INVESTIGACIÓN REVISTA MEXICANA DE FÍSICA **56** (2) 141–146 ABRIL 2010

Two methods to determine the Hermite-Gaussian beam radius by means of aperiodic rulings

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Recibido el 31 de julio de 2009; aceptado el 18 de enero de 2010

We study the diffraction of Hermite-Gaussian beams by aperiodic rulings by means of the Rayleigh-Sommerfeld theory in the scalar diffraction regime. We extend to Hermite-Gaussian beams the results of a previous paper where Gaussian beams were considered [*J. Opt. Soc. Am. A* **25** (2008) 2743]. The transmitted power and the normally diffracted energy are analyzed as a function of the beam radius. Two methods to determine the Hermite-Gaussian beam radius by means of aperiodic rulings are proposed. These two methods are based on the maximum and minimum transmitted power, and in the normally diffracted energy.

Keywords: Diffraction; gratings.

En la región escalar estudiamos la difracción de haces Hermite-Gauss con redes de difracción aperiódicas mediante la Teoría de Rayleigh-Sommerfeld. Extendemos a haces Hermite-Gauss los resultados previamente publicados para haces Gaussianos [*J. Opt. Soc. Am. A* **25** (2008) 2743]. La energía total transmitida y la energía normalmente difractada son analizadas como función del radio de los haces. Se proponen dos métodos para determinar el radio de haces Hermite-Gauss mediante redes de difracción aperiódicas. Estos dos métodos están basados en el máximo y el mínimo de la energía total transmitida y en la energía normalmente difractada.

Descriptores: Difracción; redes de difracción.

PACS: 42.25.Fx; 42.10.H.C

1. Introduction

The diffraction of Gaussian beams has been extensively treated in the past [1-6]. In this paper we are interested in the transmission and diffraction of Hermite-Gaussian beams by aperiodic rulings. These kinds of beams are described by the product of Hermite polynomials and Gaussian functions. At present, the two-dimensional Hermite-Gaussian beams can easily be excited, for instance, with end-pumped solid-state laser [7] or by inserting a cross wire into the laser cavity with the wires aligned with the nodes of the desired mode [8]. In Ref. 7 it was demonstrated that it is possible to generate two-dimensional Hermite-Gaussian modes up to the $\text{TEM}_{0,80}$ mode. In passing, we mention that these beams have been considered in relation to some other problems. The reader is referred to Ref. 9 for a more complete list of references about the applications of Hermite-Gaussian beams (28 references are given).

Some methods for determining the size of the Gaussian beams have been proposed which are based on the properties of the transmitted power by rulings [10-15]. Also, aperiodic rulings [14-16] (which are Ronchi rulings but with a large or small opaque section) have been considered. In all the mentioned papers [10-14] the beam diameters have been determined by means of the maximum and the minimum transmitted power. However, some exceptions are given in Refs. 3, 15 and 16 where the normally diffracted energy to the gratings was considered. This last method can be useful in that it uses only the diffracted energy close to the normal direction instead of the total transmitted power.

Two methods to determine the Gaussian beams radius by means of periodic and aperiodic rulings were proposed in Ref. 15. One is based on the maximum and minimum transmitted power, and the other one on the normally diffracted energy. For periodic rulings the field amplitude radius r_0/D can be determined as long as $0.02 < r_0/D < 1.2$, where D is the period of the rulings. And for aperiodic rulings r_0/D can be determined as long as $0.5 < r_0/D < 80$, in fact, this upper limit can be improved. Then, with these two methods small and large Gaussian beams radius can be treated.

In this paper we extend to Hermite-Gaussian beams the results given in Ref. 15, where Gaussian beams were considered. For periodic rulings was shown in Ref. 16 that the two methods proposed in Ref. 15 cannot be applied any more to Hermite-Gaussian beams. On the other hand, for aperiodic rulings, the two methods proposed in Ref. 15 can be extended to Hermite-Gaussian beams. It is important to notice that, to our knowledge, this is the first time that methods to determine the field amplitude radius r_0/D of Hermite-Gaussian beams by means of aperiodic rulings are proposed. Finally, we mention that in the literature, little attention has been paid to the diffraction of Hermite-Gaussian beams by gratings; some exceptions are given in Ref. 16 to 20.

2. Formulation

We have an aperiodic ruling made of alternate transparent (width l) and opaque zones (width d) with period $D = l + d$. This aperiodic ruling has a large or small opaque zone of width d', which could be equal to or different from d, *i.e.*, we have an opaque discontinuity in the ruling. In the case

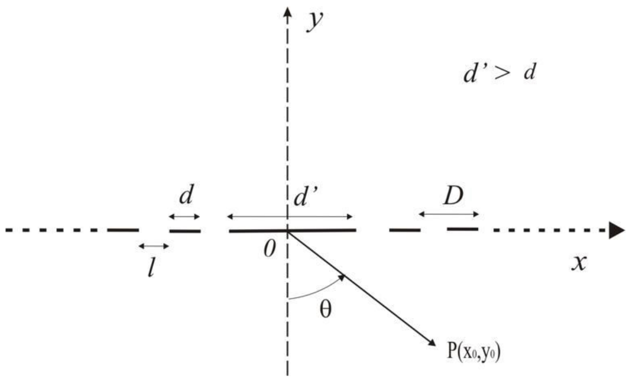

FIGURE 1. Our system. An aperiodic ruling made of alternate opaque and transparent zones of widths d and l, respectively, with an opaque zone of width d'. The ruling is parallel to the Oz axis. The observation point is given by $P(x_0, y_0)$.

where $d'{=}d$ the conventional periodic ruling is recovered. We fixed a Cartesian coordinate system at the midpoint of the opaque discontinuity of width d' with the Oz axis parallel to the ruling as shown in Fig. 1. The ruling is illuminated at normal incidence by a beam independent of the z coordinate (cylindrical incident wave). The complex representation of field quantities is used, and the complex time term $\exp(-i\,\omega\,t)$ is omitted from now on.

Since this paper can be considered to be the continuation of a previously published article, the theory of diffraction is only outlined here and the reader is referred to Ref. 15 for most details.

Let $E(x)$, $E_i(x)$, and $t(x)$ be the transmitted field, the input field or incident field, and the transmittance function, respectively, related as follows:

$$E(x) = t(x)E_i(x) \tag{1}$$

where the function $t(x)$ is null in the opaque zones and has the unity value in the transparent zones. From Eq. (1) the field $E(x)$ just below the ruling can be obtained. From the knowledge of the field $E(x)$ and the two-dimensional Rayleigh-Sommerfeld integral equation [21] the total field $E(x_0, y_0)$ at any point below the ruling can be obtained:

$$\begin{aligned} E(x_0, y_0) &= \frac{i}{2}\int_{-\infty}^{\infty} E(x)\frac{\partial}{\partial y_0}H_0^1(kr)dx \\ &= \frac{i}{2}\int_{-\infty}^{\infty} t(x)E_i(x)\frac{\partial}{\partial y_0}H_0^1(kr)dx \end{aligned} \tag{2}$$

where $k = 2\pi/\lambda$, with λ being the wavelength of the incident radiation; and $r^2 = (x - x_0)^2 + y_0^2$ with $P(x_0, y_0)$ being the observation point as illustrated in Fig. 1. H_0^1 is the Hankel function of the first kind and order zero. From Eq. (2) the far field can be obtained by looking at the asymptotic behavior of the field E when $kr \gg 1$. In this approximation the expression for the far field is given by

$$E(x_0, y_0) = f(\theta)\exp(ikR_0)/\sqrt{R_0}, \tag{3}$$

where $\sin\theta = x_0/R_0$ and $\cos\theta = -y_0/R_0$ (see Fig. 1). This is the expression of a cylindrical wave with the oblique factor $f(\theta)$ given by:

$$f(\theta) = \sqrt{k}\exp(-i\pi/4)\cos\theta\;\hat{E}(k\sin\theta), \tag{4}$$

with $\hat{E}(\alpha)$ being the Fourier transform of $E(x)$:

$$\begin{aligned} \hat{E}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} E(x)\exp(-i\alpha\,x)dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} t(x)E_i(x)\exp(-i\alpha\,x)dx \end{aligned} \tag{5}$$

The intensity $I(\theta)$ diffracted at an angle θ (see Fig. 1) is given by $C\,|f(\theta)|^2$, where C is a constant, and we have

$$\begin{aligned} I(\theta) &= C^2\frac{1}{2\pi}k \\ &\times\cos^2\theta\left|\int_{-\infty}^{+\infty} t(x)E_i(x)\exp(-ik\sin\theta x)dx\right|^2; \end{aligned} \tag{6}$$

then, the diffraction patterns can be determined from Eq. (6) if the input field $E_i(x)$ and the transmittance function $t(x)$ are given.

In what follows, our attention is focused on the transmitted power P_T and on the normally diffracted energy to the screen $I(0^0)$. The transmitted power P_T is obtained as follows:

$$P_T = \int_{-\pi/2}^{\pi/2} I(\theta)d\theta \tag{7}$$

3. Definition of hermite-gaussian beam width

It is very important to remember that the definition of the size of a beam is somewhat arbitrary. We denote by r_0 the Gaussian beam radius at which the field amplitude is 1/e times its peak value and by L the local $1/e$-intensity Gaussian beam diameter (the L-spot diameter). The values of L are related to the field amplitude radius r_0 by means of the relationship $L = \sqrt{2}\,r_0$. There are other definitions for the Gaussian spot size; for instance, the beam width can be calculated using the diameter that covers 86.5% of the energy, and in this case the beam width will be denoted by L_0. As was pointed out in Ref. 20, the relationship between the Gaussian beam widths L and L_0 is given by $L_0 = 1.057L$, so that the values of L_0 are very close to the values of L; in fact, in practice we can consider that $L_0 = L$.

As an incident wave, the two-dimensional version of the Hermite-Gaussian beam will be considered. On the screen and at normal incidence the field of the Hermite-Gaussian beam of order n is given by

$$E_i(x, y=0) = H_n\left[\frac{\sqrt{2}}{r_0}(x-b)\right]\ \exp\left[-\frac{(x-b)^2}{r_0^2}\right], \quad (8)$$

where H_n is the Hermite polynomial of order n, some of which are $H_0(t) = 1$, $H_1(t) = 2t$, $H_2(t) = 4t^2 - 2$, $H_3(t) = 8t^3 - 12t$, and so forth. The position of the incident Hermite-Gaussian beam with respect to the Oy axis is fixed by the parameter b. This parameter enables us to displace the beam along the screen.

If the beam diameter L_n for the Hermite-Gaussian beam of order n is defined by the 86.5% energy content, then L_n is related to r_0 by means of a linear relationship [20]. We have found that $L_1 = 2.3574r_0$, $L_2 = 3.0000r_0$, $L_3 = 3.5397r_0$, $L_4 = 4.0107r_0$, and so forth, so that the Hermite-Gaussian beam diameter L_n increases when n also increases (r_0 is fixed). In fact, we can consider r_0 as a common parameter for all the Hermite-Gaussian beams; however, it is necessary not to forget that the interesting and practical parameter is L_n. In what follows r_0 will be considered as the basic parameter. In addition, we call attention to the fact that the present theory is valid not only for Hermite-Gaussian beams but also for other incident beams.

4. Aperiodic ruling

In this section, we are mainly interested in studying the intensity ratio K defined as follows:

$$K = E_{\min}/E_{\max} \quad (9)$$

and the power ratio P given by

$$P = P_{\min}/P_{\max}, \quad (10)$$

where $E_{\min}$ and $E_{\max}$ are the minimum and maximum values of the normally diffracted energy $I(0^0)$ and $P_{\min}$ and $P_{\max}$ are the minimum and maximum transmitted power, both of them obtained when the spot beam is scanned by the ruling. Normally incident beams are considered in what follows.

In this section, the case of an aperiodic ruling made of alternate transparent and opaque zones will be considered (with the period $D = l + d$), but with a large or small opaque zone of width d' which could be equal or different to d, *i.e.*, we have an opaque discontinuity in the ruling. The case where $d = l$ will be treated, *i.e.*, the width of the opaque zones is equal to the width of the transparent zones. Also, the case of a great opaque discontinuity $d' > D$ is analyzed in what follows. The O_z axis will be placed halfway through the opaque discontinuity of width d'.

This aperiodic ruling has been studied by Uppal *et al.* in Ref. 14 for an incident Gaussian beam. They have divided their study into two cases: $d' > D$ (great opaque discontinuity) and $d' < D$ (small opaque discontinuity). In the first case they were able to determine a large beam radius ($1 < r_0/D < 10$) and in the second one a small beam radius ($0.05 < r_0/D < 0.5$).

The aperiodic ruling was also studied by Mata-Mendez in Ref. 15 for an incident Gaussian beam. Only the case $d' > D$ (great opaque discontinuity) was treated in Ref. 15, since two methods to determine a small beam radius ($0.02<r_0/D<1.2$) by means of the ruling were proposed in the same paper. Also, it was shown for a great opaque discontinuity that the radius r_0 can be determined from the ratios P and K as long as $0.5 < r_0/D < 80$; in fact, this last result improve that obtained by Uppal *et al.* in Ref. 14.

In Fig. 2 the transmitted power is plotted as a function of the beam position (b/D) for a normally incident Hermite-Gaussian beam of order n =1, with the following parameters: λ/l =0.0666, d/l =0.3333 and d'/l =2.0, and the field amplitude radius r_0/D=0.5 and 2.0. From the results of Fig. 2 and other results not shown, we have observed two small depressions close to the centre of the opaque discontinuity and a constant value far from this discontinuity when $r_0/D \geq$2.0 and n =1. These last two properties are very important in the determination of the field amplitude radius as we shall see below. In the case of an incident Gaussian beam only one depression was observed by Uppal *et al.* in Ref. 14 located at the center of the opaque discontinuity. We have also analyzed incident Hermite-Gaussian beams of order n =2, 3, and 4 with the same conclusions, where several depressions were observed.

Figure 3 is similar to Fig. 2 but for the normally diffracted energy. From the results of this figure and other results not shown, we have found that the value of the ratio K is null when n is an odd integer. However, K is not null ($K \neq 0$)

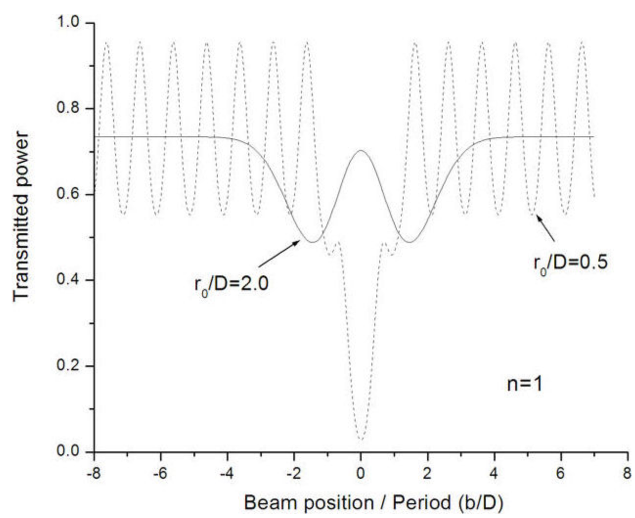

FIGURE 2. Transmitted power is plotted as a function of the spot position (b/D) for a normally incident Hermite-Gaussian beam of order n =1, with λ/l =0.0666, d/l =0.3333 and d'/l =2.0. Several values of the field amplitude radius are considered: r_0/D=0.5 and 2.0.

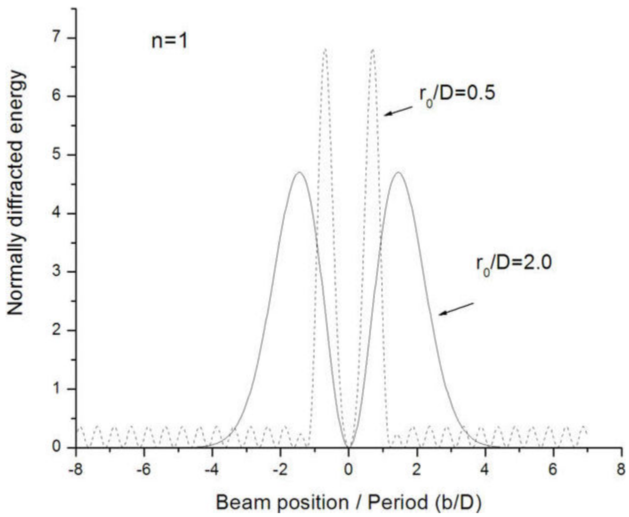

FIGURE 3. Same as Fig. 2 but for the normally diffracted energy.

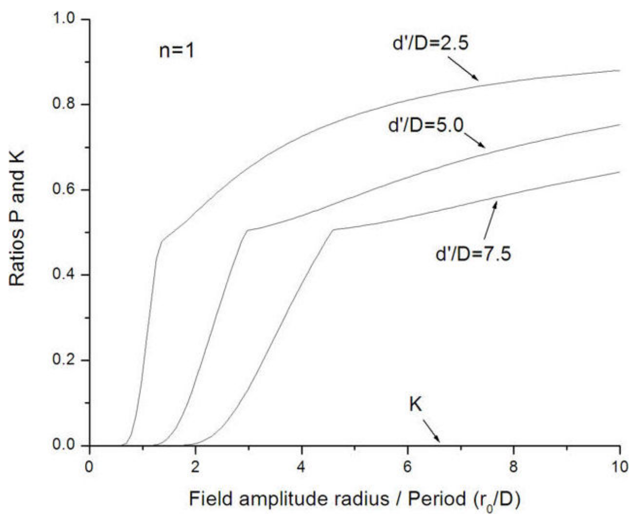

FIGURE 4. Ratios P and K are plotted as a function of the field amplitude radius (r_0/D) for a normally incident Hermite-Gaussian beam of order n =1 and several values of the opaque discontinuity (d'/D=2.5, 5.0, 7.5), when l=0.5 and d=0.5.

when n is an even integer. We have analyzed the minimum ($E_{\min}$) and maximum ($E_{\max}$) values of the normally diffracted energy for incident Hermite-Gaussian beams and the following behavior was obtained:

$$E_{\min} \propto \frac{1}{\lambda} \quad \text{and} \quad E_{\max} \propto \frac{1}{\lambda} \tag{11}$$

these results are in agreement with Eq. (14) of Ref. 15. Then, the intensity ratio K is independent of the wavelength.

In Fig. 4 the ratios P and K are plotted as a function of the field amplitude radius (r_0/D) for a normally incident Hermite-Gaussian beam of order n =1. We have the following parameters: l=0.5, d=0.5, and several values of the opaque discontinuity d'/D=2.5, 5.0, 7.5. The parameters used in Fig. 4 are the same as Figs. 8 and 9 of Ref. 15. In all cases we have an aperiodic ruling with a great opaque discontinuity ($d' > D$). We observe that the behavior of the ratio P as a function of r_0/D is changed considerably with the values of the opaque discontinuity (d'/D). A growing behavior of P is obtained in all cases in Fig. 4, while the ratio K is always null. Then, from this growing behavior we can conclude that if the ratio P is experimentally determined, the corresponding field amplitude radius r_0/D can be obtained as long as $0.66 < r_0/D < 10.0$ when d'/D=2.5, $1.25 < r_0/D < 10.0$ when d'/D=5.0, and $1.86 < r_0/D < 10.0$ when d'/D=7.5. The upper limit can be extended to great values of r_0/D as we shall see below. We have analyzed incident Hermite-Gaussian beams of order n =3, with the same conclusions.

Figures 5 and 6 are similar to Fig. 4 but for the order n =2. In Fig. 5 the ratio P is considered, while in Fig. 6 the ratio K is dealt with. In Fig. 5 a growing behavior of P is obtained for all the values of d'/D, so that the field amplitude radius r_0/D can be determined as long as $0.53 < r_0/D < 10.0$ when d'/D=2.5, $1.1 < r_0/D < 10.0$

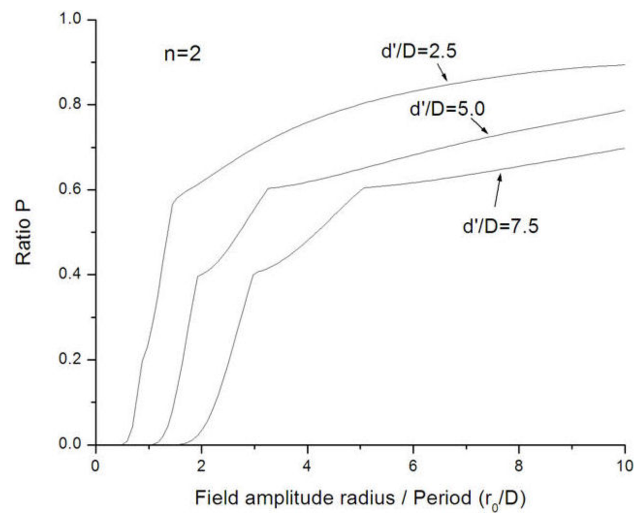

FIGURE 5. Same as Fig. 4 but for the ratio P and n =2.

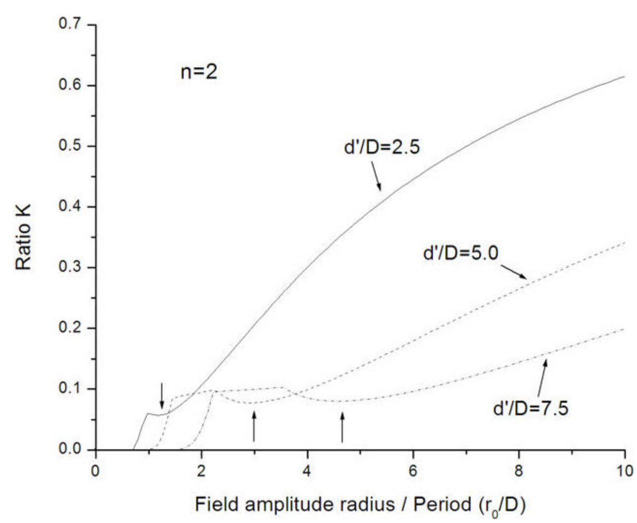

FIGURE 6. Same as Fig. 4 but for the ratio K and n =2.

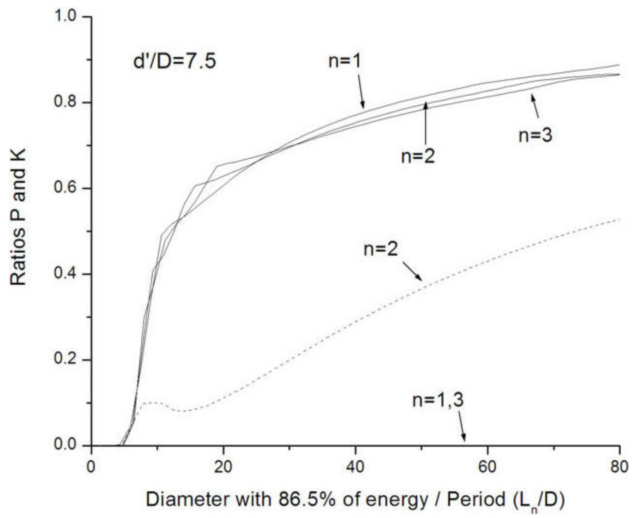

FIGURE 7. Ratios P (solid curves) and K (dashed curves) are plotted as a function of the diameter L_n/D (diameter that covers 86.5% of energy) for normally incident Hermite-Gaussian beams of order n =1, 2, 3, with the opaque continuity d'/D=7.5.

when d'/D=5.0, and $1.61 < r_0/D < 10.0$ when d'/D=7.5. From Fig. 6 we see that the growing behavior of K begins at r_0/D=1.165, 2.875, and 4.585 (pointed out by arrows) when d'/D=2.5, 5.0, and 7.5, respectively. From these last observations we conclude that if the ratio K is determined, the field amplitude radius r_0/D can be obtained as long as $1.165 < r_0/D < 10.0$ when d'/D=2.5, $2.875 < r_0/D < 10.0$ when d'/D=5.0, and $4.585 < r_0/D < 10.0$ when d'/D=7.5. We have also considered Hermite-Gaussian beams of order n =4 with the same conclusions. To our knowledge, this is the first time that two methods to determine the field amplitude radius of Hermite-Gaussian beams by means of an aperiodic ruling are proposed.

We observe from Figs. 4, 5, and 6 that the two proposed methods could determine very long values of r_0/D when the opaque discontinuity d'/D is also large. This is done in Fig. 7 where the ratios P and K are plotted as a function of the beam diameter L_n (diameter that covers 86.5% of energy), when n =1, 2, and 3, and the opaque discontinuity is given by d'/D =7.5. In fact, the upper limit $L_n/D = 80$ of Fig. 7 could be improved with a greater discontinuity. It is interesting to compare Fig. 7 with Fig. 10 of Ref. 15 where an incident Gaussian beam was considered. We consider that the results given in Figs. 4-7 are the main contributions of this paper.

In passing we mention that we have also dealt with (results not shown) the case of a small opaque discontinuity ($d' < D$), but instead of the growing behavior of P and K given in Figs. 4-6, an oscillating behavior was obtained. Finally, we mention that in a future paper the diffraction of Hermite-Gaussian beams by an aperiodic ruling is to be analyzed in detail.

5. Conclusions

The diffraction of Hermite-Gaussian beams by aperiodic rulings was studied by means of the transmitted power and the normally diffracted energy. Two methods to determine the Hermite-Gaussian beam radii are proposed. Large and very large Hermite-Gaussian beam radius could be treated with these two methods.

Acknowledgments

The author acknowledges support from Comisión de Operaciones y Fomento de Actividades Académicas del Instituto Politécnico Nacional, México.

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