Maximum efficiency of an irreversible heat engine with a distributed working fluid and linear phenomenological heat transfer law

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Maximum efficiency of an irreversible heat engine with a distributed working fluid, in which the heat transfers between the working fluid and the heat reservoirs obey the linear phenomenological heat transfer law [$q \propto \Delta(T^{-1})$], is studied in this paper by using finite-time thermodynamics based on Orlov and Berry's work[i]. Two kinds of efficiencies are defined, and the problems are divided into three cases. Optimal control theory is used to determine the upper bounds of efficiencies of the heat engines for various cases. Numerical examples of the two efficiencies for the irreversible heat engine with lumped-parameter model working between variable temperature reservoirs are provided, and the effects of changes of the reservoir's temperature on the maximum efficiency of the heat engine are analyzed. The obtained results are also compared with those obtained by Orlov and Berry[ii] with Newtonian heat transfer law [$q \propto \Delta(T)$].

Keywords: Finite-time thermodynamics; linear phenomenological heat transfer law; heat engine; distributed working fluid; maximum efficiency; optimal control.

En este artículo se estudia la eficiencia máxima de un motor térmico irreversible con un fluido de trabajo distribuido, en el cual las transferencias térmicas entre el fluido de trabajo y los depósitos térmicos obedecen la ley fenomenológica linear de transferencia térmica [$q \propto \Delta(T^{-1})$], usando la termodinámica del finito-tiempo basada en el trabajo de Orlov y Berry[i]. Se definen dos clases de eficiencias, y los problemas se dividen en tres casos. La teoría de control óptima se utiliza para determinar los límites superiores de las eficiencias de los motores térmicos para varios casos. Se proporcionan ejemplos numéricos de las dos eficiencias de motor térmico irreversible con el modelo del amontonar-parámetro trabajando entre los depósitos de temperatura variable, y se analizan los efectos de los cambios de temperatura del recipiente en la eficacia máxima del motor térmico. Los resultados obtenidos también se comparan con los obtenidos por Orlov y Barry[ii] con la ley neutoniana del transferencia térmica [$q \propto \Delta(T)$].

Descriptores: Termodinámica de tiempos finitos; ley lineal fenomenológica de transferencia de calor; motor térmico; fluido de trabajo distribuido; eficiencia máxima; control óptimo.

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1. Introduction

Determining the optimal thermodynamic process for the given optimization objectives is one of the standard problems in finite-time thermodynamics [1-12]. Rubin [13,14] derived the optimal configurations of endoreversible heat engines with the Newtonian heat transfer law [$q \propto \Delta(T)$] and different constraints, including the optimal configuration with fixed duration for maximum power output and the optimal configuration with fixed energy input for maximum efficiency [13]. The results were extended to a class of heat engines with a fixed compression ratio [14]. Ondrechen *et al.* [15] investigated the optimal configuration of a Newtonian law system variable-temperature heat reservoir heat engine for maximum power output. Chen *et al.* [16] investigated effects of the heat leakage on the optimal configuration of a Newtonian law system variable-temperature heat reservoir heat engine for maximum power output. Angulo-Brown *et al.* [17] investigated the optimal configuration of a Newtonian law system variable-temperature heat reservoir heat engine for the maximum modified ecological function. Besides, numerous studies on the optimal configurations of different theoretical and practical thermodynamic systems for different optimization criteria are performed, such as a heated working fluid in a cylinder with a moveable piston [18-25], internal-combustion engines [26-28], dissipative heat engine [29], light-driven engines [30,31], plug flow reactors [32,33] and heat exchangers [34-37]. In these publications, simplified mathematical descriptions of the processes by ordinary differential equations (*i.e.* lumped-parameter models) were used. Unlike the external environment and heat reservoir, the temperature gradient over the space in the internal system is so small that the whole system could be treated together with the same temperature. The method of lumped-parameter analysis simplifies the research object and makes the problem appear to be easily solved. Besides, the obtained results can provide some theoretical guidelines for the designs and operations of practical installations. However, the practical situation is much more complex. For example, the working fluid in the system is not spatially uniform and the molecular motion in the system has no rule. There are internal dissipation caused by viscous friction and momentum loss caused by molecular collision, so the results obtained by the method of lumped-parameter analysis are still far away from the practical ones, a situation that needs to be improved. Orlov and Berry [38] first investigated the maximum power output of an irreversible heat engine with a non-uniform working fluid and Newtonian heat transfer law. Both the lumped-parameter model with uniform temperature and the distributed-parameter model described by partial differential equations are put forward, and the results showed that the maximum power output of the heat engine in the distributed-parameter model is less than or equal to that in the lumped-parameter model. Orlov and Berry [39] further investigated the efficiency performance limit of an irreversible heat engine with distributed working fluid and the Newtonian heat transfer law. Orlov and Berry [40] further derived the upper bounds of power and efficiency of an open internal combustion engine model, taking into account the finite rate of Newtonian law heat exchange with the environment and non-zero entropy generation due to chemical combustion reactions.

In general, heat transfer is not necessarily Newtonian heat transfer law and also obeys other laws; heat transfer laws not only have significant influences on the performance of the given thermodynamic processes [41-47], but also have influences on the optimal configurations of thermodynamic processes for the given optimization objectives. Song *et al.* [48-51], Li *et al.* [52] and Chen *et al.* [53] determined the optimal configurations of endoreversible heat engines for the maximum efficiency objective and maximum power output objective with linear phenomenological heat transfer law [$q \propto \Delta(T^{-1})$] [48,52], those for maximum power output with fixed duration and radiative heat transfer law [$q \propto \Delta(T^4)$][49,50], and those for maximum power output maximum efficiency with a fixed compression ratio and generalized radiative heat transfer law [$q \propto \Delta(T^n)$] [51,53]. Yan *et al* [54] investigated the optimal configuration of a variable-temperature heat reservoir heat engine for maximum power output with the linear phenomenological heat transfer law. Chen *et al.* [55] investigated effects of the heat leakage on the optimal configuration of a variable-temperature heat reservoir heat engine for maximum power output with the linear phenomenological heat transfer law. Some studies on the optimal configuration of a variable-temperature heat reservoir heat engine for maximum power output with the generalized radiative heat transfer law [56], generalized convective heat transfer law [$q \propto (\Delta T)^m$] [57], mixed heat resistance [58], and generalized heat transfer law [$q \propto (\Delta(T^n))^m$][59] were also performed. Ares de Parga *et al.* [60] investigated the optimal configuration of a variable-temperature heat reservoir heat engine for the maximum modified ecological function with the generalized convective heat transfer law. Xia *et al.* [61] investigated the maximum power output of the irreversible heat engine with non-uniform working fluid and the linear phenomenological heat transfer law. Chen *et al.* [62] investigated the effects of different heat resistance model and rate equation of reactions on the maximum power and efficiency of the open internal combustion engine. Based on Ref. 39, this paper will study the maximum efficiency of an irreversible heat engine with distributed working fluid and linear phenomenological heat transfer law in the heat transfer process between working fluid and the reservoirs. Two kinds of efficiencies are defined, and the problems are divided into three cases. Optimal control theory is used to determine the upper bounds of efficiencies of the heat engines for various cases. Numerical examples of the two efficiencies for the irreversible heat engine with lumped-parameter model working between variable temperature reservoirs are provided, and the effects of changes of the reservoir's temperature on the maximum efficiency of the heat engine are analyzed. The obtained results are compared with those obtained with the Newtonian heat transfer law [39]. The research on the efficiency performance limit of the engines from the Newtonian heat transfer law to the linear phenomenological heat transfer law enriches the finite-time thermodynamic theory. The results presented herein can provide some guidelines for the optimal design and operation of real heat engines.

2. Model description

Assume that ξ denotes the space location of some point, τ is the cycle period, $\alpha_H(\xi)$ is the space-dependent coefficients of heat transfer between the working fluid and the hot reservoir, $\alpha_L(\xi)$ is the space-dependent coefficients of heat transfer between the working fluid and the cold reservoir, and $A_H(t)$ and $A_L(t)$ are the corresponding heat transfer surface areas. $T_H(t,\xi)$ and $T_L(t,\xi)$ are the corresponding temperatures of the engine's two reservoirs, $T(t,\xi)$ is the temperature of the working fluid inside the heat engine. The average power output of the heat engine with distributed working fluid and linear phenomenological heat transfer law in the heat transfer process between working fluid and the reservoirs has the form

$$P = \frac{1}{\tau}\int_0^\tau \Big[v_H(t) \int_{A_H(t)} \alpha_H(\xi)(T^{-1} - T_H^{-1})da + v_L(t) \int_{A_L(t)} \alpha_L(\xi)(T^{-1} - T_L^{-1})da \Big] dt \tag{1}$$

where da is the area element of contact between the working fluid and heat reservoirs. $v_H(t)$ and $v_L(t)$ are the switching functions. These switching functions regulate the finite-rate heat transfer between the working fluid and the two heat reservoirs. When $0 < v_H(t) \le 1$, the hot reservoir and the working fluid are in contact and exchange energy; when $v_H(t) = 0$, there is no exchange between them. When $0 < v_L(t) \le 1$, the cold reservoir and the working fluid are in contact and exchange energy; when $v_L(t) = 0$, there is no exchange between them. The net amount of heat that the working fluid receives from the hot reservoir is

$$Q_H = \int_0^\tau v_H \int_{A_H} \alpha_H(\xi)(T^{-1} - T_H^{-1})dadt \tag{2}$$

The corresponding efficiency, *i.e.* net work per unit of heat received from the hot reservoir, is

$$\eta_1 = \frac{\tau P}{Q_H} \tag{3}$$

In practical heat engines, situations when the local temperature of the working fluid $T(t,\xi)$ is greater than the hot reservoir's $T_H(t,\xi)$ may occur. For example, the working fluid is heated by the hot reservoir, further compressed adiabatically and then brought into contact with the hot reservoir again. In this case, part of heat in the working fluid will flow back to the hot reservoir, so the amount of heat that the working fluid receives from the hot reservoir Q_H^+ is greater than the net amount of absorbed heat Q_H. According to Ref. 39, Q_H^+ is approximately defined as follows:

$$Q_H^+ = \int_0^\tau v_H \int_{A_H} \{\alpha_H(\xi)\big[(T^{-1} - T_H^{-1}) + |(T^{-1} - T_H^{-1})|\big]/2\} da dt \tag{4}$$

Then the efficiency of the heat engine η_2 corresponding to Q_H^+ is given by:

$$\eta_2 = \tau P / Q_H^+ \tag{5}$$

It is evident that $\eta_1 \leq \eta_2$ due to $Q_H^+ \geq Q_H$.

According to Refs. 38 and 39, the total process is also assumed to be a weakly periodic process herein. It follows that $S(0) = S(\tau)$ from the weakly periodic condition, where S represents the total entropy of the working fluid. From $S(0) = S(\tau)$, one further has

$$\int_0^\tau \{v_H(t) \int_{A_H(t)} \alpha_H(\xi)(T^{-1} - T_H^{-1})/T da + v_L(t) \times \int_{A_L(t)} \alpha_L(\xi)(T^{-1} - T_L^{-1})/T da + \sigma(t)\} dt = 0 \tag{6}$$

where $\sigma(t) \geq 0$ is the integrated entropy production over the volume in the working fluid.

According to the different given conditions, there are three different types of efficiency optimization problems [39]:

(1) To find an upper bound of η_1 with constraints $\tau P = \tau P^0$ and $S(0) = S(\tau)$, *i.e.* with fixed work per cycle.

(2) To find an upper bound of η_1 with constraints $Q_H = Q_H^0$ and $S(0) = S(\tau)$, *i.e.* with fixed net heat input per cycle.

(3) To find an upper bound of η_2 with constraints $\tau P = \tau P^0$ and $S(0) = S(\tau)$, *i.e.* with fixed net work input per cycle. This is equivalent to find an upper bound of $-Q_H^+$.

The above three problems will be solved step by step in the following section.

3. Optimization

3.1. Solution procedure for problem 1

For problem 1, evaluating efficiency η_1 with constraints of $\tau P = \tau P^0$ and $S(0) = S(\tau)$ is equivalent to finding an upper bound of $-Q_H$ with the same constraints. In order to solve this problem, both the two integral constraints are multiplied by two scalar Lagrange multipliers, λ_1 and λ_2, and then they are added to $-Q_H$. An unconstrained, averaged optimization problem is given by

$$\max_{T>0,\sigma\geq 0} -Q_H + \lambda_1 \tau (P - P^0) + \lambda_2 \int_0^\tau \{v_H \int_{A_H(t)} \alpha_H (T^{-1} - T_H^{-1})/T da + v_L \int_{A_L(t)} \alpha_L (T^{-1} - T_L^{-1})/T da + \sigma(t)\} dt \tag{7}$$

where $\lambda_1 > 1$, $\lambda_2 < 0$ and control variable $T > 0$. Let $\phi_1(\lambda_1, \lambda_2)$ be

$$\phi_1(\lambda_1,\lambda_2) = \max_{T>0} \int_0^\tau \{v_H \int_{A_H(t)} \alpha_H (T^{-1} - T_H^{-1}) \times (\lambda_1 - 1 + \lambda_2/T) da + v_L \int_{A_L(t)} \alpha_L \times (T^{-1} - T_L^{-1})(\lambda_1 + \lambda_2/T) da] dt. \tag{8}$$

Maximizing the first term of the integrand of Eq. (8) with respect to T one obtains the optimal argument $\hat{T}_1 = -2\lambda_2 T_H(t)/[(\lambda_1 - 1)T_H(t) - \lambda_2]$ and the maximum value of this term

$$f_H(t,\xi,\lambda_1,\lambda_2) = -v_H(t)\alpha_H(\xi) \times [(\lambda_1 - 1)T_H(t) + \lambda_2]^2 / [4\lambda_2 T_H^2(t)]. \tag{9}$$

Maximizing the second term of the integrand of Eq. (8) with respect to T one obtains the optimal argument $\hat{T}_1 = -2\lambda_2 T_H(t)/[\lambda_1 T_H(t) - \lambda_2]$ and the maximum value of this term

$$f_L(t,\xi,\lambda_1,\lambda_2) = -v_L(t)\alpha_L(\xi) \times [\lambda_1 T_L(t) + \lambda_2]^2 / [4\lambda_2 T_L^2(t)] \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8) yields:

$$\phi_1(\lambda_1,\lambda_2) = \int_0^\tau \Big[\int_{A_H(t)} f_H(t,\xi,\lambda_1,\lambda_2)da + \int_{A_L(t)} f_L(t,\xi,\lambda_1,\lambda_2)da\Big]dt. \quad (11)$$

Combining Eqs. (7) with (8) gives the following inequality:

$$-Q_H \le \phi_1(\lambda_1,\lambda_2) - \lambda_1\tau P^0 + \lambda_2 \int_0^\tau \sigma(t)dt \quad (12)$$

where the term

$$\lambda_2 \int_0^\tau \sigma(t)dt \le 0$$

due to the Lagrange multiplier $\lambda_2 < 0$, one may omit this term from the inequality (12), in using it to get the upper bound. The problem 1 to evaluate the upper bound of η_1 can be further transformed to that of minimizing

$$\phi_1(\lambda_1,\lambda_2) - \lambda_1\tau P^0.$$

This is a two- dimensional convex optimization problem due to the fac that $\phi_1(\lambda_1,\lambda_2)$ is a convex function. Let $\hat\lambda_1$ and $\hat\lambda_2$ be the solutions to this problem; then Eq. (12) further gives:

$$Q_H \ge \hat\lambda_1\tau P^0 - \phi_1(\hat\lambda_1,\hat\lambda_2) \quad (13)$$

Combining Eqs. (3) with (13) gives the estimation of efficiency η_1:

$$\eta_1 \le \tau P^0/[\hat\lambda_1\tau P^0 - \phi_1(\hat\lambda_1,\hat\lambda_2)] \quad (14)$$

Defining function $\psi_1(\lambda_1,\lambda_2) = \phi_1(\lambda_1,\lambda_2) - \lambda_1\tau P^0$, one has $\partial\psi_1/\partial\lambda_2 = \partial\phi_1/\partial\lambda_2$. From Eq. (11), one can obtain an analytical solution for $\hat\lambda_2$. Combining Eq. (11) with $\partial\phi_1/\partial\lambda_2 = 0$, one can obtain the optimal term $\hat\lambda_2$ and the corresponding term $\psi_1(\lambda_1,\hat\lambda_2)$:

$$\hat\lambda_2 = -\sqrt{(\overline{v_H\alpha_H(\lambda_1-1)} + \overline{v_L\alpha_L\lambda_1})/(\overline{v_H\alpha_H/T_H^2} + \overline{v_L\alpha_L/T_L^2})} \quad (15)$$

$$\psi_1(\lambda_1,\hat\lambda_2) = \tau[\overline{f_H(t,\xi,\lambda_1,\hat\lambda_2)} + \overline{f_L(t,\xi,\lambda_1,\hat\lambda_2)}] - \lambda_1\tau P^0 \quad (16)$$

where

$$\overline{x_H} = 1/\tau \int_0^\tau \Big[\int_{A_H(t)} x_H(t,\xi,\lambda)da\Big]dt$$

and

$$\overline{x_L} = 1/\tau \int_0^\tau \Big[\int_{A_L(t)} x_L(t,\xi,\lambda)da\Big]dt.$$

Problem 1 could be reduced to a one-dimensional optimization problem, *i.e.* minimize $\psi_1(\lambda_1,\hat\lambda_2)$ with the constraint $\lambda_1 > 1$. From Eq. (16) and extreme condition $\partial\psi_1(\lambda_1,\hat\lambda_2)/\partial\lambda_1 = 0$, one cannot further get an analytical solution for $\hat\lambda_1$. This problem can only be solved numerically.

3.2. Solution procedure for problem 2

For problem 2, evaluating efficiency η_1 with the constraints of $Q_H = Q_H^0$ and $S(0) = S(\tau)$ is equivalent to finding an upper bound of τP with the same constraints. With the same transformation as problem 1, the final problem is to minimize $\phi_2(\lambda_1,\lambda_2) - \lambda_1 Q_H^0$ with constraints $\lambda_1 > -1$ and $\lambda_2 < 0$, where

$$\phi_2(\lambda_1,\lambda_2) = \max_{T>0} \int_0^\tau \Big[v_H \int_{A_H(t)} \alpha_H(T^{-1} - T_H^{-1}) \times (\lambda_1 + 1 + \lambda_2/T)da + v_L \int_{A_L(t)} \alpha_L \times (T^{-1} - T_L^{-1})(1+\lambda_2/T)da\Big]dt. \quad (17)$$

Maximizing the first and the second terms of the integrand of Eq. (17) with respect to T yields:

$$\phi_2(\lambda_1,\lambda_2) = \int_0^\tau \Big[\int_{A_H(t)} g_H(t,\xi,\lambda_1,\lambda_2)da + \int_{A_L(t)} g_L(t,\xi,\lambda_2)da\Big]dt \quad (18)$$

where g_H and g_L are

$$g_H(t,\xi,\lambda_1,\lambda_2) = -v_H(t)\alpha_H(\xi) \times [(\lambda_1+1)T_H(t)+\lambda_2]^2/[4\lambda_2 T_H^2(t)] \quad (19)$$

$$g_L(t,\xi,\lambda_2) = -v_L(t)\alpha_L(\xi) \times [T_L(t)+\lambda_2]^2/[4\lambda_2 T_L^2(t)] \tag{20}$$

Let $\hat{\lambda}_1$ and $\hat{\lambda}_2$ be the solutions corresponding to the minimization of $\phi_2(\lambda_1,\lambda_2) - \lambda_1 Q_H^0$. From

$$\tau P \leq \phi_2(\hat{\lambda}_1,\hat{\lambda}_2) - \hat{\lambda}_1 Q_H^0$$

and Eq. (3), one can get the estimation of efficiency η_1, as follows:

$$\eta_1 \leq [\phi_2(\hat{\lambda}_1,\hat{\lambda}_2) - \hat{\lambda}_1 Q_H^0]/Q_H^0 \tag{21}$$

Defining function $\psi_2(\lambda_1,\lambda_2) = \phi_2(\lambda_1,\lambda_2) - \lambda_1 Q_H^0$, one has $\partial\psi_2/\partial\lambda_2 = \partial\phi_2/\partial\lambda_2$. From Eq. (18), one can get an analytical solution for $\hat{\lambda}_2$. Combining Eq. (18) with $\partial\phi_2/\partial\lambda_2 = 0$ yields the optimal term $\hat{\lambda}_2(\lambda_1)$ and the corresponding term $\psi_2(\lambda_1,\hat{\lambda}_2)$:

$$\hat{\lambda}_2 = -\sqrt{(\overline{v_H\alpha_H(\lambda_1+1)} + \overline{v_L\alpha_L})/(\overline{v_H\alpha_H/T_H^2} + \overline{v_L\alpha_L/T_L^2})} \tag{22}$$

$$\psi_2(\lambda_1,\hat{\lambda}_2) = \tau[\overline{g_H(t,\xi,\lambda_1,\hat{\lambda}_2)} + \overline{g_L(t,\xi,\hat{\lambda}_2)}] - \lambda_1\tau P^0 \tag{23}$$

Problem 2 could also be reduced to a one-dimensional optimization problem, *i.e.* that of minimizing $\psi_1(\lambda_1,\hat{\lambda}_2)$ with the constraint $\lambda_1 > -1$. From Eq. (23) and extreme condition $\partial\psi_2(\lambda_1,\hat{\lambda}_2)/\partial\lambda_1 = 0$, one cannot get an analytical solution for $\hat{\lambda}_1$. This problem can only be solved numerically.

3.3. Solution procedure for problem 3

For problem 3, evaluating efficiency η_2 with the constraints of $\tau P = \tau P^0$ and $S(0) = S(\tau)$ is equivalent to finding an upper bound of $-Q_H^+$ with the same constraints. With the same transformation as problems 1 and 2, the final problem is to minimize $\phi_3(\lambda_1,\lambda_2) - \lambda_1\tau P^0$ with constraints $\lambda_1 > 1$ and $\lambda_2 < 0$, where $\phi_3(\lambda_1,\lambda_2)$ is given by

$$\phi_3(\lambda_1,\lambda_2) = \max_{T>0}\int_0^\tau \Bigg[v_H\int_{A_H(t)}\alpha_H(T^{-1}-T_H^{-1}) \times \Big[\lambda_1 - sg(T^{-1}-T_H^{-1}) + \lambda_2/T\Big]da + v_L \times \int_{A_L(t)}\alpha_L(T^{-1}-T_L^{-1})(\lambda_1+\lambda_2/T)da\Bigg]dt \tag{24}$$

where $sg(x)$ is a step function, *i.e.* $sg(x) = 1$ when $x > 0$, and $sg(x) = 0$ when $x \leq 0$. Maximizing the first term of the integrand of Eq. (24) with respect to T one gets the optimal argument

$$\hat{T}_1 = -2\lambda_2 T_H(t)/\{[\lambda_1 - sg(T_0^{-1}-T_H^{-1})]T_H(t) - \lambda_2\}$$

and the maximum value of this term

$$y_H(t,\xi,\lambda_1,\lambda_2) = -v_H(t)\alpha_H(\xi)\{[\lambda_1 - sg(T_0^{-1}-T_H^{-1})] \times T_H(t)+\lambda_2\}^2 \times [sg(l-1+\lambda/T_H) + sg(T_H^{-1}-T_0^{-1})]/[4\lambda_2 T_H^2(t)] \tag{25}$$

where $T_0 = -\lambda_2/\lambda_1$. Maximizing the second term of the integrand of Eq. (24) with respect to T one gets the optimal argument $\hat{T}_2 = -2\lambda_2 T_H(t)/[\lambda_1 T_H(t) - \lambda_2]$ and the maximum value of this term

$$y_L(t,\xi,\lambda_1,\lambda_2) = -v_L(t)\alpha_L(\xi) \times \{[\lambda_1 T_L(t)+\lambda_2\}^2/[4\lambda_2 T_L^2(t)] \tag{26}$$

Then $\phi_3(\lambda_1,\lambda_2)$ is given by

$$\phi_3(\lambda_1,\lambda_2) = \int_0^\tau \Big[\int_{A_H(t)} y_H(t,\xi,\lambda_1,\lambda_2)da + \int_{A_L(t)} y_L(t,\xi,\lambda_1,\lambda_2)da\Big]dt. \tag{27}$$

Let $\hat{\lambda}_1$ and $\hat{\lambda}_2$ be the solutions corresponding to the minimization of $\varphi_3(\lambda_1,\lambda_2) - \lambda_1\tau P^0$. Combining $Q_H^+ \geq \hat{\lambda}_1\tau P^0 - \phi_3(\hat{\lambda}_1,\hat{\lambda}_2)$ with Eq. (5), one can get the estimation of efficiency η_2, as follows:

$$\eta_2 \leq \tau P^0/\Big[\hat{\lambda}_1\tau P^0 - \phi_3(\hat{\lambda}_1,\hat{\lambda}_2)\Big] \tag{28}$$

Unlike problems 1 and 2, there is no analytical solution for $\hat{\lambda}_1$. So this two-dimensional convex optimization problem cannot be converted to a one-dimensional optimization problem. This optimization problem can only be solved numerically.

4. Numerical examples and discussions

The lumped-parameter model with an oscillating high-reservoir temperature is considered herein. In this model, the temperature of the hot reservoir is

$$T_H(t) = T_H^0 + \Delta T\sin(4\pi t),$$

while the temperature of the cold reservoir is $T_L(t) = T_L^0$. According to Ref. 39, the following parameters are set: $\tau = 1\,\text{sec}$, $t_H = 0.5\tau$, $T_H^0 = 1200$ K, $T_L^0 = 293.15$ K,

$v_H(t) = 1$ when $0 \leq t \leq t_H$ $v_H(t) = 0$ when $t_H < t \leq \tau$, $v_L(t) = 0$ when $0 \leq t \leq t_H$, and $v_L(t) = 1$ when $t_H < t \leq \tau$. For the cases with different heat transfer laws, one must change the values of $a_H \alpha_H$ in order to generate heat transfer rates that are comparable to each other for the same temperature difference, so $a_H\alpha_H = a_L\alpha_L = 100$ W/K are set for Newtonian heat transfer law [39], while

$$a_H\alpha_H = a_L\alpha_L = 3.52 \times 10^7 \mathrm{WK}$$

are set for the linear phenomenological heat transfer law. The numerical calculations are developed in a MATLAB environment, and the integral function '@quadl' and the error of 10^{-6} are chosen. Only cases that optimize two different kinds of efficiency η_1 and η_2 with the given power output P^0, *i.e.* problems 1 and 3, are considered herein. $P^0 = 0.8\tilde{P}_{\max}(0)$ is set. $\eta_1(0)$ is the value of η_1 evaluated at $\Delta T = 0$, and the Carnot efficiency is $\eta_c{=}1{-}T_L^0/(T_H^0{+}\Delta T)$ [63].

Problem 1 is to minimize $\psi_1(\lambda_1, \hat{\lambda}_2)$ with constraint $\lambda_1 > 1$. Figure 1 shows the function $-\tau P^0/\psi_1(\lambda_1, \hat{\lambda}_2)$ versus λ_1 with the linear phenomenological heat transfer law. From Fig. 1, one can see that the function $-\tau P^0/\psi_1(\lambda_1, \hat{\lambda}_2)$ has a minimum value with the growth of λ_1, and the lowest point of the curve corresponding to different temperature ratios $\Delta T/T_H^0$ is the corresponding maximum efficiency $\eta_1(\Delta T)$. Table I lists the comparison of the maximum efficiency η_1 of the heat engines with two different heat transfer laws. Both of the efficiency limits $\eta_1(\Delta T)$ of the two different cases increase with the increase in the temperature ratio $\Delta T/T_H^0$. However, they are both smaller than Carnot efficiency η_c of heat engines with hot reservoirs of constant temperature T_H^0. For the case with Newtonian heat transfer law, the variation of $\eta_1(\Delta T)$ with the increase of $\Delta T/T_H^0$ is distinct, and $\eta_1(\Delta T)/\eta_1(0) = 1.2674$ when $\Delta T/T_H^0 = 0.6$. For the case with the linear phenomenological heat transfer law, the value of $\eta_1(\Delta T)$ varies little when $\Delta T/T_H^0 < 0.4$, *i.e.* the value of $\eta_1(\Delta T)/\eta_1(0)$ is less than 1.01. When $\Delta T/T_H^0 = 0.6$, the corresponding value of $\eta_1(\Delta T)/\eta_1(0)$ is just 1.0425. This difference is due to different heat transfer laws mainly, so it is very necessary to investigate the effects of the heat transfer law on the efficiency limit of the irreversible heat engine.

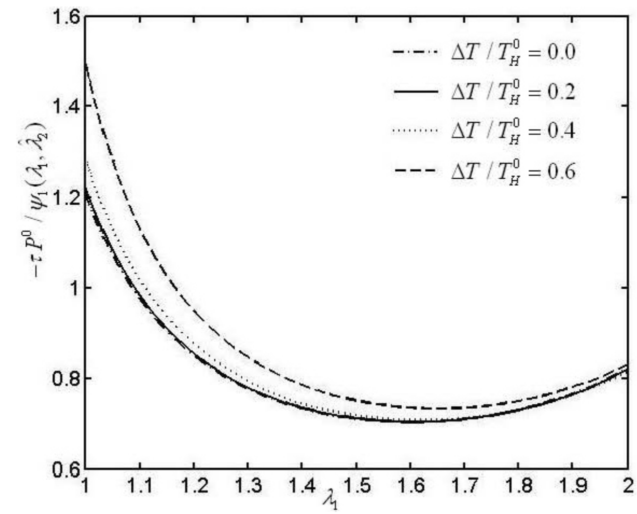

FIGURE 1. The function $-\tau P^0/\psi_1(\lambda_1, \hat{\lambda}_2)$ versus λ_1 with the linear phenomenological heat transfer law.

TABLE I. Comparison of the maximum efficiency η_1 of the heat engines with two different heat transfer laws.

——	$q \propto \Delta T^{[39]}$		$q \propto \Delta(T^{-1})$		——
$\Delta T/T_H^0$	$\eta_1(\Delta T)$	$\eta_1(\Delta T)/\eta_1(0)$	$\eta_1(\Delta T)$	$\eta_1(\Delta T)/\eta_1(0)$	η_c
0.0000	0.6412	1.0000	0.7033	1.0000	0.7557
0.2000	0.6599	1.0292	0.7037	1.0006	0.7964
0.4000	0.7157	1.1161	0.7075	1.0060	0.8255
0.6000	0.8127	1.2674	0.7332	1.0425	0.8473

TABLE II. Comparison of the maximum efficiency η_2 of the heat engines with two different heat transfer laws.

——	$q \propto \Delta T^{[39]}$		$q \propto \Delta(T^{-1})$		——
$\Delta T/T_H^0$	$\eta_2(\Delta T)$	$\eta_2(\Delta T)/\eta_2(0)$	$\eta_2(\Delta T)$	$\eta_2(\Delta T)/\eta_2(0)$	η_c
0.0000	0.6412	1.0000	0.7033	1.0000	0.7557
0.2000	0.6599	1.0292	0.7037	1.0006	0.7964
0.4000	0.7099	1.1071	0.7062	1.0041	0.8255
0.6000	0.7568	1.1800	0.7224	1.0272	0.8473

Problem 3 is to minimize $\phi_3(\lambda_1, \lambda_2) - \lambda_1 \tau P^0$ with the constraints $\lambda_1 > 1$ and $\lambda_2 < 0$. Table II lists a comparison of the maximum efficiency η_2 of the heat engines with two different heat transfer laws. Comparing with the results listed in Table I, one can see that the maximum efficiency $\eta_2(\Delta T)$ is equal to $\eta_1(\Delta T)$ when the temperature ratio $\Delta T/T_H^0$ is small, but the former is smaller than the latter when $\Delta T/T_H^0$ is larger. This is evident, because the efficiency η_1 is related to the net absorbed heat Q_H, while the efficiency η_2 is related to the practical absorbed heat Q_H^+. With the increase of the temperature variation ΔT of the hot reservoir, the situation that part of the absorbed heat in the working fluid will flow back to the hot reservoir occurs. As a result, Q_H is less than Q_H^+.

5. Conclusion

On the basis of Ref. 39, this paper studies the maximum efficiency of an irreversible heat engine with a distributed working fluid, in which heat transfers between the working fluid and the heat reservoirs obey the linear phenomenological heat transfer law [$q \propto \Delta(T^{-1})$]. Two kinds of efficiencies are defined, and the problems are divided into three cases. Optimal control theory is used to determine the upper bounds of efficiencies of the heat engine for various cases, respectively. Numerical examples of the two efficiencies for the irreversible heat engine working between variable-temperature reservoirs are given by numerical calculations, and the effects of changes of the reservoir's temperature on the maximum efficiency of the heat engine are analyzed. The results show that these two different efficiencies are equal to each other when the temperature variation of the hot reservoir is small, and when the temperature variation of the hot reservoir is larger, the difference between these two different efficiencies is large. The obtained results are also compared with those obtained with the Newtonian heat transfer law [$q \propto \Delta(T)$] [39]. The results show that heat transfer laws have significant effects on the efficiency performance limit of the irreversible heat engines. For the case with the Newtonian heat transfer law, the efficiency performance limit is sensitive to the temperature variation of the hot reservoir, while for the case with the linear phenomenological heat transfer law, the effects of temperature variation of the hot reservoir on the efficiency performance are relative smaller. In conclusion, it is very necessary to investigate the effects of heat transfer laws on the efficiency limit of the irreversible heat engine with a distributed working fluid.

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