Anisotropic cosmology in Sáez-Ballester theory: classical and quantum solutions

J. Socorro* and M. Sabido**
Departamento de Física, DCeI, Universidad de Guanajuato-Campus León, Apartado Postal E-143, 37150, Guanajuato, México.

M.A. Sánchez and M.G. Frías Palos
Facultad de Ciencias de la Universidad Autónoma del Estado de México, Instituto Literario No. 100, Toluca, 50000, Edo de Mex, México.

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We use the Sáez-Ballester theory on anisotropic Bianchi I cosmological model, with barotropic flui and cosmological constant. We obtain the classical solution by using the Hamilton-Jacobi approach. Also the quantum regime is constructed and exact solutions to the Wheeler-DeWitt equation are found.

Keywords: Classical and quantum exact solutions; cosmology.

Usamos la teoría de Sáez-Ballester en el modelo anisotrópico Bianchi I con un fluid barotrópico y constante cosmológica. Obtenemos las soluciones clásicas usando el enfoque de Hamilton-Jacobi. El regimen cuántico también es construido y soluciones exactas a la ecuación de Wheeler-DeWitt son encontradas.

Descriptores: Soluciones clásicas y cuánticas exactas; cosmología.

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1. Introduction

In the 80's Saez and Ballester [1] formulated a scalar-tensor theory of gravity in which the metric is coupled to a dimensionless scalar field several papers in the classical regime have been written [2-5], yet a study of anisotropic models, where the anisotropy is introduced in the line element, has been connected. In this theory of gravity the strength of the coupling between gravity and the scalar fiel is determined by an arbitrary coupling function ω; one particularly interesting result is the appearance of an antigravity regime, which suggests a possible connection to the missing matter problem in non-fla FRW cosmologies. In particular, Armendariz-Picon, *et al.*, related this scenario to *K-essence* [6], which is characterized by a scalar fiel with a non-canonical kinetic energy. Usually K-essence models are restricted to lagrangian densities of the form

$$\mathrm{S} = \int \mathrm{d}^4\mathrm{x}\, \sqrt{-\mathrm{g}}\, \mathrm{f}(\phi)\, (\nabla\phi)^2\,; \tag{1}$$

one of the motivations for considering this type of lagrangian originates from string theory [7] and is relation to the Dark energy problem (for more details for K-essence applied to dark energy, see Ref. 8 and reference therein).

Furthermore, the quantization program of the theory is an open problem; this is related to the difficultie of building the ADM formalism. In order to proceed with the quantization program, we transform this theory to a conventional one, by interpreting the dimensionless scalar fiel as part of the energy-momentum tensor as an exotic matter component; this is achieved by an appropriate transformation of the coupling function ω, so that we can use this modifie theory where the ADM formalism is well known [9].

In this paper we use this formulation to obtain classical and quantum solutions for the anisotropic Bianchi type I cosmological model with cosmological constant Λ. The firs step is to write Sáez-Ballester formalism in the usual manner, that is, we calculate the corresponding energy-momentum tensor to the scalar fiel and give the equivalent lagrangian density. Next, we proceed to obtain the corresponding canonical lagrangian for the Bianchi type I metric and calculate the classical hamiltonian constraint $\mathcal{H}$, and finall fin the Wheeler-DeWitt (WDW) equation for the model.

The simplest generalization of the lagrangian density for the Sáez-Ballester theory [1] with cosmological constant, is

$$\mathcal{L}_{\mathrm{geo}} = \left(\mathrm{R} - 2\Lambda - \mathrm{F}(\phi)\phi_{,\gamma}\phi^{,\gamma}\right), \tag{2}$$

where $\phi^{,\gamma} = g^{\gamma\alpha}\phi_{,\alpha}$, R is the scalar curvature, and $F(\phi)$ a dimensionless functional of the scalar field In classical fiel theory, this formalism corresponds to null potential for ϕ, but with an exotic kinetic term due to the presence of $F(\phi)$.

From the lagrangian (2) we can build the complete action

$$\mathrm{I} = \int_{\Sigma} \sqrt{-\mathrm{g}}(\mathcal{L}_{\mathrm{geo}} + \mathcal{L}_{\mathrm{mat}})\mathrm{d}^4\mathrm{x}, \tag{3}$$

where $\mathcal{L}_{mat}$ is the matter lagrangian, and g is the determinant of metric tensor. The fiel equations for this theory are

$$\mathrm{G}_{\alpha\beta}+\mathrm{g}_{\alpha\beta}\Lambda-\mathrm{F}(\phi)\left(\phi_{,\alpha}\phi_{,\beta}-\frac{1}{2}\mathrm{g}_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma}\right)=8\pi\mathcal{G}\mathrm{T}_{\alpha\beta},$$
$$2\mathrm{F}(\phi)\phi^{;\alpha}_{;\alpha}+\frac{\mathrm{dF}}{\mathrm{d}\phi}\phi_{,\gamma}\phi^{,\gamma}=0, \tag{4}$$

where $\mathcal{G}$ is the gravitational constant and as usual the semicolon means a covariant derivative.

This set of Eqs. (4) can be obtained using the general relativity lagrangian with a particular energy-momentum tensor for the fiel ϕ,

$$\mathcal{L}_\phi = \mathrm{F}(\phi)\mathrm{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta},$$

$$\mathrm{T}_{\alpha\beta}(\phi) = \mathrm{F}(\phi)\left(\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}\mathrm{g}_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma}\right), \tag{5}$$

and we can rewrite action (3) as

$$\mathrm{I} = \int_\Sigma \sqrt{-\mathrm{g}}\left(\mathrm{R} - 2\Lambda + \mathcal{L}_{\mathrm{mat}} + \mathcal{L}_\phi\right)\mathrm{d}^4\mathrm{x}; \tag{6}$$

consequently, the classical equivalence between the two theories is established through the equations of motion. We can infer that this correspondence can be carried to the quantum regime, because only the hamiltonian constraint [9] is modified

This work is arranged as follows, in Sec. 2 we construct the hamiltonian density for the cosmological model. In Sec. 3 the classical solutions using the Hamilton-Jacobi formalism are found. Here, we have used a barotropic perfect flui as a matter content and a cosmological constant, obtaining the solutions for different epochs of evolution. In Sec. 4 the quantization scheme is presented, obtaining the corresponding Wheeler-DeWitt equation and its solutions for different values of the γ parameter. Finally, the Sec. 5 is devoted to discussion.

2. The hamiltonian density

The line element for the Bianchi type I model has the form

$$\mathrm{ds}^2 = -N^2dt^2 + e^{2\Omega+2\beta_+ +2\sqrt{3}\beta_-}\left(dx^1\right)^2 + e^{2\Omega+2\beta_+ -2\sqrt{3}\beta_-}\left(dx^2\right)^2 + e^{2\Omega-4\beta_+}\left(dx^3\right)^2 \tag{7}$$

where N is the lapse function, $\beta_\pm$ are the corresponding anisotropic parameters in the scale factors, and Ω plays an analogous role as the scale factor in fla Friedmann-Robertson-Walker cosmology ($e^\Omega \equiv a$). The total volume for all diagonal Bianchi cosmological models is given by $\mathrm{V} = \mathrm{e}^{3\Omega(\mathrm{t})}$, which appears in the solutions for this theory. The corresponding lagrangian density is

$$\mathcal{L} = \frac{6\dot\Omega^2\mathrm{e}^{3\Omega}}{\mathrm{N}} - 6\frac{\dot\beta_+^2\mathrm{e}^{3\Omega}}{\mathrm{N}} - 6\frac{\dot\beta_-^2\mathrm{e}^{3\Omega}}{\mathrm{N}} + \frac{F(\phi)}{N}\dot\phi^2e^{3\Omega} + \left(16N\pi G\rho - 2N\Lambda\right)e^{3\Omega}.$$

which can be rewritten in canonical form,

$$\mathcal{L}_{can} = \Pi_\Omega\dot\Omega + \Pi_{\beta_+}\dot\beta_+ + \Pi_{\beta_-}\dot\beta_- + \Pi_\phi\dot\phi - \mathrm{N}\mathcal{H}, \tag{8}$$

with $\mathcal{H}$ as the hamiltonian density and the momenta define in the usual way: $\Pi_{q^i} = \partial\mathcal{L}/\partial\dot{\mathrm{q}}^{\mathrm{i}}$, where $\mathrm{q}^{\mathrm{i}} = (\Omega, \beta_+, \beta_-, \phi)$ are the fiel coordinates for this system,

$$\Pi_\Omega = \frac{\partial\mathcal{L}}{\partial\dot\Omega} = 12\frac{\dot\Omega\mathrm{e}^{3\Omega}}{\mathrm{N}}, \quad\to\quad \frac{\Pi_\Omega}{2} = 6\frac{\dot\Omega\mathrm{e}^{3\Omega}}{\mathrm{N}} \quad\to\quad \dot\Omega = \mathrm{Ne}^{-3\Omega}\frac{\Pi_\Omega}{12}, \tag{9}$$

$$\Pi_{\beta_+} = \frac{\partial\mathcal{L}}{\partial\dot\beta_+} = -12\frac{\dot\beta_+\mathrm{e}^{3\Omega}}{\mathrm{N}}, \quad\to\quad \frac{\Pi_{\beta_+}}{2} = -6\frac{\dot\beta_+\mathrm{e}^{3\Omega}}{\mathrm{N}} \quad\to\quad \dot\beta_+ = -\mathrm{Ne}^{-3\Omega}\frac{\Pi_{\beta_+}}{12}, \tag{10}$$

$$\Pi_{\beta_-} = \frac{\partial\mathcal{L}}{\partial\dot\beta_-} = -12\frac{\dot\beta_-\mathrm{e}^{3\Omega}}{\mathrm{N}}, \quad\to\quad \frac{\Pi_{\beta_-}}{2} = -6\frac{\dot\beta_-\mathrm{e}^{3\Omega}}{\mathrm{N}} \quad\to\quad \dot\beta_- = -\mathrm{Ne}^{-3\Omega}\frac{\Pi_{\beta_-}}{12}, \tag{11}$$

$$\Pi_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = 2\frac{\mathrm{F}(\phi)\dot\phi\mathrm{e}^{3\Omega}}{\mathrm{N}}, \quad\to\quad \frac{\Pi_\phi}{2} = \frac{\mathrm{F}(\phi)\dot\phi\mathrm{e}^{3\Omega}}{\mathrm{N}} \quad\to\quad \dot\phi = \frac{\mathrm{N}\Pi_\phi}{2\mathrm{F}(\phi)\mathrm{e}^{3\Omega}}. \tag{12}$$

The matter content is introduced as a barotropic perfect flui $P = \gamma\rho$ where γ is a constant between $-1 < \gamma < 1$, with its energy-momentum tensor given by $\mathrm{T}_{\mu\nu} = (\rho + \mathrm{P})\mathrm{U}_\mu\mathrm{U}_\nu - \mathrm{g}_{\mu\nu}\mathrm{P}$ where U_μ is the four-velocity, ρ is the energy density and P is the thermodynamic pressure in the fluid from this tensor we obtain the differential equation $3\dot\Omega\rho + 3\dot\Omega\mathrm{P} + \dot\rho = 0$ and the solution $\rho = \mu_\gamma\mathrm{e}^{-3\Omega(1+\gamma)}$. with μ_γ a constant for the corresponding scenario. Finally we obtain the following expression for the canonical lagrangian:

$$\mathcal{L}_{can} = \Pi_{\mathrm{q}^{\mathrm{i}}}\dot{\mathrm{q}}^{\mathrm{i}} - \frac{\mathrm{Ne}^{-3\Omega}}{24}\left[\Pi_\Omega^2 - \Pi_{\beta_+}^2 - \Pi_{\beta_-}^2 + \frac{6\Pi_\phi^2}{\mathrm{F}(\phi)} - 384\pi\mathrm{G}\mu_\gamma\mathrm{e}^{3\Omega(1-\gamma)} + 48\Lambda\mathrm{e}^{6\Omega}\right],$$

and

$$\mathcal{H} = \frac{\mathrm{e}^{-3\Omega}}{24}\left(\Pi_\Omega^2 - \Pi_{\beta_+}^2 - \Pi_{\beta_-}^2 + \frac{6\Pi_\phi^2}{\mathrm{F}(\phi)} - 384\pi\mathrm{G}\mu_\gamma\mathrm{e}^{3\Omega(1-\gamma)} + 48\Lambda\mathrm{e}^{6\Omega}\right), \tag{13}$$

and using the lagrange equation for the fiel N, we get the Hamiltonian constraint

$$\mathcal{H} = \Pi_\Omega^2 - \Pi_{\beta_+}^2 - \Pi_{\beta_-}^2 + \frac{6\Pi_\phi^2}{\mathrm{F}(\phi)} - 384\pi\mathrm{G}\mu_\gamma\mathrm{e}^{3\Omega(1-\gamma)} + 48\Lambda\mathrm{e}^{6\Omega} = 0. \tag{14}$$

TABLE I. Solutions for Ω in the scenarios $\gamma = -1, 0, 1$

Case		$\Omega(\tau)$
Inflation $\gamma = -1$	$b_{-1} = 384\pi G\mu_{-1} - 48\Lambda > 0$	$\frac{1}{3}\,\mathrm{Ln}\left[\frac{1}{4\sqrt{b_{-1}}}\left(e^{\frac{\sqrt{b_{-1}}}{4}\Delta\tau} - 4\xi^2 e^{-\frac{\sqrt{b_{-1}}}{4}\Delta\tau}\right)\right]$
Dust, $\gamma = 0$	$b_0 = 384\pi G\mu_0,\ \Lambda < 0$	$\frac{1}{3}\mathrm{Ln}\left[\frac{\left(e^{\sqrt{-3\Lambda}\Delta\tau} - \frac{b_0}{4\sqrt{-3\Lambda}}\right)^2 - 4\xi^2}{16\sqrt{-3\Lambda}e^{\sqrt{-3\Lambda}\Delta\tau}}\right]$
Stiff matter, $\gamma = 1$	$b_1 = \xi^2 + 384\pi G\mu_1, \Lambda < 0$	$\frac{1}{3}\,\mathrm{Ln}\left[\frac{1}{16\sqrt{-3\Lambda}}\left(e^{\sqrt{-3\Lambda}\Delta\tau} - 4b_1 e^{-\sqrt{-3\Lambda}\Delta\tau}\right)\right]$

TABLE II. Solutions for the anisotropic variables $\beta_\pm$ and fiel ϕ in the scenarios $\gamma = -1, 0, 1$

Case	$\beta_\pm(\tau)$	$\phi(\tau)$
$\gamma = -1, \omega > 0$	$\mp\frac{2\kappa_\pm}{3\xi}\tanh^{-1}\left(\frac{e^{\frac{\sqrt{b_{-1}}}{4}\Delta\tau}}{2\xi}\right)$	$\left[\mp\frac{2\theta(m+2)}{\xi\sqrt{6\omega}}\tanh^{-1}\left(\frac{\mathrm{Exp}\left(\frac{\sqrt{b_{-1}}}{4}\Delta\tau\right)}{2\xi}\right)\right]^{\frac{2}{m+2}}, \quad m \neq -2$
$k_+ = -\sqrt{3}k_-$		$\mathrm{Exp}\left[\mp\frac{4\theta}{\xi\sqrt{6\omega}}\tanh^{-1}\left(\frac{\mathrm{Exp}\left(\frac{\sqrt{b_{-1}}}{4}\Delta\tau\right)}{2\xi}\right)\right]$, m=-2
$\gamma = 0$	$\pm\frac{2\kappa_\pm}{3\xi}\tanh^{-1}\left(\frac{-b_0 + 4\sqrt{-3\Lambda}e^{\sqrt{-3\Lambda}\Delta\tau}}{8\sqrt{-3\Lambda}\xi}\right)$	$\left[\pm\frac{2\theta(m+2)}{\xi\sqrt{6\omega}}\tanh^{-1}\left(\frac{-b_0 + 4\sqrt{-3\Lambda}e^{\sqrt{-3\Lambda}\Delta\tau}}{8\sqrt{-3\Lambda}\xi}\right)\right]^{\frac{2}{m+2}}, \quad m \neq -2$
$\omega > 0$		$\mathrm{Exp}\left[\pm\frac{4\theta}{\xi\sqrt{6\omega}}\tanh^{-1}\left(\frac{-b_0 + 4\sqrt{-3\Lambda}e^{\sqrt{-3\Lambda}\Delta\tau}}{8\sqrt{-3\Lambda}\xi}\right)\right]$, m=-2,
$\gamma = 1, \omega > 0$	$\mp\frac{2\kappa_\pm}{3\sqrt{b_1}}\tanh^{-1}\left(\frac{e^{\sqrt{-3\Lambda}\tau}}{2\sqrt{b_1}}\right)$	$\left[\mp\frac{2\theta(m+2)}{\sqrt{6b_1\omega}}\tanh^{-1}\left(\frac{e^{\sqrt{-3\Lambda}\Delta\tau}}{2\sqrt{b_1}}\right)\right]^{\frac{2}{m+2}}, \quad m \neq -2$
$k_+ = -\sqrt{3}k_-$		$\mathrm{Exp}\left[\mp\frac{4\theta}{\sqrt{6b_1\omega}}\tanh^{-1}\left(\frac{e^{\sqrt{-3\Lambda}\Delta\tau}}{2\sqrt{b_1}}\right)\right]$, m=-2

3. Classical regimen: Hamilton-Jacobi approach

Employing the Hamilton-Jacobi formulation, the momenta are $\Pi_q = \partial S_q/\partial q$ and S_q is the superpotential function; then the hamiltonian takes the following form:

$$\left(\frac{\partial S_\Omega}{\partial\Omega}\right)^2 - \left(\frac{\partial S_{\beta_+}}{\partial\beta_+}\right)^2 - \left(\frac{\partial S_{\beta_-}}{\partial\beta_-}\right)^2 + \frac{6}{F(\phi)} \times \left(\frac{\partial S_\phi}{\partial\phi}\right)^2 - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega} = 0, \quad (15)$$

and solving for Ω we have

$$\left(\frac{\partial S_\Omega}{\partial\Omega}\right)^2 - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega} = \left(\frac{\partial S_{\beta_+}}{\partial\beta_+}\right)^2 + \left(\frac{\partial S_{\beta_-}}{\partial\beta_-}\right)^2 - \frac{6}{F(\phi)}\left(\frac{\partial S_\phi}{\partial\phi}\right)^2 = \xi^2.$$

Using (9), we obtain the following integral equation that depends on the time parameter $N dt = d\tau$, and so we have

$$\frac{dS_\Omega}{d\Omega} = \sqrt{\xi^2 + 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} - 48\Lambda e^{6\Omega}} = 12e^{3\Omega}\frac{d\Omega}{d\tau} \tag{16}$$

$$\Delta\tau = \int \frac{12}{\sqrt{\xi^2 e^{-6\Omega} + 384\pi G\mu_\gamma e^{-3\Omega(1+\gamma)} - 48\Lambda}} d\Omega \tag{17}$$

where ξ is a separation constant. This equation does not have a general solution, but we can solve for the particular values of the γ; these solutions are presented in the Table I.

The corresponding solutions for the anisotropic functions and the fiel ϕ appear in Table II. For fiel ϕ we consider

$$\frac{6}{F(\phi)}\left(\frac{\partial S_\phi}{\partial \phi}\right)^2 = -\xi^2 + \kappa_+^2 + \kappa_-^2 = \theta^2$$

$$\frac{dS_\phi}{d\phi} = \sqrt{\frac{F(\phi)}{6}}\theta \equiv -2F(\phi)\frac{d\phi}{d\tau}e^{3\Omega}$$

with $\theta^2 = -\xi^2 + \kappa_+^2 + \kappa_-^2$, and with the cuadrature solutions

$$\int \sqrt{F(\phi)}\, d\phi = \frac{\theta}{2\sqrt{6}} \int e^{-3\Omega}(\tau)\, d\tau, \tag{18}$$

Now considering the original Sáez-Ballester theory, $F(\phi) = \omega\phi^m$, the corresponding solutions for all m are show in Table II.

This set of solutions satisfie Einstein's fiel equations (4) (these solutions were checked using the REDUCE package for symbolic calculations).

4. Quantum regime: Wheeler-DeWitt equation

For the quantum regime, we calculate the Wheeler-DeWitt equation

$$\hat{H}\Psi = \left\{\hat{\Pi}_\Omega^2 - \hat{\Pi}_{\beta_+}^2 - \hat{\Pi}_{\beta_-}^2 + \frac{6\hat{\Pi}_\phi^2 \phi^{-m}}{\omega} - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega}\right\}\Psi = 0$$

where the momentum operators are given by $\hat{\Pi}_q = -i\hbar\partial/\partial q$, Ψ is the wave function of the universe, and we have chosen $\hbar = 1$; thus

$$\hat{H}\Psi = \left\{\left(-\frac{\partial^2}{\partial\Omega^2} + Q\frac{\partial}{\partial\Omega}\right) + \frac{\partial^2}{\partial\beta_+^2} + \frac{\partial^2}{\partial\beta_-^2} - \frac{6\phi^{-m}}{\omega} \times \frac{\partial^2}{\partial\phi^2} - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega}\right\}\Psi = 0, \tag{19}$$

where we have used $\left(-\partial^2/\partial\Omega^2 + Q\partial/\partial\Omega\right)$ to address the factor ordering problem. Applying the separation method, $\Psi = \mathcal{A}(\Omega)\,\mathcal{B}(\beta_+)\,\mathcal{C}(\beta_-)\,\mathcal{D}(\phi)$, we obtain

$$\left\{-\frac{1}{\mathcal{A}}\frac{d^2\mathcal{A}}{d\Omega^2} + Q\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\Omega} + \frac{1}{\mathcal{B}}\frac{d^2\mathcal{B}}{d\beta_+^2} + \frac{1}{\mathcal{C}}\frac{d^2\mathcal{C}}{d\beta_-^2} - \frac{1}{\mathcal{D}}\frac{6\phi^{-m}}{\omega}\frac{d^2\mathcal{D}}{d\phi^2} - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega}\right\} = 0,$$

yielding the following set of differential equations:

$$-\frac{1}{\mathcal{A}}\frac{d^2\mathcal{A}}{d\Omega^2} + Q\frac{1}{\mathcal{A}}\frac{d\mathcal{A}}{d\Omega} - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Lambda e^{6\Omega} = -a_1^2, \tag{20}$$

$$\frac{1}{\mathcal{B}}\frac{d^2\mathcal{B}}{d\beta_+^2} = a_2^2, \tag{21}$$

$$\frac{1}{\mathcal{C}}\frac{d^2\mathcal{C}}{d\beta_-^2} = a_3^2, \tag{22}$$

$$\frac{1}{\mathcal{D}}\frac{6\phi^{-m}}{\omega}\frac{d^2\mathcal{D}}{d\phi^2} = a_4^2, \tag{23}$$

where the following relation between the separation variables arises: $a_4^2 = -a_1^2 + a_2^2 + a_3^2$, where the sign for these constants is arbitrary.

The solution to Eqs. (21) and (22) has the generic form

$$\mathcal{B} = e^{\pm a_2\beta_+}, \qquad \mathcal{C} = e^{\pm a_3\beta_-}, \tag{24}$$

The solution to Eq. (23) is more complicated, due to the dependence on m, which is a parameter in Sáez-Ballester theory. This equation is rewritten as $d^2\mathcal{D}/d\phi^2 - \omega a_4^2\phi^m/6\mathcal{D} = 0$, which is analogous to the equation found in Ref. 10, $y'' - ax^n y = 0$, with $a = \omega a_4^2/6$ and $n = m$.

1. Case m=-2 corresponds to the Euler equation, whose solution has the following structure [10]:

$$\mathcal{D} = \sqrt{\phi}\begin{cases} c_1\,\phi^\mu + c_2\phi^{-\mu} & \text{si} \quad a > -\frac{1}{4} \\ c_1\, + c_2 \mathrm{Ln}\phi & \text{si} \quad a = -\frac{1}{4} \\ c_1 \sin(\mu \mathrm{Ln}\phi) + c_2\cos(\mu \mathrm{Ln}\phi) & \text{si} \quad a < -\frac{1}{4} \end{cases} \tag{25}$$

where

$$\mu = (1/2)\sqrt{|1+4a|} >$$

2. In the case m=-4, we introduce the following transformation: $z = 1/\phi$ and $u/z = \mathcal{D}$. The resulting equation has the following solution:

$$\mathcal{D} = \phi\begin{cases} c_1\sinh(\sqrt{a}\phi) + c_2\cosh(\sqrt{a}\phi) & \text{si} \quad a>0 \\ c_1\sin\left(\sqrt{|a|}\phi\right) + c_2\cos\left(\sqrt{|a|}\phi\right) & \text{si} \quad a<0 \end{cases} \tag{26}$$

the case a=0 is discarded as it implies $\omega = 0$ (GR limit).

3. When m satisfie $2/(m+2) = 2n+1$, where n is an integer, the general solution takes the form

$$\mathcal{D}=\phi\left\{\begin{array}{ll}\left(\phi^{1-2\mathrm{q}}\frac{\mathrm{d}}{\mathrm{d}\phi}\right)^{\mathrm{n}+1}\left[\mathrm{D}_6\mathrm{Exp}\left(\sqrt{\frac{\omega}{6}}\frac{\phi^{\mathrm{q}}}{\mathrm{q}}\right)+\mathrm{D}_7\mathrm{Exp}\left(-\sqrt{\frac{\omega}{6}}\frac{\phi^{\mathrm{q}}}{\mathrm{q}}\right)\right] & \mathrm{si}\quad \mathrm{n}\geq 0\\ \left(\phi^{1-2\mathrm{q}}\frac{\mathrm{d}}{\mathrm{d}\phi}\right)^{-\mathrm{n}}\left[\mathrm{D}_6\mathrm{Exp}\left(\sqrt{\frac{\omega}{6}}\frac{\phi^{\mathrm{q}}}{\mathrm{q}}\right)+\mathrm{D}_7\mathrm{Exp}\left(-\sqrt{\frac{\omega}{6}}\frac{\phi^{\mathrm{q}}}{\mathrm{q}}\right)\right] & \mathrm{si}\quad \mathrm{n}<0\end{array}\right. \tag{27}$$

where D_6, D_7 are integration constants and $q=(m+2)/2=1/(2n+1)$.

4. General solution for any m: the solution is expressed in terms of the Bessel and modifie Bessel functions; for ϕ,

$$\mathcal{D}=\sqrt{\phi}\,\mathrm{Z}_\nu\left(\frac{\sqrt{\mathrm{a}}}{\mathrm{q}}\phi^{\mathrm{q}}\right), \tag{28}$$

where Z_ν is a generic Bessel function, $\nu=1/2q$ is the order of the Bessel function and $\mathrm{q}=(\mathrm{m}+2)/2$. If $a<0$ implies that $\omega<0$, Z_ν becomes the modifie Bessel function, (I_ν, K_ν). When $a>0, w>0$, $\mathrm{Z}_\nu\to(\mathrm{J}_\nu,\mathrm{Y}_\nu)$.

On the other hand, Eq. (20) does not have a general solution, but solutions for particular values of γ can be found; this is achieved by rewriting in the following form:

$$\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\Omega^2}-Q\frac{d\mathcal{A}}{d\Omega}+\Big(384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} - 48\Lambda e^{6\Omega}-a_1^2\Big)\mathcal{A}=0, \tag{29}$$

1. Any factor ordering Q and the inflatio phenomenon $\gamma=-1$:

$$\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\Omega^2}-Q\frac{d\mathcal{A}}{d\Omega}+\left[b_{-1}e^{6\Omega}-a_1^2\right]\mathcal{A}=0,$$
$$b_{-1}=384\pi G\mu_{-1}-48\Lambda; \tag{30}$$

thus by making the transformations $z=\sqrt{b_1}/3e^{3\Omega}$ and $\mathcal{A}=z^{Q/6}\Phi(z)$ we arrive at the Bessel differential equation for the function Φ. With this the general solution becomes [10]

$$\mathcal{A}=\left(\frac{\sqrt{b_{-1}}}{3}e^{3\Omega}\right)^{\frac{Q}{6}}Z_\nu\left(\frac{\sqrt{b_{-1}}}{3}e^{3\Omega}\right),$$
$$\nu=\pm\frac{1}{6}\sqrt{Q^2+4a_1^2} \tag{31}$$

where Z_ν is a generic Bessel function. If $\mathrm{b}_{-1}>0$, we have the ordinary Bessel function; otherwise, we have the modifie Bessel function.

2. Factor ordering Q=0 and $\gamma=0$:

$$\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\Omega^2}-\left(48\Lambda\mathrm{e}^{6\Omega}-\mathrm{b}_0\mathrm{e}^{3\Omega}+\mathrm{a}_1^2\right)\mathcal{A}=0, \tag{32}$$

by making the transformation $\mathrm{z=e^{3\Omega}}$ and $R{=}z^{-(a_1/3)}\mathcal{A}$, the solution is constructed by the degenerate hypergeometric functions $\mathrm{F_1(a,b;z)}$ [10]:

$$\mathcal{A}=e^{a_1\Omega}Exp\left[\frac{2}{3}\sqrt{6\Lambda}e^{a_1\Omega}\right]\times F_1\left(\frac{B(k)}{18k},\frac{2}{3}a_1+1;\frac{e^{3\Omega}}{\lambda}\right), \tag{33}$$

where

$$\lambda{=}-\frac{1}{2k},\quad \mathrm{k}{=}\frac{2}{3}\sqrt{6\Lambda},\quad \mathrm{B(k)}{=}9\mathrm{k}\left(\frac{2}{3}\mathrm{a}_1+1\right)+\mathrm{b}_0.$$

3. Any factor ordering and stiff matter $\gamma=1$:

$$\frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\Omega^2}-Q\frac{d\mathcal{A}}{d\Omega}+\left[b_1-48\Lambda e^{6\Omega}\right]\mathcal{A}=0,$$
$$b_1=384\pi G\mu_1-a_1^2 \tag{34}$$

in a similar way to the firs case, the transformations $z=4\sqrt{-\Lambda/3}\,e^{3\Omega}$ and $\mathcal{A}=z^{Q/6}\Phi(z)$, the differential Bessel function appears for the function Φ, then we have the general solution

$$\mathcal{A}=\left(4\sqrt{-\frac{\Lambda}{3}}\,e^{3\Omega}\right)^{\frac{Q}{6}}Z_\nu\left(4\sqrt{-\frac{\Lambda}{3}}\,e^{3\Omega}\right),$$
$$\nu=\pm\frac{1}{6}\sqrt{Q^2-4b_1}, \tag{35}$$

with $\Lambda<0$, having the ordinary Bessel function. In the case when the factor ordering is Q=0, we have the same Bessel function, but the cosmological constant could be positive or negative, yielding a modifie Bessel function and ordinary Bessel function, respectively, but of imaginary order in both cases $\nu=\pm i\sqrt{b_1}/3$.

5. Conclusions

An equivalent lagrangian density was constructed in order to apply the quantum regime in the Sáez-Ballester theory, where the constant ω can be used in order to have a lorenzian or pseudo-lorenzian signature for the Wheeler-DeWitt equation. The values for this parameter in the classical regime are dictated when we apply the reality conditions on the functions, and are encoded in the parameter a, Eqs. (25,26). In this sense, classical and quantum exact solutions were found for the cosmological Bianchi type I model in the frame of Sáez-Ballester theory, for the different values of the γ parameter

$\{-1, 0, 1\}$. The presence of the exotic fiel ϕ does not delay the anisotropic behavior in this model; the classical behavior of this fiel for large value in the parameter ω it is similar to the anisotropic parameters $\beta_{\pm}$.

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∗ e-mail: socorro@fisica.ugto.m

∗∗ e-mail: msabido@fisica.ugto.m

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