Geometric associative memories applied to pattern restoration

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Two main research areas in Pattern Recognition are pattern classification and pattern restoration. In the literature, many models have been developed to solve many of the problems related to these areas. Among these models, Associative Memories (AMs) can be highlighted. An AM can be seen as a one-layer Neural Network. Recently, a Geometric Algebra based AM model was developed for pattern classification, the so-called Geometric Associative Memories (GAMs). In general, AMs are very efficient for restoring patterns affected BY either additive or subtractive noise, but in the case of mixed noise their efficiency is very poor. In this work, modified GAMs are used to solve the problem of pattern restoration. This new modification makes use of Conformal Geometric Algebra principles and optimization techniques to completely and directly restore patterns affected by (mixed) noise. Numerical and real examples are presented to test whether the modification can be efficiently used for pattern restoration. The proposal is compared with other reported approaches in the literature. Formal conditions are also given to ensure the correct functioning of the proposal.

Keywords: Associative memories; pattern restoration; mixed noise; conformal geometric algebra.

Dos áreas de investigación muy importantes en reconocimiento de patrones son la clasificación y la restauración de patrones. En la literatura, se han propuesto muchos modelos para resolver varios de los problemas relacionados con estas dos áreas. Entre estos modelos, hay que resaltar a las memorias asociativas (MA). Una MA puede ser vista como red neuronal de una sola capa. Recientemente, un nuevo modelo de MA basado en la llamada álgebra geométrica fue desarrollado para la clasificación de patrones: las llamadas memorias asociativas geométricas (MAG). En general, las MA son muy eficientes en la restauración de patrones afectados por ruido ya sea aditivo o substractivo, pero en el caso de ruido mezclado su eficiencia es muy pobre. En este trabajo se utilizan MAGS modificadas para resolver el problema de la restauración de patrones. Esta nueva modificación hace uso de principios del álgebra geométrica conforme y de técnicas de optimización para restaurar patrones afectados con ruido mezclado en forma directa y completa. Se presentan, además, ejemplos numéricos y con datos reales para probar la propuesta. Finalmente, se presenta una comparación con otras reportadas en la literatura. También se proporcionan algunas condiciones que garantizan el funcionamiento de la propuesta.

Descriptores: Memorias asociativas; restauración de patrones; ruido mixto; álgebra geométrica conforme.

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1. Introduction

Two important problems in image processing are pattern classification and pattern restoration. Pattern restoration is an essential part of many signal and image processing applications. There is a strong need for the development of practical algorithms able to ensure restoration of patterns corrupted by noise.

In this paper we are going to focus our efforts on providing a solution to the problem of pattern restoration in the presence of noise.

Many approaches have been proposed during the last century in an attempt to solve the restoration problem. In particular the well-known Neural Networks have been used for this purpose (see for example Chinarov and Metzinger 2003, Fukishima 2005, and Cruz, Sossa and Barrón 2007).

One approach that has attracted the attention of the scientific community during the last few years as providing a solution to the problem of restoring patterns from noisy versions of them is the application of the so-called Associative Memories (AMs). It is worth mentioning that in the case of using AMs to restore patterns, in general, the restored pattern turns out to be the original pattern. This is because the original patterns are encoded somehow into the memory.

This does not happen with filters; they take the patterns as inputs and try to reduce or to completely eliminate the added distortion. To work properly, on the other hand, AMs need to have somehow a codification of the pattern or patterns that need to be restored.

Initial AMs models were based on vector algebra for their operations. Later models of AMs based their functioning on so-called Mathematical Morphology operations. These associative memories were called in the literature Morphological Associative Memories (MAMs) (Ritter, Sussner and Diaz de Leon 1998).

The associative memory models developed so far can be categorized in two groups, those based on traditional algebraic operations and those based on mathematical morphology operations. Recently a new type of Associative Memory has been developed based on the Geometric Algebra (GA) paradigm, the so-called Geometric Associative Memory (GAM) (Cruz, Barrón and Sossa 2008).

Originally, GAMs were developed for pattern classification. The basic idea is to create class spheres as decision surfaces, to decide to which class a given pattern belongs. In this case it is necessary to know where the pattern should be located: inside or outside of the sphere. To do so, an inner product between the pattern and the trained GAM is applied to obtain a vector. A minimum function is then applied to that vector to get the corresponding index class. An important feature of GAMs is that they can deal with patterns affected by mixed noise.

In this work, a change to the original algorithm for building GAMs is performed. With this change, as we will see, GAMs can be used to restore patterns and not only to classify patterns. The result will be a restored pattern. In the experimentation section we shall show that the new proposal can be used to restore patterns affected by subtractive, additive, mixed noise and misfocusing (in the case of images). We shall also compare the performance of the new proposal with other reported approaches.

2. Associative memories

An AM is a device whose main function consists in associating input patterns with output patterns, as depicted in Eq. (1). When an input pattern (vector) $a$is presented to the memory, it returns an output pattern (vector) $b$previously associated with a. AMs can be seen as a one-layer neural network.

$$a \to M \to b. \tag{1}$$

The notation for an association between two patterns a and b can be seen as an ordered pair (a, b). The whole set of all associations that form the AM is called *fundamental pattern set* or simply *fundamental set* (FS).

Associations are completely stored in a weighted matrix. This matrix (the AM) can be used to generate output patterns using the corresponding input patterns. The process by which an AM is built is called the *learning or building phase*, while the process by which an output pattern is classified or restored using an input pattern is called *the classification or the restoration phase*.

Fundamental patterns could be presented to the input of the memory but corrupted with noise or other distortions. A corrupted or distorted version of a pattern x will be denoted as $\tilde{x}$, where $\tilde{x} = x + r$; in this case, vector r is the noise. If all the components in vector r are positive then $\tilde{x}$ presents additive noise, if all the components are negative then $\tilde{x}$ presents subtractive noise. If instead the components are positive or negative, then $\tilde{x}$ presents mixed noise.

The robustness of an AM depends on the kind of distortion against which it can function. And largely, it depends of the level of distortion that a pattern can permit to obtain the original pattern by using the AM.

Examples of AM models in the literature can be mentioned: the well-known *learning matrix* of Steinbuch (Steinbuch 1961), the so-called *correlograph* (Willshaw, Buneman and Longuet-Higgins 1969), the *Linear Associator* model for AM (Anderson 1972 and Kohonen 1972) and the so-called *Hopfield Memory* (Hopfield 1982).

In the 90's a new set of lattice algebra based associative memories appeared, the so-called Morphological Associative Memories (MAMs) (Ritter, Sussner and Diaz de Leon 1998). Minima or maxima of sums were used for the operation of these memories, in contrast to the sums of products used in earlier models.

There are two types of MAMs, the *min* memories that can cope with patterns altered with subtractive noise, and the *max* memoires that can cope with patterns altered with additive noise. However, contrary to what one might think, against patterns altered with mixed noise (most common in real situations), their performance is highly deficient.

Three alternatives have been developed to solve the problem of mixed noise. The first one makes used of the so-called *kernels* (Sussner 2000), by means of the so-called *median memories* (Sossa and Barrón 2003), and by decomposing a pattern into parts by means of the so-called sub-patterns (Cruz, Sossa and Barrón 2007).

In this work a modification of the Geometric Algebra Memory recently described in Cruz, Barrón and Sossa 2008 is presented. It will be shown how this adaptation can be used to provide an efficient solution to the important problem of restoring distorted patterns.

3. Basics on conformal geometric algebra

The Geometric Algebra (GA) is a priori coordinate-free geometric schema (Hestenes and Sobczyk 1984) developed by William Clifford in 1878 (Clifford 1878). In GA, the geometric objects and the operators over these objects are treated in a single algebra (Hitzer 2004).

The Conformal Geometric Algebra (CGA) is an a (3,2)-dimensional coordinate free theory; this model provides a way to encode naturally points, lines, planes, etc. off the origin (Hitzer 2004). Also, these objects are easily represented as *multi-vectors* (Hestenes 2001). A multi-vector is the product of various vectors (Hestenes, Li and Rockwood 2001).

In addition, CGA provides a great variety of basic geometric entities to compute with (Hildenbrand 2005). Intersections between lines, circles, planes and spheres are directly generated. The creation of such elementary geometric objects simply occurs by algebraically joining a number of points (Hitzer 2004).

Geometric Algebra provides three main products: the inner product (commonly used to compute angles and distances); the outer product (usually used for the creation of high-order geometric objects using others low-order objects); and the geometric (or Clifford) product that encodes the two previous products in a single product. In this work, the conformal domain is used for the algebraic operations while the Euclidean domain is used for the geometric semantics.

Let p be a Euclidean point in R^n; then, it is extended to a conformal representation such as (Hestenes, Li, and Rockwood 2001):

$$P = p + \frac{1}{2}(p)^2 e_\infty + e_0, \tag{2}$$

where p is a linear combination of the nEuclidean base vectors. In this case e_0 represents the Euclidean origin and e_∞ is the point at infinity, such that $(e_0)^2 = (e_\infty)^2 = 0$ and $e_0 \cdot e_\infty = -1$ (Hitzer 2004) and $(p)^2$ is the quadratic norm of the Euclidean point.

Also, spheres take a representation (Hitzer 2004):

$$S = C - \frac{1}{2}(\gamma)^2 e_\infty = c - \frac{1}{2}\left((c)^2 - (\gamma)^2\right) e_\infty + e_0, \tag{3}$$

where C is the center of the sphere in conformal notation as defined in (2), γ is the corresponding radius, and c is the Euclidean point of C. Also, a sphere can be easily obtained by four points that lie on it (Hitzer 2004):

$$S = P^1 \wedge P^2 \wedge P^3 \wedge P^4. \tag{4}$$

In this case, (3) and (4) are both *dual* representations of each other. Likewise, a plane can be defined by three points that lie on it and the point at infinity (Hitzer 2004):

$$T = P^1 \wedge P^2 \wedge P^3 \wedge e_\infty. \tag{5}$$

From (4) and (5) we can note that a plane is a sphere with infinite radius (Hestenes 2001).

The inner product can be used to measure distances between objects (Hildenbrand 2005). For example, when the inner product between two conformal points $P$and Q is found, it turns out to be the square of the Euclidean distance:

$$\begin{aligned} P \cdot Q &= p \cdot q - \frac{1}{2}(p)^2 - \frac{1}{2}(q)^2 \\ &= -\frac{1}{2}(p-q)^2 \Leftrightarrow (p-q)^2 \\ &= -2(P \cdot Q). \end{aligned} \tag{6}$$

Also, when the inner product between one conformal point $P$and a sphere S is computed, we get:

$$\begin{aligned} P \cdot S &= p \cdot c - \frac{1}{2}\left((c)^2 - (\gamma)^2\right) - \frac{1}{2}(p)^2 \\ &= \frac{1}{2}\left((\gamma)^2 - (c-p)^2\right). \end{aligned} \tag{7}$$

Note that expression (7) can be simplified as:

$$2(P \cdot S) = (\gamma)^2 - (c-p)^2. \tag{8}$$

Based on (8), if $P \cdot S > 0$ then p is inside of the sphere, if $P \cdot S < 0$ then p is outside of the sphere, and if $P \cdot S = 0$ then p is on the sphere (Hildenbrand 2005). Therefore, for pattern restoration a CGA spherical neighborhood can be used. In this case the quantity of noise that a pattern can admit is given by the radius of the sphere. With the help of the inner product, it is possible to know if a given noise pattern is inside of a specific sphere; the restored pattern is the center of the sphere that contains it.

4. Geometric associative memories

A Geometric Associative Memory (GAM) is a pattern classification tool that uses CGA operators for its functioning (Cruz, Barron and Sossa 2009). A GAM is precisely a matrix whose components are spheres. This can be appreciated in (9), where m is the total number of classes. It uses spherical neighborhoods as decision regions:

$$M = \begin{bmatrix} S^1 \\ S^2 \\ \vdots \\ S^m \end{bmatrix}. \tag{9}$$

When two sets of points in R^n can be completely separated by a hyper-plane, they are said to be linearly separable. Linear separation is important for pattern classification. That hyper-plane works as a decision surface; it can be used for deciding to which class an unclassified pattern will be assigned by finding on which side of the hyper-plane the pattern is located. Many classification models (*i.e.* neural networks) have better results when the patterns are linearly separable.

In the same way, when two sets of points in R^n can be completely separated by a hyper-sphere, they are said to be spherically separable. In this case the decision is made by finding if the pattern is located inside or outside of the sphere. Any two sets of linearly-separable points in R^n are spherically-separable also (Cruz, Barron and Sossa 2009). It is worth mentioning that two sets of spherically-separable points could not be linearly-separable.

GAMs can perfectly operate when the classes are spherically separable (Cruz, Barron and Sossa 2009).

5. Geometric associative memories for pattern restoration

Based on the ideas from the previous section (Cruz, Barron and Sossa 2009) a new operating mode of GAMs will be introduced in this section, in this case for pattern restoration.

The basic idea will be to assign a spherical neighborhood to each pattern of the FS. As we shall later see, the main advantages to the new proposal are that it can cope with mixed noise directly and that the quantity of noise is given by the radius of the sphere. In the case of images, we shall also show that the proposal can be used to restore images distorted by misfocusing.

5.1. Learning phase of GAM's

The learning phase consists in building a spherical neighborhood for each pattern; this can be done by taking the pattern itself as the center of the sphere so that the radius is then computed as the distance measured between it and each of the other patterns. In the following paragraphs a change to this will be described that will allow us to restore a pattern given a distorted version of it, either corrupted by noise or by misfocusing in the case of images.

Let $\{p^1, p^2, \ldots, p^m\}$ be a FS of patterns in R^n. The problem is to find an optimal (in quadratic terms) sphere with the least square error, such that p^i is inside S and $p^j | j = 1, \ldots, m\ j \neq i$ are outside of it. This can be done by solving expression (10),

$$\min_S \sum_{i=1}^{m} \left(P^i \cdot S\right)^2, \tag{10}$$

subject to (11) for the point inside of the sphere and (12) for points outside of it.

$$P^i \cdot S \geq 0 \tag{11}$$

$$P^j \cdot S < 0\ for\ j = l+1, \ldots, m, j \neq i. \tag{12}$$

In order to find an optimal solution a quadratic programming algorithm was adopted; expression (10) is changed to expression (13), where $x = \left[\begin{array}{cccc} S_1 & S_2 & \ldots & S_{n+1} \end{array}\right]$ is the vector to be optimized, and $S_k | k = 1, \ldots, n+1$ are the components of the sphere, $\mathbf{H} = m$and $y = \sum_{i=1}^{m} \left(p^i\right)^2$:

$$\min_x = x^t \mathbf{H} x + y^t x. \tag{13}$$

The constraint for the point inside of the sphere will be:

$$S_{n+1} \leq \frac{1}{2} \left(p^i\right)^2. \tag{14}$$

The constraint for the points outside of the sphere will be:

$$-S_{n+1} < \frac{1}{2} \left(p^j\right)^2 - p^j \cdot p^i. \tag{15}$$

Note that, as the center of the sphere is known (is the pattern itself), then the optimization function must operate only over S_{n+1}. This procedure must be done for all patterns in FS. Let a be a vector where $a_i = 2$ and $a_j = -2$, and let b be a vector such that $b_i = \left(p^i\right)^2$ and $b_j = \left(p^j\right)^2 - 2\left(p^j \cdot p^i\right) - \varepsilon$. ε is a smallest positive quantity used to change the "$<$" of (14) to a "$\leq$". Finally, let $x = S_{n+1}$; then both constraints can be joined as follows:

$$ax \leq b. \tag{16}$$

The resulting spheres represent spherical separation surfaces and they can be used as support regions. The quantity of noise that they can admit is given by the radius of the corresponding sphere.

Note that it can be seen that expression (13) was transformed into a quadratic equation; the problem can be solved by finding the minimum of this function, but taking into account the corresponding constraints.

The GAM M is thus a matrix whose i-th component is the i-th sphere for $i = 1, \ldots, m$; this can be seen in expression (17), where C^i and γ^i are the center and the radius of the i-th sphere respectively:

$$\mathbf{M} = \left[\begin{array}{c} S^1 \\ S^2 \\ \vdots \\ S^m \end{array}\right] = \left[\begin{array}{l} C^1 + \frac{1}{2}\left(\gamma^1\right)^2 e_\infty \\ C^2 + \frac{1}{2}\left(\gamma^2\right)^2 e_\infty \\ \vdots \\ C^m + \frac{1}{2}\left(\gamma^m\right)^2 e_\infty \end{array}\right] \tag{17}$$

5.2. Restoration Phase of GAM's

Restoration of a pattern is performed as follows: First, an inner product between an unclassified pattern q and $\mathbf{M}$ must be applied thus obtaining a vector v by applying expression (18), where Q is the conformal representation of $q$and γ^i is the radius of the i-th sphere:

$$v_i = \begin{cases} -\infty & if\ S^i \cdot Q < 0 \\ S^i \cdot Q - \left(\gamma^i\right)^2 & \text{otherwise} \end{cases} \tag{18}$$

Finally, restored pattern is given by the returning of the center of the position of the maximum value of the vector v.

In other words, the restored pattern will be the center of the sphere. This operation mode can be seen as an attractor for the input distorted pattern.

5.3. Conditions for perfect restoration and robust restoration

In associative memories, when an AM **M** restores the fundamental set correctly, it is said that **M**presents perfect restoration. And, when an AM **M** restores noise pattern, it is said that **M** presents robust restoration. Let **M** a GAM built as explained in section 5.1.

Theorem 1. *Let m be patterns in* R^n*, and let* M be *a GAM for those patterns, built as in as in Sec. 4.2.1, then* M*presents perfect restoration.*

Proof. Let $\{p^i\}_{i=1}^m$ be a FS of patterns in R^n and **M** be the corresponding GAM built as in Sec. 4.2 where S^i is the sphere of the i-th pattern (p^i). S^i was obtained by means of expression (10). Then according to conditions (11) and (12), $P^i \cdot S^i \geq 0$ and $P^i \cdot S^k \geq 0$ for $k = 1, \ldots, m$ and $k \neq i$.

When expression (18) is applied, vector v has a positive number in position i and $-\infty$ in the other positions. Thus, if a maximum function is applied to vector v, it will return i, and finally the restored pattern is thereforep^i.

Theorem 2. *Let **M** be a GAM built as in Sec. 4.2.1 where* S^i *is the sphere of the i-th pattern (*p^i*). Let* $\tilde{p}^i$ *be a fundamental pattern affected by some type of noise such that:* $\tilde{p}^i = p^i + r$*. If expression* (19) *is true, then pattern* $\tilde{p}^i$ *can be restored by* **M**.

$$(r)^2 \leq \left(\gamma^i\right)^2 \tag{19}$$

Proof. Let $\{p^i\}_{i=1}^m$ be a FS of patterns in R^n and let $S = C - 1/2\,(\gamma)^2 e_\infty$ be the sphere of the pattern p; S was found using the method described in Sec. 4.2. Let $\tilde{p}$ be the same pattern but affected by some type of noise such that:

$\tilde{p} = p + r$. The center of the sphere is the pattern itself, this is $c = p$. The noise of the pattern can be expressed as:

$$\begin{aligned} \tilde{p} &= p + r \\ r &= \tilde{p} - p \\ -r &= p - \tilde{p} \\ (r)^2 &= (p - \tilde{p})^2 . \end{aligned} \tag{20}$$

When the inner product between S and the conformal representation of $\tilde{p}$ is calculated, by expression (8), the following expression is obtained:

$$\begin{aligned} 2\left(\tilde{P} \cdot S\right) &= (\gamma)^2 - (c - \tilde{p})^2 \\ &= (\gamma)^2 - (p - \tilde{p})^2 \\ &= (\gamma)^2 - (r)^2 . \end{aligned} \tag{21}$$

Then:
If $(r)^2 > (\gamma)^2$ *then* $\left(\tilde{P} \cdot S\right) < 0$.
If $(r)^2 \leq (\gamma)^2$ *then* $\left(\tilde{P} \cdot S\right) \geq 0$.
Therefore, by a), the pattern $\tilde{p}$ will be restored if and only if $(r)^2 \leq (\gamma)^2$.■

Note that Theorem 2 does not depend on the type of noise used.

6. Numerical Example

In this section some illustrative examples are given. Numerical results, are presented for the problem of restoring sets of patterns. For simplicity, and in order to clarify the results, a 2D Euclidean space for the geometric problem will be used.

Example 1. Let the following be a linearly-separable fundamental set of patterns in R^2:

$$\begin{aligned} x^1 &= \left[\begin{array}{cc} 1 & 1 \end{array}\right], \\ x^2 &= \left[\begin{array}{cc} -1 & 1 \end{array}\right], \\ x^3 &= \left[\begin{array}{cc} 1 & -1 \end{array}\right], \\ x^4 &= \left[\begin{array}{cc} -1 & -1 \end{array}\right], \end{aligned} \tag{22}$$

Figure 1 shows a graphical representation of these patterns.

Using the method given in section 4.2, and the value of $\varepsilon = 0.0001$, the corresponding spheres are obtained. Their centers and radius are respectively:

$$\begin{aligned} c^1 &= \left[\begin{array}{cc} 1 & 1 \end{array}\right], & \gamma^1 &= 1.99 \\ c^2 &= \left[\begin{array}{cc} 1 & -1 \end{array}\right], & \gamma^2 &= 1.99 \\ c^3 &= \left[\begin{array}{cc} -1 & 1 \end{array}\right], & \gamma^3 &= 1.99 \\ c^4 &= \left[\begin{array}{cc} -1 & -1 \end{array}\right], & \gamma^4 &= 1.99 \end{aligned} \tag{23}$$

Note that the centers of the spheres are their own patterns. Another way to find these values is by finding the minimum of function (13) but taking into account the corresponding constraints. A graphical solution can be seen in Fig. 2.

The GAM **M** is:

$$\mathbf{M} = \left[\begin{array}{c} S^1 = C^1 - \left(\gamma^1\right)^2 e_\infty \\ S^2 = C^2 - \left(\gamma^2\right)^2 e_\infty \\ S^3 = C^3 - \left(\gamma^3\right)^2 e_\infty \\ S^4 = C^4 - \left(\gamma^4\right)^2 e_\infty \end{array}\right] = \left[\begin{array}{c} S^1 = e_1 + e_2 - 0.98e_\infty + e_0 \\ S^2 = e_1 - e_2 - 0.98e_\infty + e_0 \\ S^3 = -e_1 + e_2 - 0.98e_\infty + e_0 \\ S^4 = -e_1 - e_2 - 0.98e_\infty + e_0 \end{array}\right] \tag{24}$$

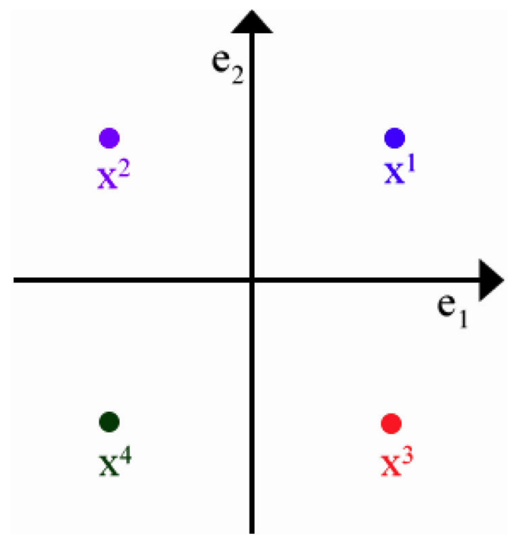

FIGURE 1. Graphical representation for the sets of patterns of Example 1.

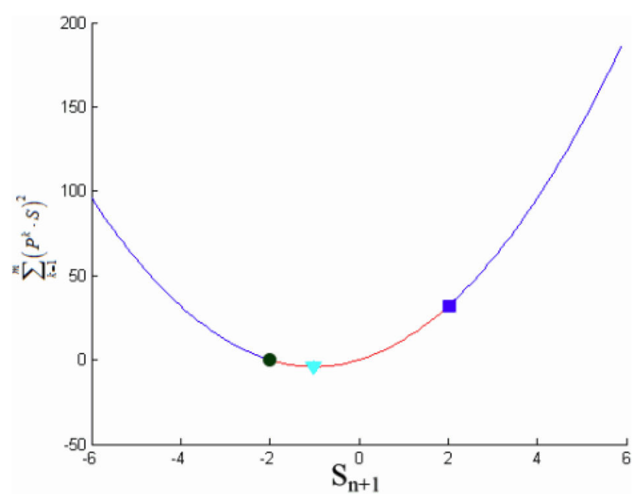

FIGURE 2. Graphical solution to the minimization problem, the circle and the square are constraints for the points inside and outside of the sphere respectively. The triangle is the minimum point.

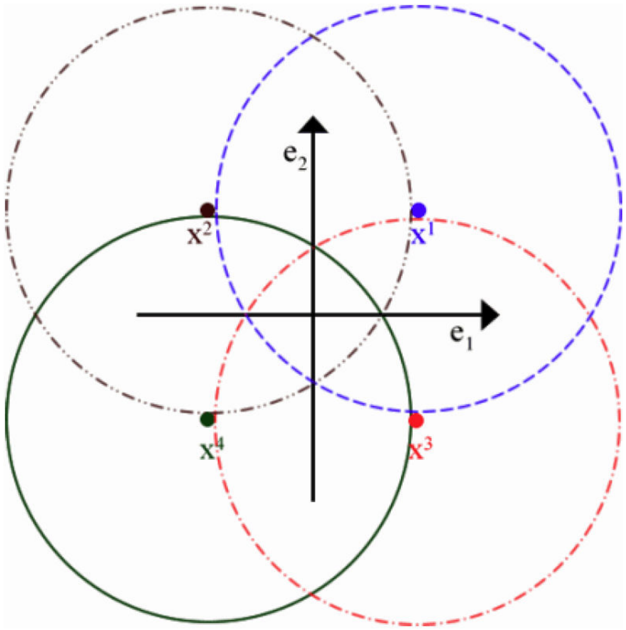

FIGURE 3. Spheres for the example 1, they were obtained using the method from Sec. 4.2.

FIGURE 4. Samples of the images for the first test set.

In Fig. 3 the corresponding spheres (in this case circles) of each pattern are presented.

Let the following be a set of patterns to be restored (these are patterns that belong to FS but they were affected with additive, subtractive, and mixed noise):

$$\begin{aligned}
\tilde{x}^1 &= x^1 + \left[\begin{array}{cc} 0 & 1.5 \end{array}\right] = \left[\begin{array}{cc} 1 & 2.5 \end{array}\right], \\
\tilde{x}^2 &= x^2 + \left[\begin{array}{cc} 0 & -1.9 \end{array}\right] = \left[\begin{array}{cc} 1 & -2.9 \end{array}\right], \\
\tilde{x}^3 &= x^3 + \left[\begin{array}{cc} 0.5 & -0.5 \end{array}\right] = \left[\begin{array}{cc} -0.5 & 0.5 \end{array}\right], \\
\tilde{x}^4 &= x^4 + \left[\begin{array}{cc} -1 & -0.5 \end{array}\right] = \left[\begin{array}{cc} -2 & -1.5 \end{array}\right],
\end{aligned} \quad (25)$$

Note that, for all of these examples, condition (19) is true. If Eq. (18) is applied, the result is:

$$\mathbf{M}\cdot\tilde{X}^1 = \left[\begin{array}{c} -1.13 \\ -\infty \\ -\infty \\ -\infty \end{array}\right], \qquad \mathbf{M}\cdot\tilde{X}^2 = \left[\begin{array}{c} -\infty \\ -1.81 \\ -\infty \\ -\infty \end{array}\right],$$

$$\mathbf{M}\cdot\tilde{X}^3 = \left[\begin{array}{c} -1.25 \\ -\infty \\ -0.25 \\ -1.25 \end{array}\right], \qquad \mathbf{M}\cdot\tilde{X}^4 = \left[\begin{array}{c} -\infty \\ -\infty \\ -\infty \\ -0.63 \end{array}\right], \quad (26)$$

Final step consists in applying a maximization function, then for $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$, and $\tilde{x}^4$, $j = 1, 2, 3, 4$ respectively. Therefore, returned centers for each patterns are x^1, x^2, x^3, and x^4, respectively, which is correct.

7. Real Examples

In this section the performance of the proposal is tested. For this, experiments were conducted with two sets of images. The first set is composed of 25 photos of animals (Fig. 4); these images were obtained from the internet. The second set is composed of 21 images (Fig. 5); these images were taken with a standard camera at the laboratory. The camera has the capability of manually focusing and misfocusing, the targets.

The 25 images of animals are gray scale pictures 300×300 pixels in size. The 21 images of the second set are also gray scale pictures 314 × 235 pixels in size.

The first set was used to test the proposal when images were distorted by adding some kind of noise to them. A comparison with other reported methods is also given. The second set was used to test the proposal when images were distorted by misfocusing.

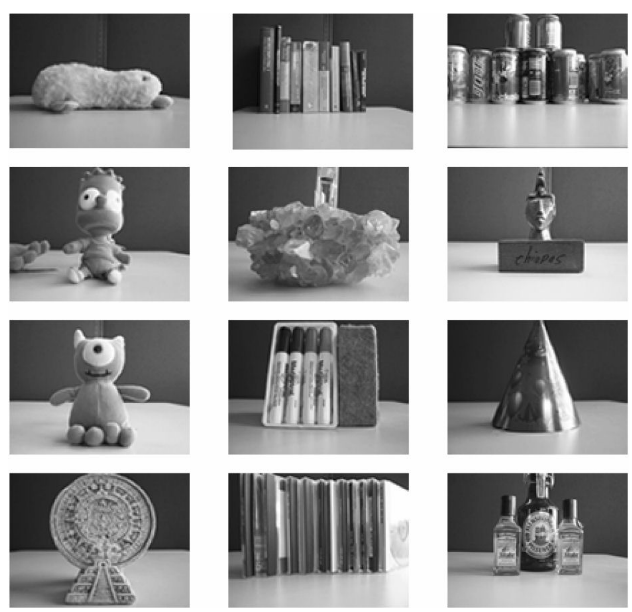

FIGURE 5. Samples of the images for the second test set.

TABLE I. Number of images correctly restored in restoration phase.

Set	0%	5%	10%	25%	30%	40%	50%
Animals	25	25	25	25	24	15	6

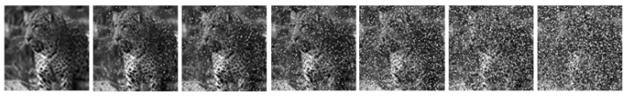

FIGURE 6. From left to right: original image, image with 5, 10, 25, 30, 40, and 50% of noise, respectively.

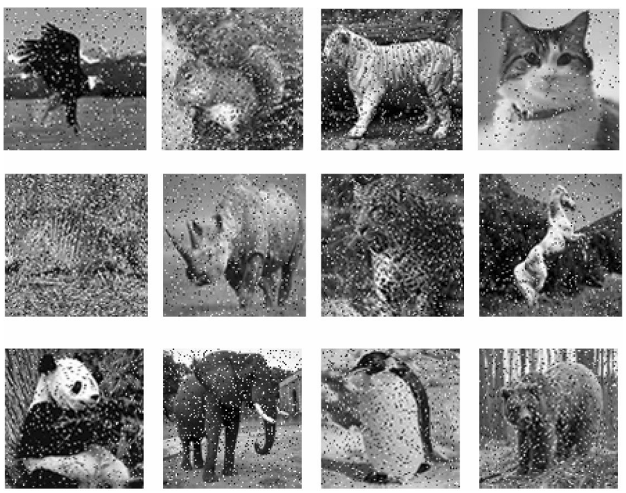

FIGURE 7. Samples of noisy images for the first test set.

FIGURE 8. Restored images of the Fig. 7 for the first test.

7.1. Performance of the proposal in the presence of noise

In this case, image pre-processing was omitted. Patterns were built as image-vectors by taking the first row of a given image, then joining at the end of this row the second row of the image, and so on until ending with the last row of the image. This way, image-patterns were made up of 90000 elements. These image-patterns formed the fundamental set.

Two test sets were formed by taking the 25 images of set number one, by adding 5, 10, 25, 30, 40, and 50% of randomly mixed noise at them. Figure 6 shows six noisy versions of one of the images of that set.

The corresponding restoration results for all images are shown in Table I. The first column presents the name of the set. The next columns show the number of restored images for the complete set of images with 0 (fundamental set), 5, 10, 25, 30, 40, and 50% of noise, respectively.

As can be observed from this table, image restoration was correct until the case of 25% of noise. For the case of 30% of noise only one image was not restored. As can be appreciated from this same table, as the level of noise is increased, fewer images were correctly restored.

This experiment was repeated 100 times with different noise distribution over the images (see Fig. 7). The results were the same as shown in Table I from 0 to 30% of noise. This happened because Theorem 2 is true for the images with that quantity of noise. For 40 and 50% of noise the average of the number of images restored is given again in 6.2. Examples of restored images are given in Fig. 8.

7.2. Comparison with other methods and discussion

In this section, restoration results for the same two sets used to test the proposal when the patterns are distorted with noise are presented. In this case the same noisy test set of images were used to test the functioning of the average filter, the median filter, the morphological filter (open-close) (Gonzalez and Woods 2008), dust and points removal tool for Paint Shop Pro X2, sub-patterns and AM's described in (Cruz, Sossa and Barrón 2007).

Figure 9 shows a comparison for one image with 5, 10, 25, 30, 40 and 50% of noise and the corresponding restored image by using the proposed, and the mentioned techniques. Table II and Table III show the mean square error for two different images. The columns show the mean square error obtained between the corresponding restored images of each method used. Each row of these tables correspond to a test image with 5, 10, 25, 30, 40, and 50% of randomly mixed noise respectively.

As can be observed, the means square error for GAMs and sub-patterns is zero in all cases; this is because all the images are encoded into the memory. This is the main feature of an associative memory. The other models show good results, but when the image has completely lost information, these methods do not work correctly. For example in the case of the the images in Fig 10, they cannot be restored by means of those methods, but the GAM is capable of do it.

7.3. Performance of the proposal in the presence of missing information

Another experiment was implemented to test the potential of the GAMs; in this case the images in Fig. 4 were affected by

TABLE II. Mean square error for a restored image

Noise	Average Filter	Dust Removal	Point Removal	Median Filter	Morph. Filter	Sub-Patterns	GAM
5%	1165529	632256	328043	461169	3352608	0	0
10%	1833253	894465	503496	509266	4980297	0	0
25%	2858467	1312237	1312237	717805	6712505	0	0
30%	6306509	3498588	2893229	1730484	12392659	0	0
40%	8877902	6272391	4837072	3252628	15759837	NA	0
50%	12031332	10442603	8683319	6375399	19111305	NA	0

TABLE III. Mean square error for another restored image

Noise	Average Filter	Dust Removal	Point Removal	Median Filter	Morph. Filter	Sub-Patterns	GAM
5%	738906	673584	331396	516934	754959	0	0
10%	926361	686920	435422	562626	1058108	0	0
25%	1187129	703110	512046	572439	1397366	0	0
30%	2109969	963170	1058996	801835	2692502	NA	0
40%	2988743	1410100	1534965	1079258	3620676	NA	0
50%	3911128	2018000	2263242	1578445	4812627	NA	0

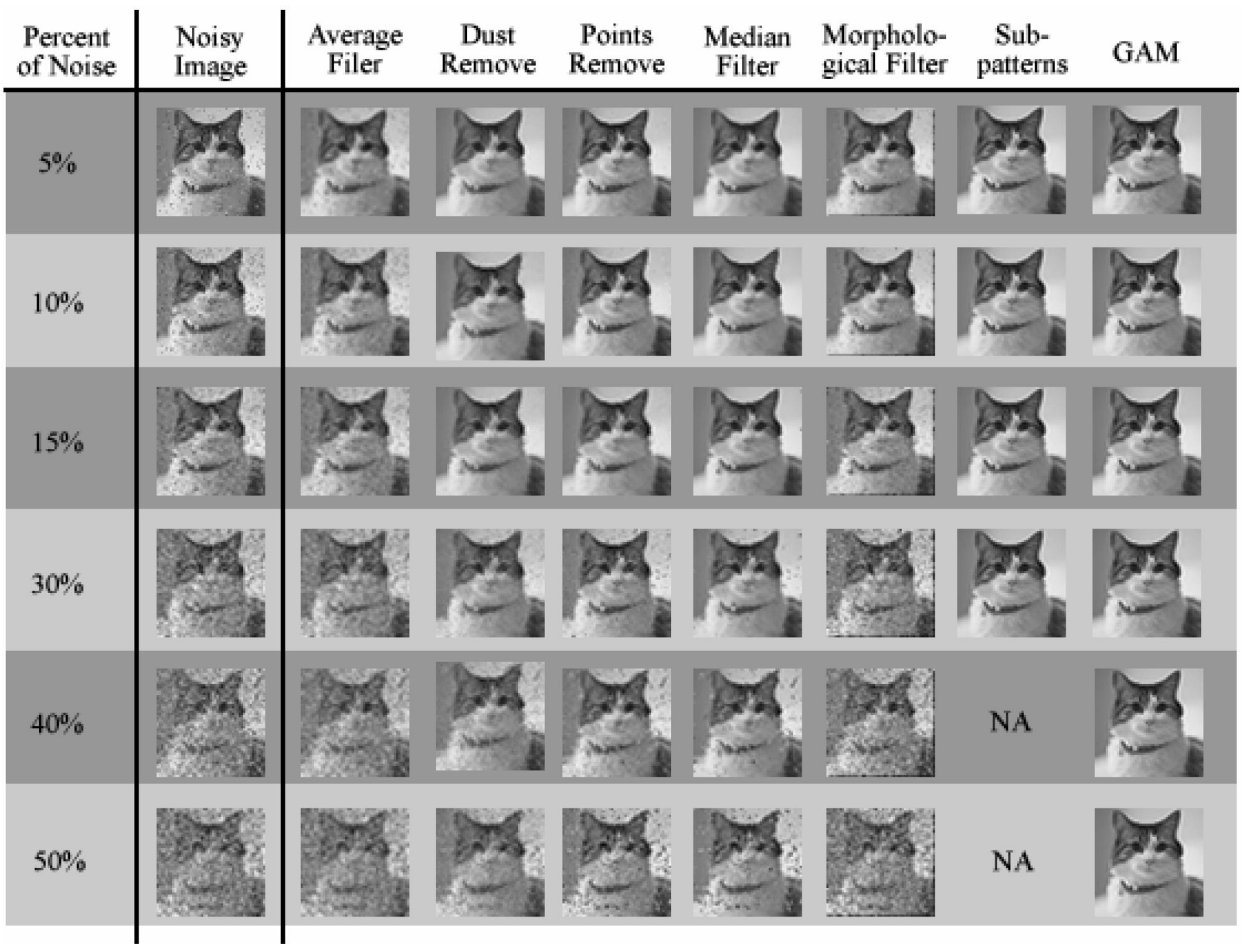

FIGURE 9. Comparison among the proposed memory and other image restoration techniques.

FIGURE 10. Samples of images with some of its information missed.

FIGURE 11. Restored images of 7.3. for the second test.

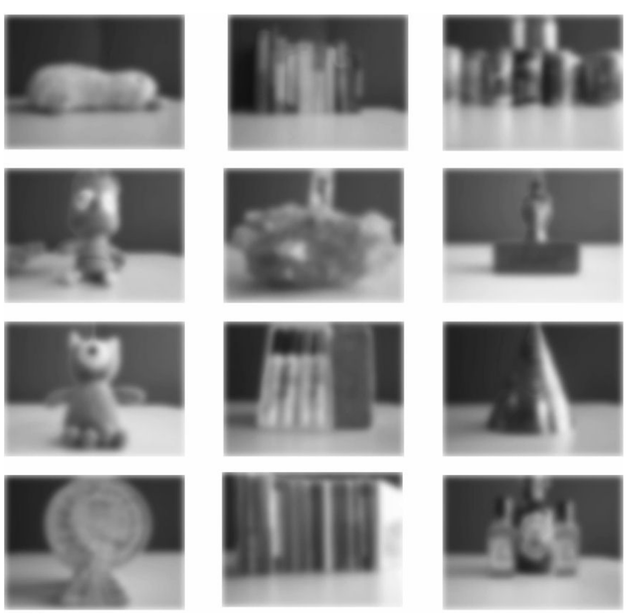

FIGURE 12. Samples of images with misfocusing.

FIGURE 13. Three kind of images affected with misfocusing.

noise for to covering a part or parts of the images; samples of those distorted images can be seen in Fig 10. The same trained GAM of the previous experiments was also used.

In all cases the corresponding images were restored correctly with the GAM (see Fig. 11). Other tests were done with similar images and with images of the scientist set with similar kinds of distortions.

We saw that while the condition for robust restoration was met, the GAM was capable of restoring the corresponding image from the noisy version. A simple application of this experiment could be the recovering of images by using part of them.

Note that the images in Fig. 10 cannot be restored using the traditional methods.

7.4. Performance of the proposal in the presence of misfocusing

In this section the functioning of the proposal was tested when the patterns were distorted by misfocusing. For this a set of 21 images was used. Some of them are shown in Fig. 5.

These images were taken with a standard photographic camera. Distorted versions of these images were obtained by manually changing the focus of the lens. Examples of these distorted images by misfocusing are shown in Fig. 12.

The idea was to adjust the camera's focusing device to get some misfocussing versions of the target, see Fig. 13. The original image set was used to build the GAM.

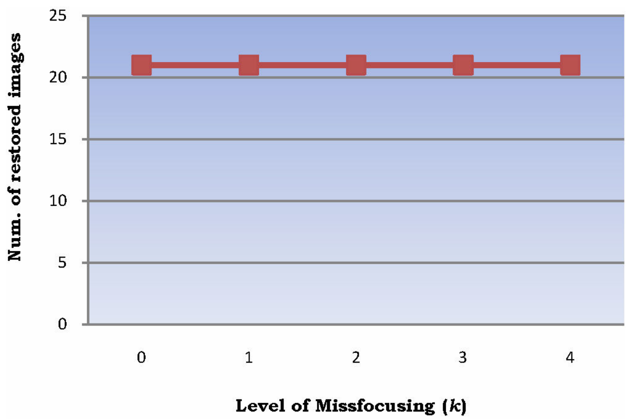

FIGURE 14. Results of the restored images affected with misfocusing.

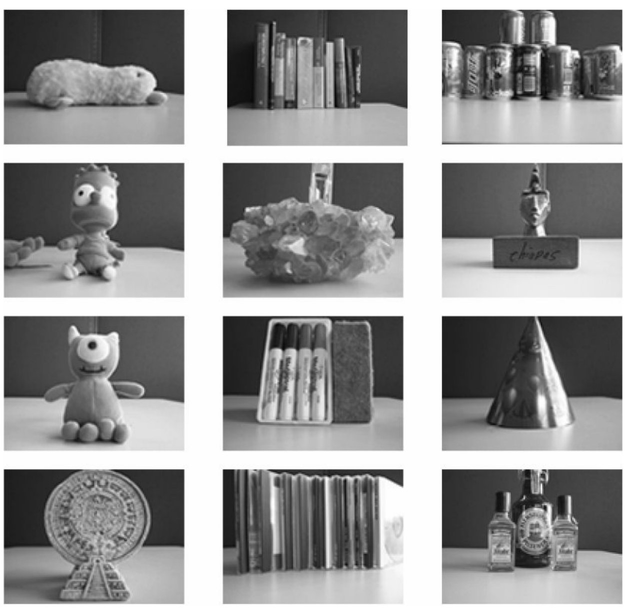

FIGURE 15. Samples of restored images for the third test.

All the set of misfocusing images were used to test the functioning of the already trained GAMs; the results are shown in the graph of Fig. 14. As can be observed, all 21 images with the four levels of misfocusing were correctly restored.

Note that all the images restored were completely equal to the original images; the least square error obtained was zero in all cases (see Fig. 15).

8. Conclusions and future work

Geometric Associative Memories (GAMs) were, firstly, developed for pattern classification. In the learning phase, optimization techniques were used. In that operation mode, for the classification phase an inner product between the unclassified pattern and the GAM itself must be applied. In that case the result is a class index.

In this work a new operation mode for the GAMs was described, in this case for pattern restoration. The optimization problem of the learning phase was reformulated for the operation in restoration mode. The restoration phase is similar to the classification mode, but in this case a restored pattern is returned.

Numerical and real examples were presented to demonstrate the potential of the proposal. Graphical solutions for the optimization problem were shown also. The proposed model can efficiently cope with patterns affected by additive, subtractive, mixed noise and distortions caused by misfocusing and missed information. A comparison with different pattern restoration methods such as linear and morphological filters, and the so-called sub-pattern method, was carried out.

Our proposal performed considerably better than the other methods, because whole patterns were encoded into the memory itself; then restored patterns were precisely the original ones. Also, GAM's were capable of restoring images with missing information.

In comparison with other pattern recognition methods like neural networks is that the training phase of the GAM's presented in this work converge in one step. This happened because the way of they are built (see Sec. 5.1.)

Formal conditions under which the proposed model can work were also given and proven. In Particular for the cases of perfect and robust restoration were presented.

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