Diffraction of beams by infinite or finite amplitude-phase gratings

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In this paper a theory for the diffraction of beams by thin amplitude-phase gratings in the scalar diffraction regime is given. The grating can be strictly periodic and therefore of infinite spatial extent (infinite grating) or can be a grating with a finite number of periods (finite grating). The main result of this paper is that we can write down mathematical expressions for the diffraction of beams by these kinds of gratings. General expressions for the diffraction patterns at the far-field region are derived. As a numerical application of the theory presented in this paper the diffraction of Hermite-Gaussian and distorted beams by a Ronchi ruling (infinite and finite) is studied.

Keywords: Diffraction; gratings.

En este artículo se presenta una teoría para la difracción de haces por redes de difracción delgadas de amplitud y fase en la región escalar. La red puede ser periódica y de extensión infinita (redes infinitas) o puede ser una red con un número finito de periodos (red finita). El principal resultado de este artículo es que es posible obtener expresiones matemáticas para la difracción por este tipo de redes. Fórmulas generales para los patrones de difracción en el campo lejano son obtenidas. Como una aplicación numérica de la teoría de este artículo estudiamos la difracción de haces Hermite-Gauss y haces distorsionados por una red de difracción de Ronchi (finita e infinita).

Descriptores: Difracción; redes de difracción.

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1. Introduction

In the present paper the diffraction of beams by thin amplitude-phase gratings is theoretically considered. The grating can be strictly periodic and therefore of infinite spatial extent (infinite grating) or can be a grating with a finite number of periods (finite grating). In the past, the diffraction of beams by infinite gratings has been extensively analyzed in the scalar regime. In particular, the diffraction by amplitude and phase gratings has been studied. For instance, phase sinusoidal gratings [1-4], holographic gratings [5-7], amplitude gratings consisting of equidistant slits [8], double-layer rectangular phase gratings [9], Ronchi gratings (grating with alternate clear and dark fringes of square profile per period) [10], square and hexagonal phase gratings [11-12], have been considered. More recently, attention has been paid to the diffraction of beams by finite gratings. However, to the best of our knowledge, the majority of published papers are dedicated to the study of the diffraction by amplitude finite gratings. Thus, the diffraction by N equidistant slits [13-17] and finite strip grating [18] has been analyzed. The existence of constant-intensity angles in the far-field diffraction patterns of N equally spaced slits, when the spot position of the incident beam is changed on the screen, was shown [15].

In this paper a general theory for the diffraction of two-dimensional beams by finite or infinite amplitude-phase grating in the scalar diffraction regime is given. We consider one-dimensional thin amplitude-phase gratings where the optical thickness of the recording medium is much less than the fringe spacing [5]. The main result of this paper is that we can write down analytical expressions for the diffraction of beams by these kinds of gratings. As a numerical application, the diffraction of Hermite-Gaussian and distorted beams by a Ronchi ruling (infinite and finite) is treated.

2. Basic concepts

We have a one-dimensional thin amplitude-phase grating modulated by a complex transmittance function $t(x)$. We consider that the optical thickness of the recording medium is much less than the fringe spacing (thin holographic grating). The grating can be strictly periodic and therefore of infinite spatial extent (infinite grating) or can be a grating with a finite number of periods (finite grating). The grating is placed in a vacuum, and the position of a point in space is given by its Cartesian coordinates x, y, and z. Our configuration is illustrated in Fig. 1 for the particular case of an infinite (a) and finite (b) Ronchi ruling made of alternate transparent (width l) and opaque zones (width d). However, it is important to remember that in what follows a general transmittance function $t(x)$ is considered. The thin amplitude-phase grating is illuminated by a beam independent of the z coordinate (cylindrical incident wave). The complex representation of field quantities is used, and the complex time term $\exp(-i\,\omega\,t)$ is omitted from now on.

Let $E(x)$, $E_i(x)$ and $t(x)$ be: the transmitted field, the input field or incident field, and the transmittance function, respectively, related as follows

$$E(x) = t(x)E_i(x)\,. \tag{1}$$

From this equation, the field $E(x)$ just below the grating can be obtained. Since we are interested in incident beams of finite cross section, then, the function $E(x)$ will be different

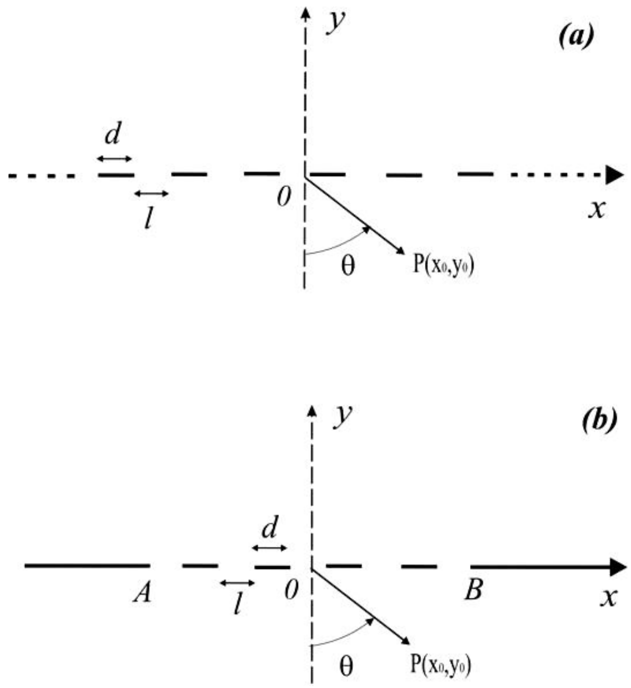

FIGURE 1. A Ronchi ruling made of alternate opaque and transparent zones of widths d and l, respectively. (a) infinite ruling and (b) finite ruling. The ruling is parallel to the Oz axis. The observation point is given by $P(x_0, y_0)$.

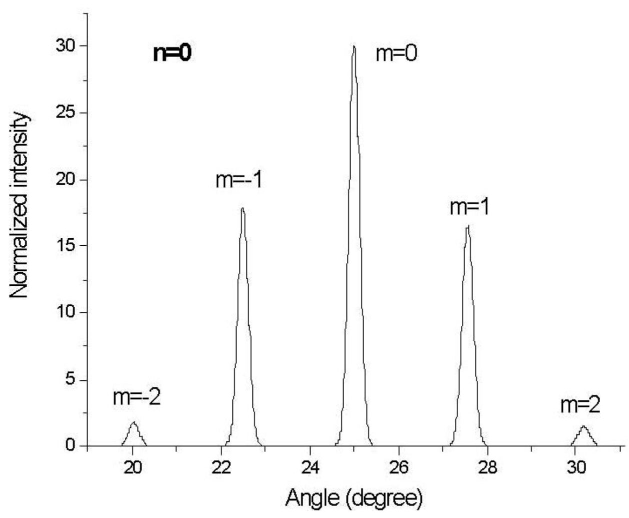

FIGURE 2. Diffraction pattern when a Gaussian beam (n=0) is incident on an infinite Ronchi ruling. With the following parameters: $L/l = 15/\sqrt{2}$, $\lambda/l = 0.1$, $\theta_0 = 25^{\circ}$, $d/l = 1.5$ and $D/l = 2.5$.

from zero within a finite interval $[a, b]$ and zero outside of it (or very close to zero). It is interesting to mention that the theory presented in this section can be utilized not only for the particular case of finite or infinite gratings, but also for the general case of transmittance functions $t(x)$, which could be periodic or not.

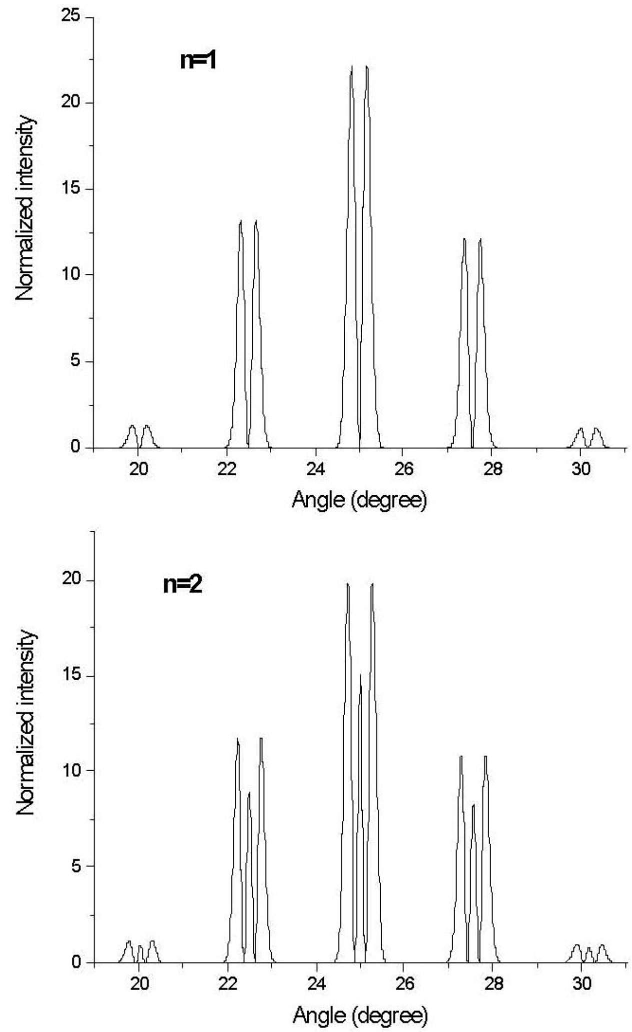

FIGURE 3. Same as Fig. 2 but for Hermite-Gaussian beams of order n=1, 2.

The diffracted field E^d for $y < 0$ can be expressed by means of the following angular plane-wave expansion [13]:

$$E^d(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{E}(\alpha)\ \exp[i(\alpha\, x - \beta\, y)]\, d\alpha, \qquad (2)$$

where we have the definition $\beta^2 = k^2 - \alpha^2$ with $\beta \geq 0$ or $\beta/i > 0$, and $k = 2\pi/\lambda$ is the module of the wave vector in the vacuum. The term $\hat{E}(\alpha)$ represents the amplitude of the transmitted waves, composed of two parts: downward-propagating waves ($|\alpha| \leq k$) and evanescent waves ($|\alpha| > k$). So that, the diffracted field is determined by means of the amplitude function $\hat{E}(\alpha)$. The determination of this function is our main problem in what follows.

From Eq. (2) at $y = 0$ and utilizing the inverse Fourier transform, the amplitude function $\hat{E}(\alpha)$ is given as follows:

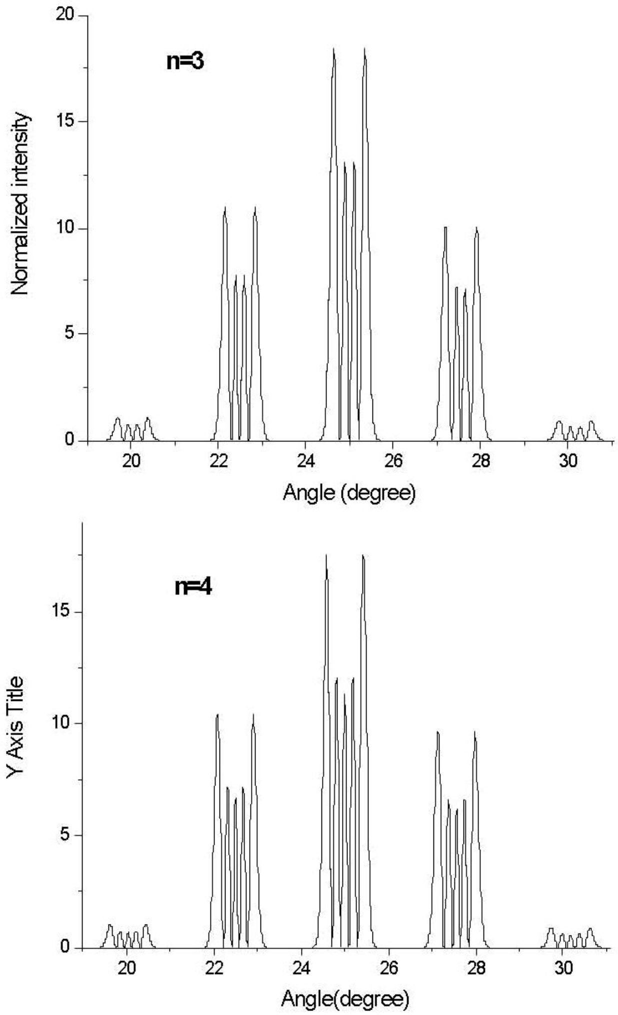

FIGURE 4. Same as Fig. 2 but for Hermite-Gaussian beams of order n=3, 4.

$$\hat{E}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E^d(x,0)\exp(-i\alpha x)dx\,. \tag{3}$$

On the other hand, we have that $E^d(x,0) = E(x)$ at $y = 0$, then, from Eqs. (1) and (3) we get that:

$$\hat{E}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t(x)E_i(x)\exp(-i\alpha x)dx\,. \tag{4}$$

This is a new result and it is the theoretical base of this paper. Then, the amplitude function $\hat{E}(\alpha)$ can be determined from the knowledge of the transmittance function $t(x)$ and the incident field $E_i(x)$. From this result and considering Eq. (2), the diffracted field can be determined anywhere. So that, our fundamental problem in what follows is to determine the amplitude $\hat{E}(\alpha)$ in several interesting cases.

It is important to determine the general expression for the far-field. As was mentioned, from Eq. (1) the field $E(x)$ just below the amplitude-phase grating can be obtained. From the knowledge of the field $E(x)$ and the two-dimensional Rayleigh-Sommerfeld integral equation [19] the total field $E(x_0, y_0)$ at any point below the ruling can be obtained

$$\begin{aligned} E(x_0,y_0) &= \frac{i}{2}\int_{-\infty}^{\infty} E(x)\frac{\partial}{\partial y_0}H_0^1(kr)dx \\ &= \frac{i}{2}\int_{-\infty}^{\infty} t(x)E_i(x)\frac{\partial}{\partial y_0}H_0^1(kr)dx \end{aligned} \tag{5}$$

where $r^2 = (x-x_0)^2 + y_0^2$ with $P(x_0, y_0)$ being the observation point as illustrated in Fig. 1. H_0^1 is the Hankel function of the first kind and of order zero. From Eq. (5) the far field can be obtained by looking at the asymptotic behavior of the field E when $kr \gg 1$ (Fraunhofer approximation). In this approximation the expression for the far field is given by [20]

$$E(x_0, y_0) = f(\theta)\exp(ikr_0)/\sqrt{r_0}\,, \tag{6}$$

where $\sin\theta = x_0/r_0$ and $\cos\theta = -y_0/r_0$ (see Fig. 1). This is the expression of a cylindrical wave with the oblique factor $f(\theta)$

$$f(\theta) = \sqrt{k}\exp(-i\pi/4)\cos\theta\,\hat{E}(k\sin\theta) \tag{7}$$

with $\hat{E}(\alpha)$ given by Eq. (4).

As we are now concerned with the scalar region, the polarization effects can be neglected. Then without loss of generality we can assume that the incident beam is TE-polarized, *i.e.*, the incident electric field is parallel to the Oz axis. Using the complex Poynting vector, we can straightforwardly obtain from Eq. (6) that the intensity $I(\theta)$ diffracted at an angle θ (see Fig. 1) is given by $C\,|f(\theta)|^2$, where C is a constant given by $1/2\Gamma_0\omega$, with μ_0 the magnetic permeability of the vacuum. In the TM polarization case, *i.e.*, the incident magnetic field is parallel to the Oz axis, we obtain the same result, but with C=$1/2\varepsilon_0\omega$, with ε_0 the dielectric constant of the vacuum. So that, the diffracted intensity is given by

$$\begin{aligned} I(\theta) &= k^2\cos^2\theta\left|\hat{E}(k\sin\theta)\right|^2 \\ &= \frac{1}{2\pi}k^2\cos^2\theta\left|\int_{-\infty}^{\infty} t(x)E_i(x)\exp(-ik\sin\theta x)dx\right|^2, \end{aligned} \tag{8}$$

where $I(\theta)$ has been normalized to C since we are interested only in relative quantities. Then the diffraction patterns can be determined from Eq. (8) if the input field $E_i(x)$ and the transmittance function $t(x)$ are given.

3. Grating

Let us consider the case of a periodic transmittance function $t(x)$ with period D, which extends from $x = -\infty$ to $x = +\infty$, *i.e.*, we have a traditional grating with an infinite number of periods.

Now transform Eq. (4) as follows. From the periodicity of the grating we have

$$\hat{E}(\alpha)=\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{n=\infty}\int_{nD}^{(n+1)D} t(x)E_i(x)\exp(-i\alpha x)dx; \quad (9)$$

if we take $x' = -nD + x$, this equation becomes

$$\hat{E}(\alpha) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{n=\infty}\int_{0}^{D} t(x'+nD) \times E_i(x'+nD)\exp\left[-i\alpha(x'+nD)\right]dx' , \quad (10)$$

but from the periodicity of $t(x)$ we have $t(x'+nD) = t(x')$, then

$$\hat{E}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{0}^{D} dx'\; t(x')\exp\left[-i\alpha\, x'\right] \times \left[\sum_{n=-\infty}^{n=\infty} E_i(x'+nD)\exp(-i\alpha\, nD)\right]. \quad (11)$$

Given this result, let us to define the auxiliary function $U_i(x,\alpha)$ as follows

$$U_i(x,\alpha) = \sum_{n=-\infty}^{n=\infty} E_i(x+nD)\exp(-i\alpha nD)\,, \quad (12)$$

so that Eq. (9) takes the form

$$\hat{E}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{0}^{D} t(x)U_i(x,\alpha)\exp\left[-i\alpha x\right]dx\;. \quad (13)$$

Before considering Eq. (13) in detail let us to mention that the function $U_i(x,\alpha)$ as defined in Eq. (12) is a pseudoperiodic function

$$U_i(x+D,\alpha) = \exp(i\alpha\, D)U_i(x,\alpha). \quad (14)$$

In order to simplify Eq. (13) it is more convenient to express the incident field as an angular spectrum of plane waves [13], given by

$$E_i(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-k}^{k} A(\alpha)\exp\left[i(\alpha\, x - \beta\, y)\right]d\alpha\,, \quad (15)$$

where $\alpha^2+\beta^2 = k^2$ with $\beta \geq 0$ and $A(\alpha)$ is the spectral amplitude. We notice that no evanescent waves are considered in Eq. (15) because they do not take part in the formation of the far field of the incident wave [13,21]. For the particular case of an incident plane wave, the amplitude $A(\alpha)$ is given by the Dirac delta function. After replacing Eq. (15) into Eq. (12) we get

$$U_i(x,\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-k}^{k} A(\alpha')\exp(i\alpha' x) \times \left[\sum_{n=-\infty}^{n=\infty}\exp\left[i(\alpha'-\alpha)nD\right]\right]d\alpha'\,, \quad (16)$$

but, if we consider the following property of the Dirac delta function

$$\sum_{n=-\infty}^{\infty}\exp(i2\pi\, nx) = \sum_{n=-\infty}^{\infty}\delta(x-n), \quad (17)$$

and the fact that $\delta(kx) = \delta(x)/\left|k\right|$, we obtain

$$U_i(x,\alpha) = \frac{\sqrt{2\pi}}{D}\sum_{n=-\infty}^{\infty} A\left(\alpha+\frac{2\pi}{D}n\right)\exp\left[i\left(\alpha+\frac{2\pi}{D}n\right)x\right]. \quad (18)$$

With these operations we have expressed the pseudoperiodic function $U_i(x,\alpha)$ given in Eq. (12) in terms of the spectral amplitude $A(\alpha')$ of the incident field of Eq. (15). If we now replace Eq. (18) into Eq. (13) we get the amplitude of the diffracted waves when a thin amplitude-phase grating is illuminated by a beam

$$\hat{E}(\alpha) = \sum_{n=-\infty}^{\infty} A(\alpha+\frac{2\pi}{D}n)t_{-n}\,, \quad (19)$$

where the t_n are the Fourier coefficients of the grating transmittance function $t(x)$

$$t(x) = \sum_{n=-\infty}^{\infty} t_n\exp(i2\pi\, nx/D), \quad (20)$$

given by

$$t_n = \frac{1}{D}\int_{0}^{D} t(x)\;\exp\left[-i\frac{2\pi}{D}nx\right]dx. \quad (21)$$

Finally, if we replace Eq. (19) into Eq. (2) the diffracted field at any point below the grating can be obtained

$$E^d(x,y) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} t_{-n} \times \int_{-\infty}^{\infty} A\left(\alpha+\frac{2\pi}{D}n\right)\exp[i(\alpha x-\beta y)]d\alpha\,, \quad (22)$$

in order to determine the meaning of this result, let us define the diffracted wave in the n-order $E_n(x,y)$ as follows

$$E_n(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A \left(\alpha + \frac{2\pi}{D} n\right) \exp[i(\alpha x - \beta y)] d\alpha \qquad (23)$$

so that Eq. (22) takes the simple form

$$E^d(x,y) = \sum_{n=-\infty}^{\infty} t_n \; E_{-n}(x,y)\,, \qquad (24)$$

where n has been changed by -n. This is an important and general result which gives the diffracted field when a beam is incident on a thin amplitude-phase grating. Notice that the transmittance function could be a complex function. Also, we notice the influence of the Fourier coefficients t_n of the transmittance function $t(x)$ into the process of diffraction.

The physical meaning of Eqs. (23) and (24) is the following one: the form of the several transmission orders $E_n(x,y)$ is similar to the form of the incident field. In Sec. 5, of Numerical Results, this conclusion is discussed in connection with the diffraction of Hermite-Gaussian and distorted beams. On the other hand, we are interested in incident beams of finite cross section, then the incident field $E_i(x)$ will be different from zero within a finite interval [a,b] and zero outside of it (or very close to zero). In consequence, from Eq. (15) we get that the function $A(\alpha)$ will also be different from zero (or very close to zero) within a finite interval [α_1,α_2] where $-k \leq \alpha_1 < \alpha_2 \leq k$. So that, due to the argument of the function $A(\alpha+2\pi\, n/D)$ in Eq. (23), only a finite number of $E_n(x,y)$ are different from zero. This last conclusion will be verified in Sec. 5, of Numerical Results, where the diffraction of Hermite-Gaussian and distorted beams will be treated. To our knowledge, this is the first time that this concept of transmission order in the diffraction of beams is presented in the literature.

We consider Eq. (24) as the generalization for general incident beams of the particular case of incident plane waves on the grating at the angle θ_0, with the angle of incidence θ_0 measured from the normal. As is known, the result of this interaction is the generation of plane waves propagating at the angles θ_n, given by $\sin\theta_n = \sin\theta_0 + n\lambda/D$; this is the famous grating equation. From our theory we can obtain these facts as follows. For incident plane waves we have the spectral amplitude $A(\alpha) = \sqrt{2\pi}\; \delta(\alpha-\alpha_0)$, where $\alpha_0 = k\sin\theta_0$, so that, from Eq. (24) we have

$$E^d(x,y) = \sum_{n=-\infty}^{\infty} t_n \exp\left[i(\alpha_n x - \beta_n y)\right]\,, \qquad (25)$$

with $\alpha_n{=}\alpha_0{+}(2\pi n/D)$ (grating equation) and $\beta_n{=}\sqrt{k^2{-}\alpha_n^2}$. This is the very well-known expression of the Rayleigh expansion for thin amplitude-phase gratings. Then, we have obtained the known result that the amplitude of each diffraction order n is given by the Fourier coefficient t_n of the transmittance function $t(x)$ [5]. Finally, if we replace Eq. (19) into Eq. (8) the diffraction pattern can be obtained

$$I(\theta) = k^2 \cos^2\theta \left| \sum_{n=-\infty}^{\infty} A(k\sin\theta + \frac{2\pi}{D} n) t_{-n} \right|^2 . \qquad (26)$$

A. Diffraction by a Ronchi ruling

We have a periodic ruling made of alternate transparent and opaque zones. The function $t(x)$ is null in the opaque zones and has the unitary value in the transparent zones. The period of the ruling is given by $D = l + d$, where d is the width of the opaque zones and l the width of the transparent zones. The Fourier coefficients t_n of the grating are given by:

$$t_n = \frac{1}{D} \int_0^l \exp[-i\frac{2\pi}{D} nx] dx = \frac{l}{D} \frac{\sin(\pi nl/D)}{(\pi nl/D)} \exp\left[-i\frac{\pi nl}{D}\right]\,, \qquad (27)$$

then, for this particular ruling the pattern diffraction can be obtained from Eq. (26)

$$I(\theta) = \frac{l^2}{D^2} k^2 \cos^2\theta \left| \sum_{n=-\infty}^{\infty} A \left(k\sin\theta + \frac{2\pi}{D} n\right) \frac{\sin(\pi nl/D)}{(\pi nl/D)} \exp\left[i\frac{\pi nl}{D}\right] \right|^2 , \qquad (28)$$

when $d = l$ a Ronchi ruling is given and in this case Eq. (28) is considerably simplified.

Finally, we mention that with the theory given in Sec. 3 other gratings can be considered, for instance, cosinusoidal amplitude grating, cosinusoidal modulation of the absorbance grating, binary amplitude grating, cosinusoidal phase grating, and so forth. A numerical study of these gratings will be carried out in a future paper.

B. Diffraction by an amplitude-phase grating

Let us consider a complex transmittance function $t(x)$, given by

$$t(x) = t_1(x) \exp\left[i\varphi(x)\right] = t_1(x) t_2(x)\,, \qquad (29)$$

where $t_1(x)$ and $t_2(x)$ are periodic functions with the same period D. When $t_1(x) = 1$ or $t_2(x) = 1$, a phase or amplitude grating is considered, respectively. In the general case, we have a thin amplitude-phase grating.

We have the following Fourier expansion for $t_1(x)$ and $t_2(x)$

$$t_1(x) = \sum_{j=-\infty}^{\infty} t_{1j} \exp(i2\pi jx/D);$$
$$t_2(x) = \sum_{m=-\infty}^{\infty} t_{2m} \exp(i2\pi mx/D), \qquad (30)$$

from these results the Fourier transform $\hat{t}_1(\alpha)$ and $\hat{t}_2(\alpha)$ of $t_1(x)$ and $t_2(x)$, respectively, are given by

$$\hat{t}_1(\alpha) = \sqrt{2\pi}\sum_j t_{1j}\delta\left(\alpha - \frac{2\pi}{D}j\right);$$
$$\hat{t}_2(\alpha) = \sqrt{2\pi}\sum_m t_{2m}\delta\left(\alpha - \frac{2\pi}{D}m\right), \qquad (31)$$

then, the Fourier transform $\hat{t}(\alpha)$ of $t(x)$ is obtained

$$\hat{t}(\alpha) = \frac{1}{\sqrt{2\pi}}\left(\hat{t}_1 \otimes \hat{t}_2\right)(\alpha)$$
$$= \sqrt{2\pi}\sum_{j,m} t_{1j}t_{2m}\delta\left(\alpha - \frac{2\pi}{D}(m+j)\right), \qquad (32)$$

where $\otimes$ means the convolution product.

On the other hand, from Eq. (15) we have $\hat{E}_i = A(\alpha)$ at $y = 0$ and from Eq. (4) we get

$$\hat{E}(\alpha) = \frac{1}{\sqrt{2\pi}}(\hat{t} \otimes A)(\alpha). \qquad (33)$$

Finally, from Eqs. (32) and (33) the amplitude of the diffracted waves when a thin amplitude-phase grating is illuminated by a beam is obtained

$$\hat{E}(\alpha) = \sum_{j,m} t_{1j}t_{2m}A\left(\alpha - \frac{2\pi}{D}(m+j)\right), \qquad (34)$$

and the diffracted intensity is given by

$$I(\theta)=k^2\cos^2\theta\left|\sum_{j,m} t_{1j}t_{2m}A\left(k\sin\theta - \frac{2\pi}{D}(m+j)\right)\right|^2. \qquad (35)$$

It is interesting to consider the particular case of an incident plane wave with the angle of incidence θ_0. Keeping in mind that $A(\alpha) = \sqrt{2\pi}\delta(\alpha - \alpha_0)$, with $\alpha_0 = k\sin\theta_0$; from Eqs. (2) and (34) we have

$$E^d(x,y) = \sum_{j,m} t_{1j}t_{2m}\exp[i(\alpha_{m+j}x - \beta_{m+j}y)], \qquad (36)$$

where $\alpha_{m+j}=\alpha_0+(2\pi/d)(m+j)$ and $(\alpha_{m+j})^2+(\beta_{m+j})^2=k^2$. As it is known, the Rayleigh expansion in transmission is given by

$$E^d(x,y) = \sum_{n=-\infty}^{\infty} A_n \exp[i(\alpha_n x - \beta_n y)], \qquad (37)$$

where the amplitude of each diffraction order n is given by the unknown coefficient A_n. Then, after comparing Eqs. (36) and (37) we get the important result

$$A_n = \sum_{\substack{j,m \\ n=m+j}} t_{1j}t_{2m}. \qquad (38)$$

It is interesting to mention that in Ref. 5 the particular case of a cosinusoidal amplitude-phase grating was treated, where only three diffracted orders are assumed, and only the coefficients A_0 and A_1 have been obtained. Now, from Eq. (38) we have a general equation for the calculation of the coefficient A_n, for any value of n. The diffraction efficiency of order n for this grating is given by

$$e_n = \left|\sum_{\substack{j,m \\ n=j+m}} t_{1j}t_{2m}\right|^2 \frac{\cos\theta_n}{\cos\theta_0}. \qquad (39)$$

4. Finite grating

In this section the case of a periodic transmittance function $t(x)$ with N periods will be considered, *i.e.*, a grating with a finite number of periods. The finite grating extends from $x = a$ to $x = a + ND$, with $t(x)$ null outside this interval.

From Eq. (4) we have

$$\hat{E}(\alpha) = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{N-1}\int_{a+nD}^{a+(n+1)D} t(x)E_i(x)\exp[-i\alpha x]dx; \qquad (40)$$

if we take $x' = -nD + x$ and we consider the periodicity $t(x') = t(x' + nD)$, we get:

$$\hat{E}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_a^{a+D} t(x')U_i^N(x',\alpha)\exp[-i\alpha x']dx', \qquad (41)$$

where the auxiliary function $U_i^N(x,\alpha)$, equivalent to Eq. (12) for the infinite grating, is given by

$$U_i^N(x,\alpha) = \sum_{n=0}^{N-1} E_i(x+nD)\,\exp[-i\alpha\, nD]\,. \qquad (42)$$

We can express the auxiliary function U_i^N in terms of the spectral amplitude $A(\alpha)$. From Eqs. (15) and (42) we have

$$U_i^N(x,\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(\alpha')$$
$$\times\exp[i(\alpha' x - (\alpha-\alpha')(N-1)D/2)]$$
$$\times\frac{\sin[(\alpha-\alpha')ND/2]}{\sin[(\alpha-\alpha')D/2]}d\alpha'; \qquad (43)$$

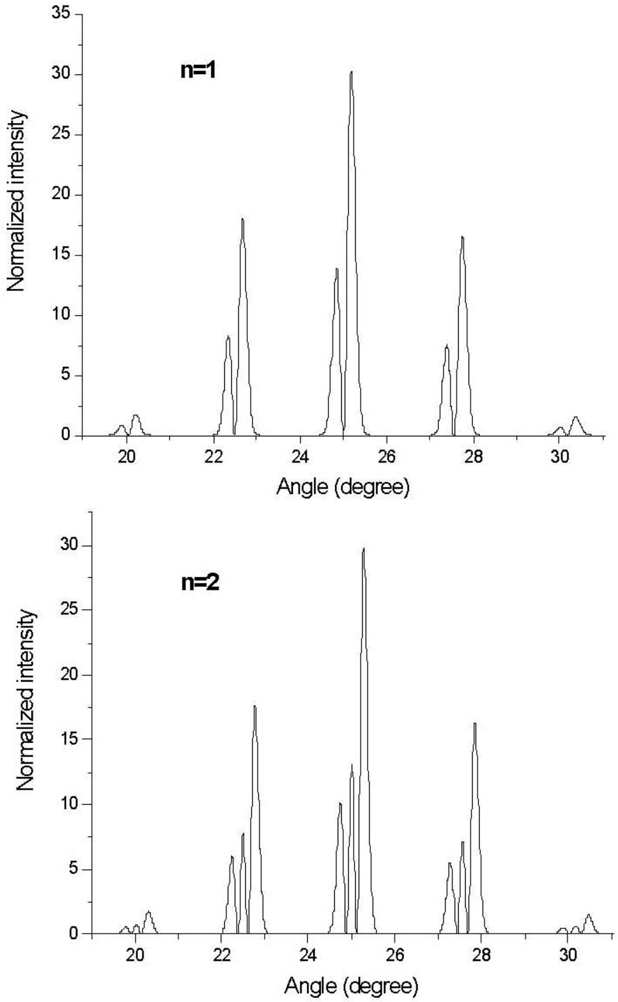

FIGURE 5. Same as Fig. 2 but for distorted Hermite-Gaussian beams of orders n= 1 and 2.

after replacing Eq. (43) into Eq. (41) we get the amplitude of the diffracted waves when a finite amplitude-phase grating is illuminated by a beam

$$\hat{E}(\alpha)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} d\alpha' T(\alpha,\alpha') \times A(\alpha')\exp[-i(\alpha-\alpha')(N-1)D/2] \times \frac{\sin[(\alpha-\alpha')ND/2]}{\sin[(\alpha-\alpha')D/2]}, \tag{44}$$

where the function $T(\alpha,\alpha')$ is defined as follows

$$T(\alpha,\alpha') = \int\limits_{a}^{a+D} t(x')\ \exp[i(\alpha'-\alpha)x']dx'; \tag{45}$$

if we replace Eq. (44) into Eq. (2) the diffracted field can be determined anywhere, for this, it is necessary to know the

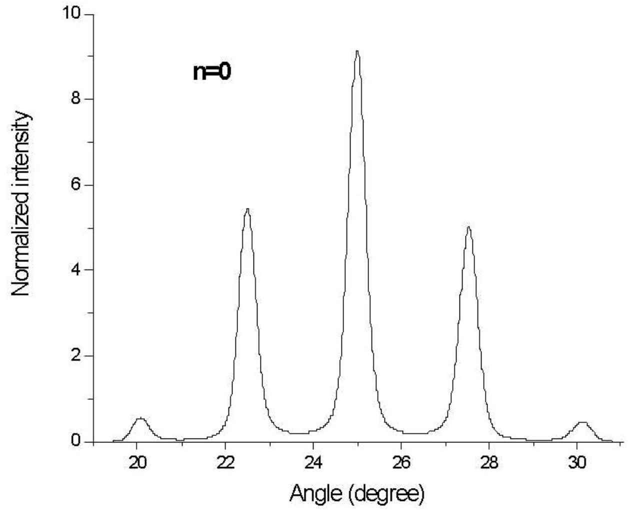

FIGURE 6. Diffraction pattern of a Gaussian beam (n=0) incident on a finite Ronchi ruling. Same parameters as that of Fig. 2 but for a finite Ronchi ruling with 20 periods.

periodic transmittance function $t(x)$ and the spectral amplitude $A(\alpha)$.

Finally, the diffracted intensity is given by

$$I(\theta) = \frac{k^2}{4\pi^2}\cos^2\theta \left| \int\limits_{-\infty}^{\infty} d\alpha' T(k\sin\theta,\alpha') \times A(\alpha')\exp[-i(k\sin\theta-\alpha')(N-1)D/2] \times \frac{\sin[(k\sin\theta-\alpha')ND/2]}{\sin[(k\sin\theta-\alpha')D/2]}\right|^2 \tag{46}$$

we must to remember that the functions $A(\alpha)$ and $t(x)$ are arbitrary functions, for instance, $t(x)$ may be a finite cosinusoidal amplitude grating, a finite binary amplitude grating, a finite cosinusoidal phase grating, and so forth. A numerical study of these kinds of finite gratings will be carried out in a future paper.

A. Diffraction by a finite Ronchi ruling

For a finite Ronchi Ruling (N identical clear fringes of width l) the function $T(\alpha,\alpha')$ of Eq. (45) is given by

$$T(\alpha,\alpha')=l\exp\left[i(\alpha'-\alpha)(a+l/2)\right]\frac{\sin\left[(\alpha'-\alpha)l/2\right]}{(\alpha'-\alpha)l/2}; \tag{47}$$

from this result and Eq. (46) the diffraction pattern when a beam is incident on a finite Ronchi ruling can be obtained. When the beam is a plane wave with the angle of incidence θ_0, we obtain from Eq. (44)

$$\hat{E}(\alpha)=\frac{1}{\sqrt{2\pi}}\exp[-i(\alpha-\alpha_0)(N-1)D/2] \times \frac{\sin[(\alpha-\alpha_0)ND/2]}{\sin[(\alpha-\alpha_0)D/2]}T(\alpha,\alpha_0), \tag{48}$$

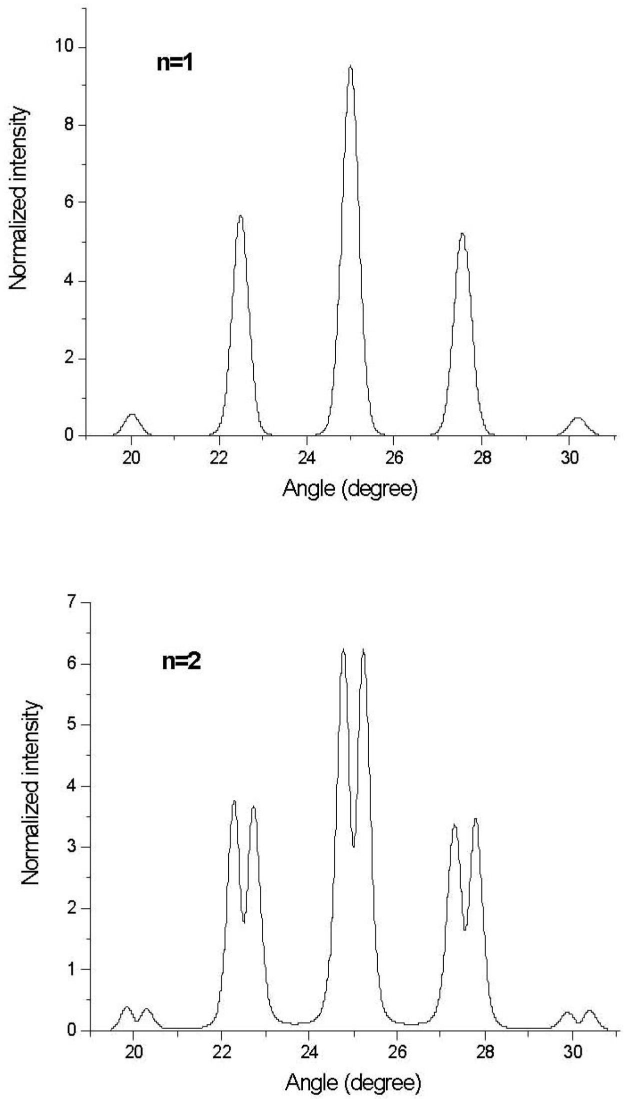

FIGURE 7. Same as Fig. 3 but for a finite Ronchi ruling with 20 periods. Hermite-Gaussian beams of order n=1, 2 are considered.

where $\alpha_0 = k\sin\theta_0$. Then, from Eqs. (47) and (48) we get the amplitude of the diffracted waves when a plane wave is incident on a finite Ronchi ruling

$$\hat{E}(\alpha) = \frac{l}{\sqrt{2\pi}} \exp\left[-i(\alpha-\alpha_0)\left\{(N-1)D/2 + a + l/2\right\}\right] \times \frac{\sin[(\alpha-\alpha_0)ND/2]}{\sin[(\alpha-\alpha_0)D/2]} \frac{\sin[(\alpha-\alpha_0)l/2]}{(\alpha-\alpha_0)l/2}, \quad (49)$$

where the last term gives the contribution of the diffraction by one slit and the penultimate term is the interference of N sources with period D. If the function $\left|\hat{E}(\alpha)\right|^2$ is calculated from Eq. (48) and it is normalized to $l^2N^2/2\pi$, then, Eq. (2.25) of [34] is obtained.

5. Numerical results

In this section, as a numerical application of the theory presented in this paper, the diffraction of Hermite-Gaussian and distorted beams by a Ronchi ruling (infinite and finite, see Fig. 1) is studied. The Hermite-Gaussian beams are described by the product of Hermite polynomials and Gaussian functions. The two-dimensional Hermite-Gaussian beams can easily be excited with an end-pumped solid-sate laser [23]. These beams have been considered in relation to some diffraction problems [15-16, 24-25]. For a more complete list of references about the applications of these beams see [26].

On the screen and at normal incidence, the field of the Hermite-Gaussian beam of order n is given by

$$E_i(x, y=0) = H_n\left[\frac{2}{L}(x-b)\right] \exp\left[-\frac{2(x-b)^2}{L^2}\right], \quad (50)$$

where H_n is the Hermite polynomial of order n and $L/2$ the local 1/e intensity Gaussian beam radius. The position of the incident Hermite-Gaussian beam with respect to the Oy axis is fixed by parameter b. This parameter enables us to displace the beam along the screen.

In order to numerically consider Hermite-Gaussian beams at oblique incidence θ_0, it is convenient to determine the spectral amplitude $A(\alpha)$ for normal incidence from Eqs. (15) and (50), and to perform a rotation of an angle θ_0 about the Oz axis. In this procedure the following identity [27] must to be utilized

$$\int_{-\infty}^{\infty} \exp(ixy) \exp(-x^2/2) H_n(x) dx = (2\pi)^{1/2} (i)^n \exp(-y^2/2) H_n(y); \quad (51)$$

finally, by a translation to the point $x = b$ the following amplitude $A(\alpha)$ is obtained

$$A(\alpha) = \frac{L}{2}(i)^n H_n\left[-Lq_1(\theta_0)/2\right] q_2(\theta_0) \times \exp(-i\alpha b) \exp\left[-q_1(\theta_0)^2 L^2/8\right], \quad (52)$$

where θ_0 is the angle of incidence of the beam with respect to the Oy axis and $q_1(\theta_0) = \alpha\cos\theta_0 - \beta\sin\theta_0$, and $q_2(\theta_0) = \cos\theta_0 + (\alpha/\beta)\sin\theta_0$.

For distorted incident waves we take the field whose spectral amplitude A_d is given by

$$A_d(\alpha) = A(\alpha)\exp(\alpha), \quad (53)$$

where $A(\alpha)$ is given by Eq. (52). The nonsymmetrical factor $\exp(\alpha)$ modifies the angular spectrum of plane waves of Eq. (15) in such a way that distorted Hermite-Gaussian beams take place. These distorted beams are very interesting and, they show the potential of our theory.

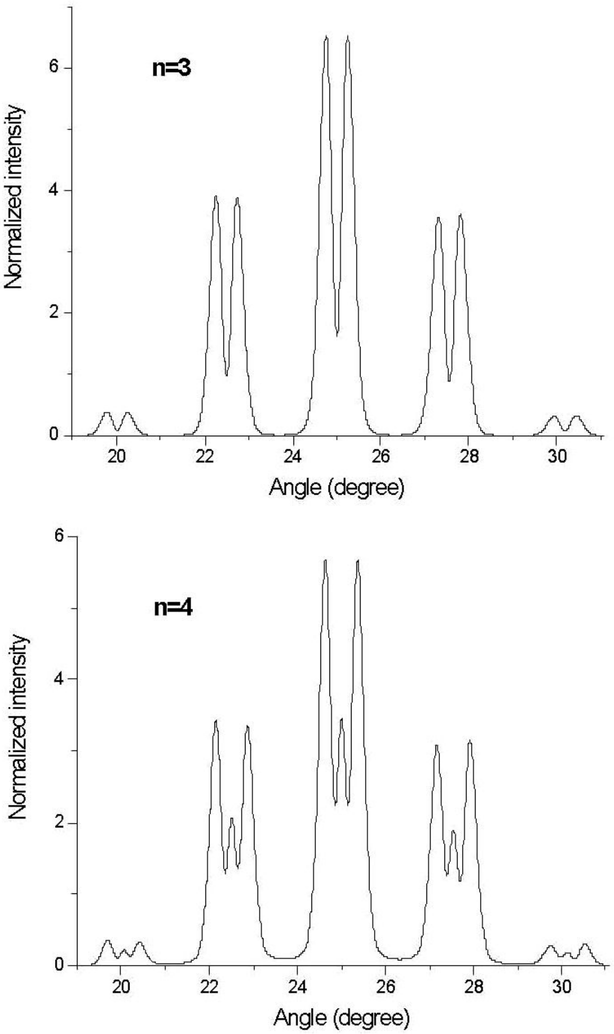

FIGURE 8. Same as Fig. 4 but for a finite Ronchi ruling with 20 periods. Hermite-Gaussian beams of order n=3, 4 are considered.

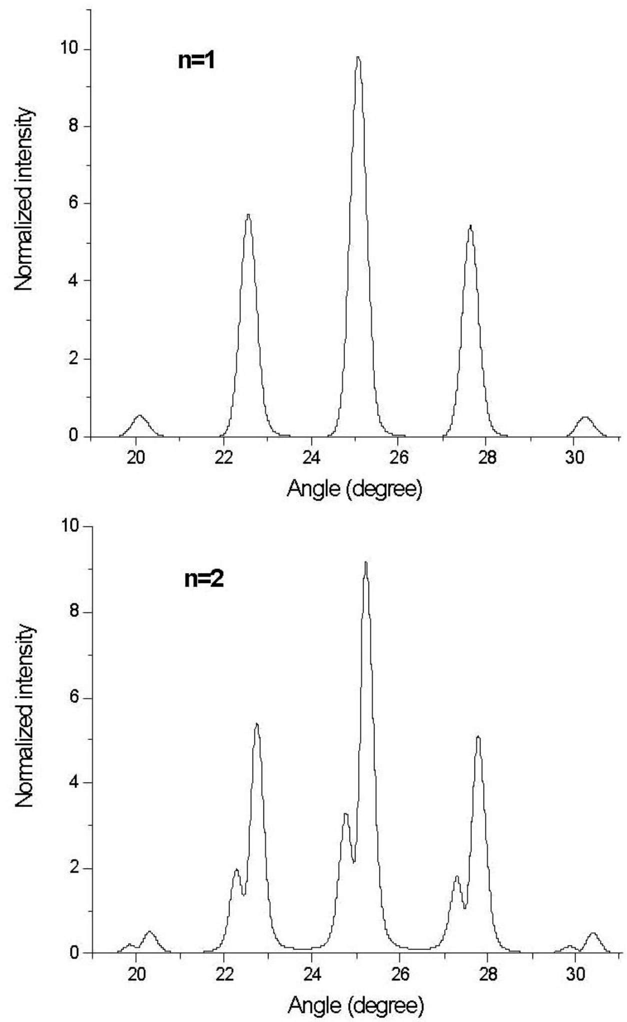

FIGURE 9. Same as Fig. 5 but for a finite Ronchi ruling with 20 periods. Distorted Hermite-Gaussian beams of order n=1, 2 are considered.

In what follows we consider the diffraction of Hermite-Gaussian and distorted beams for the particular case of infinite and finite Ronchi rulings made of alternate transparent (width l) and opaque zones (width d), as illustrated in Fig. 1. Some other gratings will be treated in a future paper.

A. Infinite Ronchi ruling

In Fig. 2, the diffraction pattern when a Gaussian beam (n=0) is incident on an infinite Ronchi ruling is shown. We have the following parameters: $L/l = 15/\sqrt{2}$, $\lambda/l = 0.1$, and the angle of incidence $\theta_0 = 25°$. A Ronchi ruling with $d/l = 1.5$ and period $D/l = 2.5$ is considered. In this case the diameter ratio of the spot/period of the ruling (L/D) is 4.24, *i.e.*, the beam is very far from a plane wave. From Fig. 2 several transmission orders (five orders are shown in figure) can be observed; the existence of these orders has been predicted by Eqs. (23) and (24). We note that the several orders are very well separated (angularly). These orders resemble those of the diffraction of plane waves by infinite gratings, where the angular positions are determined by means of the classical grating equation given by $\sin\theta_m = \sin\theta_0 + m\lambda/D$, m=0, ±1, ±2,... By using the parameters of our ruling Ronchi and the grating equation, the following angular positions are obtained: $\theta_{-2} = 20.0364°$, $\theta_{-1} = 22.4959°$, $\theta_0 = 25°$, $\theta_1 = 27.5561°$ and $\theta_2 = 30.1733°$. From Fig. 2, the angular positions of the several orders (determined by our theory) are: $\theta_{-2} = 20.0369°$, $\theta_{-1} = 22.4990°$, $\theta_0 = 25°$, $\theta_1 = 27.5590°$, and $\theta_2 = 30.1763°$. We have compared the angular positions of the several orders of Fig. 2 with those calculated directly from the grating equation and a maximum relative error of 0.013% is found. Then, the agreement is very good. In order to explain this agreement it is convenient to express the grating equation in terms of parameter α. Thus, we have $k\sin\theta_m = k\sin\theta_0 + m2\pi/D$, which can be written in terms of the parameter α as follows $\alpha_m = \alpha_0 + 2\pi\, m/D$, then, the incident wave determined by α_0 is "translated" to the several orders determined by α_m by means of the term $2\pi\, m/D$. Then, after comparing Eqs. (15) and (23), we see that the incident wave determined by means of the amplitude $A(\alpha)$ is "translated" to the several orders determined by the amplitude $A(\alpha + 2\pi\, n/D)$ by means of the term $2\pi\, n/D$.

In Figs. 3 and 4 the diffraction patterns when Hermite-Gaussian beams are incident on an infinite Ronchi ruling, for n=1, 2, and n=3, 4, respectively, are plotted. We have used the same parameters as in Fig. 2. In these figures the existence of one, two, three and four dips at each maximum (transmission order), which reach the bottom (null intensity), are shown. Also, we observe that the number of deep dips at the maxima is the same as the order n of the Hermite polynomial in the incident beam. In fact, these dips in the diffraction patterns are caused by the zeros of the Hermite polynomials. It is interesting to mention that Kojima [28] has considered the scattering of TE-polarized Hermite-Gaussian beams of only order one ($n = 1$) from a sinusoidal conducting grating, and one deep dip at each of the maxima in the reflection patterns were found. So that our results for transmission ruling extend the result of Kojima for $n = 1$ to higher orders of the Hermite-Gaussian beam. It is interesting to see that the several orders of Figs. 3 and 4 are similar to $|E_i|^2$ calculated from Eq. (50), *i.e.*, the transmission orders and the incident beam as the same form as was established in Eqs. (23) and (24). Finally, we mention that the angular position of the transmission orders are in accord ance with the grating equation; in a future paper we will analyze in more detail this last result.

Figure 5 is also similar to Fig. 2 but for distorted Hermite-Gaussian beams of orders n= 1 and 2. In this figure the effect of the exponential function in Eq. (53) is evident; each transmission order of the diffraction pattern is also distorted. We believe that it is the first time that this kind of beam is treated in the literature of diffraction of beams.

B. Finite ronchi ruling

In Fig. 6 the diffraction pattern of a Gaussian beam (n=0) incident on a finite Ronchi ruling (see Fig. 1b) is considered. The parameters considered are the same as those of Fig. 2 but for a finite Ronchi ruling with 20 periods. We have placed the center of the incident Gaussian beam on the edge of the finite ruling (point A in Fig. 1b) by means of parameter b. Great differences with respect to the case of an infinite Ronchi ruling, as shown in Fig. 2, are observed. In Fig. 2 the several orders are very well separated (angularly), but in Fig. 6 an interaction between them is noted (see bottom of figure).

Figures 7 and 8 are similar to Fig. 3 and 4 but for a finite Ronchi ruling. The center of the incident Hermite-Gaussian beams is placed on the edge of the finite ruling (point A in Fig. 1b). The parameters are the same as those of Fig. 2 but for a finite Ronchi ruling with 20 periods. We observe that the dips of Figs. 3 and 4 for an infinite ruling have suffered a great change when a finite ruling is considered. We note also that the number of dips in these figures is different from the order n of the Hermite polynomial. With these results it is shown that the infinite ruling case is very different to that of the finite ruling case. It was a surprise to verify that the angular positions of the several transmission orders for a finite ruling are in accord once with the grating equation. These conclusions will be analyzed in a future paper.

Figure 9 is similar to Fig. 5 but for a finite Ronchi ruling with 20 periods. The center of the distorted Hermite-Gaussian beams is incident on point A of Fig. 1b. We have considered the same parameters as Fig. 2 but for a finite Ronchi ruling with 20 periods. Again, very big changes are observed among Fig. 9 for a finite ruling and Fig. 5 for an infinite ruling.

6. Conclusions

We present a theory for the diffraction of two-dimensional beams by one-dimensional thin amplitude-phase gratings. Mathematical expressions for the diffraction of beams were proposed. We have considered finite and infinite gratings. In the case of infinite gratings the existence of transmission orders of diffraction, whose form is similar to the form of the incident beam, are shown. As a numerical application of the theory presented in this paper the diffraction of Hermite-Gaussian beams and distorted beams by a Ronchi ruling (infinite and finite) is studied. We have obtained the result that the classical grating equation predicts very well the angular positions of the transmission orders predicted by our theory.

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