calibrated with the asymptotic value of the real part of the QNFs. The facts mentioned in the above paragraph imply that Hod's conjecture is not valid for the D-dimensional MTBHs (as for the D-dimensional de Sitter spacetime [81]), since in this conjecture we must assume that the real part of the QNFs depends only on the physical parameters of the black hole [6,7], but this does not happen in D-dimensional MTBHs. Thus we think that for the D-dimensional MTBHs we must investigate whether the recent proposal of Maggiore [7] can be used to determine the area quantum of its event horizon. Work along this line is in progress.

Finally we notice that formulas (31) also give the QNFs of the Dirac field propagating in three-dimensional static BTZ black hole with mass $M = 1$. The QNFs of the Dirac field evolving in the static BTZ were previously calculated in Refs. 5 and 11.

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i. We notice that in Ref. 29, Saavedra presented an exact expression for the QNFs of Unruh's acoustic black hole. The expression used in that reference for the effective metric of Unruh's acoustic black hole is valid near the horizon. For the asymptotic region of Unruh's acoustic black hole, it is probable that we need to use a different approximation of the effective metric. Thus we believe that this problem deserves additional study. This issue was pointed out to the Author by the Referee.

ii. For the related case of the so-called plane symmetric black hole, it is convenient to notice that the spectrum of the Dirac operator on the higher dimensional flat tori has been calculated (see Theorem 4.1 of Ref. 69 and Theorem 2.1.1 of Ref. 73). We point out that the flat tori admits several spin structures and the spectrum of the Dirac operator depends on the spin structure [73]. For other examples of flat manifolds for which the spectrum of the Dirac operator is calculated exactly see Chapter 2 of Ref. 73.

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