INVESTIGACIÓN REVISTA MEXICANA DE FÍSICA **57** (1) 53–59 FEBRERO 2011

Relativistic charged particle in a uniform electromagnetic field

G.F. Torres del Castillo
Departamento de Física Matemática, Instituto de Ciencias, Universidad Autónoma de Puebla, Puebla, Pue., 72570, México.

C. Sosa Sánchez
Facultad de Ciencias Físico Matemáticas, Universidad Autónoma de Puebla, Apartado postal 1152 Puebla, Pue.,72001, México.

Recibido el 14 de mayo de 2010; aceptado el 16 de noviembre de 2010

The equations of motion for a relativistic charged particle in a uniform electromagnetic field are solved in a covariant form by calculating the exponential of the matrix corresponding to the electromagnetic field tensor. It is shown that owing to the antisymmetry of the electromagnetic field tensor, the exponential mentioned above can be easily calculated. Some results are then applied to study the algebraic properties of the energy-momentum tensor of the electromagnetic field and the orthogonal transformations in spaces of dimension 4.

Keywords: Electromagnetic field; matrix exponentials; dominant energy condition; orthogonal transformations.

Se resuelven las ecuaciones de movimiento de una partícula relativista cargada en un campo electromagnético uniforme en forma covariante, calculando la exponencial de la matriz correspondiente al tensor del campo electromagnético. Se muestra que debido a la antisimetría del tensor del campo electromagnético, dicha exponencial se puede calcular fácilmente. Algunos resultados se aplican para estudiar las propiedades algebraicas del tensor de energía-momento del campo electromagnético y las transformaciones ortogonales en espacios de dimensión 4.

Descriptores: Campo electromagnético; exponenciales de matrices; condición de energía dominante; transformaciones ortogonales.

PACS: 03.30.+p; 03.50.De; 02.20.Qs

1. Introduction

The motion of a relativistic charged particle in a constant uniform electromagnetic field has been studied in a large number of papers and textbooks (see, *e.g.*, Refs. 1 to 6 and the references cited therein), making use of various procedures. The corresponding equations of motion can be solved in a manifestly covariant form, expressing the four-velocity or the space-time coordinates of the particle as a function of the proper time of the particle, or making use of a convenient reference frame in which the electromagnetic field takes some simplified form; for instance, when $\mathbf{E}\cdot\mathbf{B} = 0$ and $\mathbf{E}^2 - \mathbf{B}^2$ is different from zero, there exists an inertial frame where one of the fields, $\mathbf{E}$ or $\mathbf{B}$, vanishes and the equations of motion can be easily solved (see, *e.g.*, Ref. 6).

The equations of motion of a charged particle in a uniform electromagnetic field constitute a system of four homogeneous linear first-order ordinary differential equations with constant coefficients for the four-velocity of the particle as a function of the proper time, whose solution can be expressed in terms of the exponential of a 4×4 matrix. The aim of this paper is to show that, owing to the antisymmetry of the electromagnetic field tensor, this exponential can be calculated directly by appropriately decomposing the argument of the exponential into two matrices corresponding to the self-dual and the anti-self-dual parts of the electromagnetic field tensor (see also Ref. 1). Some of the basic relations thus established are then used to study the algebraic properties of the energy-momentum tensor of the electromagnetic field and the orthogonal transformations of any four-dimensional space.

In Sec. 2 the four-velocity of a relativistic charged particle in a uniform electromagnetic field is obtained in a covariant form by directly calculating the exponential of a matrix formed by the components of the electromagnetic field. The procedure followed here essentially coincides with that employed in Taub's paper [1]; however, here we present the mechanism behind the algebraic relations that simplify the calculation of the exponential mentioned above. We also demonstrate, in Sec. 3, that some of the relations obtained in Sec. 2 can be employed to study the algebraic properties of the energy-momentum tensor of the electromagnetic field and to find the orthogonal transformations of the four-dimensional spaces.

2. Solution of the equations of motion

In the framework of special relativity, the equations of motion of a charged particle with rest mass m and electric charge q in an arbitrary electromagnetic field can be written in the covariant form

$$\frac{\mathrm{d}U^\alpha}{\mathrm{d}\tau} = \frac{q}{mc} F^\alpha_\beta U^\beta, \tag{1}$$

where U^α ($\alpha = 0, 1, 2, 3$) is the four-velocity of the particle, τ is its proper time, c is the speed of light in vacuum, and $F_{\alpha\beta}$ is the electromagnetic field tensor, with sum over repeated indices (see, *e.g.*, Ref. 6). Equivalently, this system of equations can be expressed in the more elementary matrix form

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\tau} = \frac{q}{mc} F\mathbf{U}, \tag{2}$$

where $\mathbf{U}$ is the column matrix with entries U^0, U^1, U^2, U^3, and F is the 4×4 matrix

$$\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \tag{3}$$

where (E_x, E_y, E_z) and (B_x, B_y, B_z) are the Cartesian components of the electric and magnetic fields, respectively, with respect to some inertial reference frame.

If one assumes that the electromagnetic field is uniform, the entries of F are all constant, and the solution of Eq. (2) is given by (see, *e.g.*, Refs. 7 and 8)

$$\mathbf{U}(\tau) = \exp\left(\frac{q\tau}{mc}F\right)\mathbf{U}(0), \tag{4}$$

with the exponential of a matrix being defined by means of the series expansion

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!}A^n.$$

In spite of the fact that F is a 4×4 matrix, this exponential can be very easily computed if one writes the matrix F as the sum of two complex matrices (*cf.* Ref. 1)

$$F = \frac{1}{2}(S + A), \tag{5}$$

with

$$S \equiv \begin{pmatrix} 0 & F_x & F_y & F_z \\ F_x & 0 & -\mathrm{i}F_z & \mathrm{i}F_y \\ F_y & \mathrm{i}F_z & 0 & -\mathrm{i}F_x \\ F_z & -\mathrm{i}F_y & \mathrm{i}F_x & 0 \end{pmatrix}, \tag{6}$$

and

$$A \equiv \begin{pmatrix} 0 & F_x^* & F_y^* & F_z^* \\ F_x^* & 0 & \mathrm{i}F_z^* & -\mathrm{i}F_y^* \\ F_y^* & -\mathrm{i}F_z^* & 0 & \mathrm{i}F_x^* \\ F_z^* & \mathrm{i}F_y^* & -\mathrm{i}F_x^* & 0 \end{pmatrix}, \tag{7}$$

where F_x, F_y, F_z are the Cartesian components of the complex vector field

$$\mathbf{F} \equiv \mathbf{E} + \mathrm{i}\mathbf{B}$$

and * denotes complex conjugation (note that A is the complex conjugate of S).

The usefulness of this decomposition comes from the fact that S and A commute with each other, and their squares are proportional to the identity matrix,

$$SA = AS, \quad S^2 = \mathbf{F}^2 I, \quad A^2 = \mathbf{F}^{*2} I, \tag{8}$$

where I is the 4×4 identity matrix, and

$$\mathbf{F}^2 \equiv {F_x}^2 + {F_y}^2 + {F_z}^2 = (\mathbf{E} + \mathrm{i}\mathbf{B})^2 = \mathbf{E}^2 - \mathbf{B}^2 + 2\mathrm{i}\mathbf{E}\cdot\mathbf{B}$$

(see the discussion in Sec. 2.2, below). Hence, the powers of S are given by

$$S^{2n} = (\mathbf{F}^2)^n I, \qquad S^{2n+1} = (\mathbf{F}^2)^n S, \tag{9}$$

for $n = 0, 1, 2, \ldots$. It may be remarked that the two Lorentz invariants [6] of the electromagnetic field, $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$, arise here in a natural way.

Since S and A commute with each other, from Eq. (2) we have

$$\begin{aligned}\mathbf{U}(\tau) &= \left[\exp\frac{q\tau}{2mc}(S + A)\right]\mathbf{U}(0) \\ &= \left(\exp\frac{q\tau}{2mc}S\right)\left(\exp\frac{q\tau}{2mc}A\right)\mathbf{U}(0),\end{aligned}$$

with

$$\exp\left(\frac{q\tau}{2mc}A\right)$$

being the complex conjugate of

$$\exp\left(\frac{q\tau}{2mc}S\right).$$

Then, with the definition

$$|\mathbf{F}| \equiv \sqrt{\mathbf{F}^2} = \sqrt{\mathbf{E}^2 - \mathbf{B}^2 + 2\mathrm{i}\mathbf{E}\cdot\mathbf{B}} \tag{10}$$

(so that $\mathbf{F}^2 = |\mathbf{F}|^2$), making use of Eq. (9) we obtain

$$\exp\frac{q\tau}{2mc}S = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{q|\mathbf{F}|\tau}{2mc}\right)^{2n} I + \frac{1}{|\mathbf{F}|}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{q|\mathbf{F}|\tau}{2mc}\right)^{2n+1} S = \left(\cosh\frac{q|\mathbf{F}|\tau}{2mc}\right) I + \frac{1}{|\mathbf{F}|}\left(\sinh\frac{q|\mathbf{F}|\tau}{2mc}\right) S, \tag{11}$$

provided that $|\mathbf{F}| \neq 0$. In the case where $|\mathbf{F}| = 0$, only the first two terms of the series contribute [see Eqs. (8) and (9)] and we have

$$\exp\frac{q\tau}{2mc}S = I + \frac{q\tau}{2mc}S. \tag{12}$$

Note that $|\mathbf{F}|$ is a square root of the complex number $\mathbf{E}^2 - \mathbf{B}^2 + 2\mathrm{i}\mathbf{E}\cdot\mathbf{B}$, and that the final expression in Eq. (11) is an even function of $|\mathbf{F}|$; therefore, since the two square roots of a complex number differ by a -1 factor, both square roots produce the same result. Note also that Eq. (12) can be obtained from Eq. (11) in the limit $|\mathbf{F}| \to 0$, and therefore it is enough to consider Eq. (11) only.

From Eq. (11) we then obtain

$$\exp\left(\frac{q\tau}{mc}F\right) = \left(\cosh\frac{q|\mathbf{F}|\tau}{2mc}\right)\left(\cosh\frac{q|\mathbf{F}|^*\tau}{2mc}\right)I + \frac{1}{|\mathbf{F}|}\left(\sinh\frac{q|\mathbf{F}|\tau}{2mc}\right)\left(\cosh\frac{q|\mathbf{F}|^*\tau}{2mc}\right)S$$
$$+ \frac{1}{|\mathbf{F}|^*}\left(\sinh\frac{q|\mathbf{F}|^*\tau}{2mc}\right)\left(\cosh\frac{q|\mathbf{F}|\tau}{2mc}\right)A + \frac{1}{|\mathbf{F}||\mathbf{F}|^*}\left(\sinh\frac{q|\mathbf{F}|\tau}{2mc}\right)\left(\sinh\frac{q|\mathbf{F}|^*\tau}{2mc}\right)SA, \qquad (13)$$

which only involves the matrices I, S, A, and SA, and according to Eq. (4) gives the solution to Eq. (2) for any initial condition.

2.1. Some particular cases

As shown in Eqs. (11) and (12), the expression for

$$\exp\left(\frac{q\tau}{2mc}S\right)$$

depends essentially on $|\mathbf{F}|$, and this expression is simplified if $\mathbf{E}\cdot\mathbf{B}=0$, in which case $|\mathbf{F}|$ is real or pure imaginary [see Eq. (10)]. As is well known, if $\mathbf{E}\cdot\mathbf{B}=0$ then there exist inertial frames in which $\mathbf{B}$ or $\mathbf{E}$ is equal to zero, provided that $\mathbf{E}^2-\mathbf{B}^2$ is greater than zero or less than zero, respectively (see, *e.g.*, Ref. 6).

If $\mathbf{E}\cdot\mathbf{B}=0$ and $|\mathbf{E}|>|\mathbf{B}|$, then $|\mathbf{F}|$ is real and Eq. (13) reduces to

$$\exp\left(\frac{q\tau}{mc}F\right) = \frac{1}{2}\left(1+\cosh\frac{q|\mathbf{F}|\tau}{mc}\right)I + \frac{1}{|\mathbf{F}|}\left(\sinh\frac{q|\mathbf{F}|\tau}{mc}\right)F - \frac{1}{2|\mathbf{F}|^2}\left(1-\cosh\frac{q|\mathbf{F}|\tau}{mc}\right)SA, \qquad (14)$$

with $|\mathbf{F}|=\sqrt{\mathbf{E}^2-\mathbf{B}^2}$. (Note that, since $\mathbf{E}^2-\mathbf{B}^2$ is invariant under the Lorentz transformations, in this case $|\mathbf{F}|$ is the magnitude of the electric field in a reference frame where $\mathbf{B}=0$.) On the other hand, from Eqs. (5) and (8) we find that

$$F^2 = \frac{1}{4}(S^2+2SA+A^2) = \frac{1}{2}(|\mathbf{F}|^2 I + SA), \qquad (15)$$

hence, eliminating SA in favor of F^2, Eq. (14) can also be written in the form

$$\exp\left(\frac{q\tau}{mc}F\right) = I + \frac{1}{|\mathbf{F}|}\left(\sinh\frac{q|\mathbf{F}|\tau}{mc}\right)F - \frac{1}{|\mathbf{F}|^2}\left(1-\cosh\frac{q|\mathbf{F}|\tau}{mc}\right)F^2. \qquad (16)$$

Both invariants of the electromagnetic field are equal to zero if and only if $|\mathbf{F}|=0$; in that case from Eq. (16) we obtain

$$\exp\left(\frac{q\tau}{mc}F\right) = I + \frac{q\tau}{mc}F + \frac{1}{2}\left(\frac{q\tau}{mc}\right)^2 F^2, \qquad (17)$$

which correspond to the first three terms of the series expansion of the exponential on the left-hand side of the equation; in fact, making use of Eq. (15) and that both S^2 and A^2 are equal to zero one finds that $F^3=0$ and therefore $F^n=0$ for $n\geqslant 3$. As we shall prove in Sec. 3.2, in all cases (for each value of τ)

$$\exp\left(\frac{q\tau}{mc}F\right)$$

is a Lorentz transformation; the transformations of the form (17) are known as *null rotations* (or *parabolic* Lorentz transformations).

In a similar manner one finds that if $\mathbf{E}\cdot\mathbf{B}=0$ and $|\mathbf{B}|>|\mathbf{E}|$, then $|\mathbf{F}|$ is pure imaginary and writing $|\mathbf{F}|=\mathrm{i}B_0$, with B_0 real and positive (the magnitude of $\mathbf{B}$ in a reference frame where $\mathbf{E}=0$), Eq. (13) reduces to

$$\exp\left(\frac{q\tau}{mc}F\right) = I + \frac{1}{B_0}\left(\sin\frac{qB_0\tau}{mc}\right)F + \frac{1}{{B_0}^2}\left(1-\cos\frac{qB_0\tau}{mc}\right)F^2 \qquad (18)$$

which also leads to Eq. (17) in the limit as B_0 tends to zero. Equation (18) is a periodic function of τ with angular frequency $\omega=qB_0/mc$, which produces a rotation of the particle's four-velocity with this angular velocity (see also Ref. 1).

2.2. Origin of the algebraic simplifications

As we have seen, the properties (8) simplify the calculation of the exponential of the matrix F; however, the derivation presented above, or that given in Ref. 1, does not elucidate the origin of these key relations. However, as we shall presently show, by using an appropriate basis, relations (8) become trivial. Indeed, making use of the unitary 4×4 matrix

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & \mathrm{i} & 0 \\ 0 & 1 & -\mathrm{i} & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad (19)$$

(which represents a change of basis, with the new basis containing two complex vectors; the new basis, called a *null tetrad*, is formed by four null four-vectors), one finds that

$$K \equiv MSM^{-1} = \begin{pmatrix} F_z & F_x - \mathrm{i}F_y & 0 & 0 \\ F_x + \mathrm{i}F_y & -F_z & 0 & 0 \\ 0 & 0 & F_z & F_x - \mathrm{i}F_y \\ 0 & 0 & F_x + \mathrm{i}F_y & -F_z \end{pmatrix} \tag{20}$$

and

$$L \equiv MAM^{-1} = \begin{pmatrix} F_z^* & 0 & F_x^* + \mathrm{i}F_y^* & 0 \\ 0 & F_z^* & 0 & F_x^* + \mathrm{i}F_y^* \\ F_x^* - \mathrm{i}F_y^* & 0 & -F_z^* & 0 \\ 0 & F_x^* - \mathrm{i}F_y^* & 0 & -F_z^* \end{pmatrix}. \tag{21}$$

Making use of the *tensor product* (or *direct product*) of two 2×2 matrices, defined by (see, *e.g.*, Ref. 10)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} a\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & b\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ c\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & d\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta & b\alpha & b\beta \\ a\gamma & a\delta & b\gamma & b\delta \\ c\alpha & c\beta & d\alpha & d\beta \\ c\gamma & c\delta & d\gamma & d\delta \end{pmatrix}, \tag{22}$$

we see that the matrices K and L can be expressed in the form

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} F_z & F_x - \mathrm{i}F_y \\ F_x + \mathrm{i}F_y & -F_z \end{pmatrix}$$

and

$$L = \begin{pmatrix} F_z^* & F_x^* + \mathrm{i}F_y^* \\ F_x^* - \mathrm{i}F_y^* & -F_z^* \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then, using the fact that $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, it trivially follows that K and L commute with each other. Similarly, noting that

$$\begin{pmatrix} F_z & F_x - \mathrm{i}F_y \\ F_x + \mathrm{i}F_y & -F_z \end{pmatrix}^2 = \mathbf{F}^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we readily obtain $K^2 = \mathbf{F}^2 I$ and, similarly, $L^2 = \mathbf{F}^{*2} I$, which are equivalent to the last two Eqs. (8). It may be also noticed that

$$\begin{pmatrix} F_z & F_x - \mathrm{i}F_y \\ F_x + \mathrm{i}F_y & -F_z \end{pmatrix} = \mathbf{F} \cdot \boldsymbol{\sigma},$$

where $\boldsymbol{\sigma}$ is the vector formed with the standard Pauli matrices and, therefore, K and L can be conveniently expressed in terms of the Pauli matrices.

The matrices S and A (or, equivalently, K and L) correspond to the self-dual and anti-self-dual parts of $F_{\alpha\beta}$, respectively. If the dual of an antisymmetric tensor $t_{\alpha\beta}$ is defined by

$$^*t_{\alpha\beta} \equiv \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}t^{\gamma\delta}, \tag{23}$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is totally antisymmetric with $\varepsilon_{0123} \equiv 1$ and the tensor indices are raised or lowered with the aid of the metric tensor

$$(g_{\alpha\beta}) = \mathrm{diag}\,(-1, 1, 1, 1) \tag{24}$$

and its inverse, it follows that $^*(^*t_{\alpha\beta}) = -t_{\alpha\beta}$. Hence the eigenvalues of the duality operator are $\pm\mathrm{i}$. The entries of the matrices (6) and (7) are $S^\alpha{}_\beta$ and $A^\alpha{}_\beta$, respectively, with

$$S_{\alpha\beta} = F_{\alpha\beta} - \mathrm{i}\,^*F_{\alpha\beta}, \quad A_{\alpha\beta} = F_{\alpha\beta} + \mathrm{i}\,^*F_{\alpha\beta},$$

so that $^*S_{\alpha\beta} = \mathrm{i}\,S_{\alpha\beta}$, and $^*A_{\alpha\beta} = -\mathrm{i}\,A_{\alpha\beta}$. Similar results hold for all possible metrics in four-dimensional spaces (see Sec. 3.2.1, below) as can readily be shown making use of the spinor formalism [9].

2.3. Comparison with other covariant approaches

In Ref. 3, the series corresponding to

$$\exp\left(\frac{q\tau}{mc}F\right)$$

is calculated by showing first that all powers of F are linear combinations of the four matrices I, F, F^2, and the matrix corresponding to the dual of $F_{\alpha\beta}$ [equivalent to $(1/2)\mathrm{i}(S-A)$], which implies that the desired exponential is also a linear combination of these matrices, and then finding the coefficients by calculating traces. In the approach followed in Ref. 2,

$$\exp\left(\frac{q\tau}{mc}F\right)$$

is calculated by establishing some algebraic relations between F and the matrix formed with the dual of $F_{\alpha\beta}$, and then translating the problem into that of solving a homogeneous fourth-order linear ordinary differential equation with constant coefficients. By contrast, the calculation presented in Ref. 1 and in this section is much shorter and direct, by virtue of Eqs. (8), and has some other useful applications, as we shall show in Sec. 3. The algebraic relations employed in Refs. 2 and 3 follow at once from Eqs. (8).

On the other hand, in Ref. 5, the solution of the matrix equation (1) is obtained making use of the exponential of a differential operator and a decomposition analogous to Eq. (5) (see also Ref. 4).

3. Further applications

In this section we show that the results derived above can be employed in the study of the algebraic properties of the energy-momentum tensor of the electromagnetic field, and to find the orthogonal transformations of a four-dimensional space.

3.1. The energy-momentum tensor of the electromagnetic field

The components of the energy-momentum tensor of the electromagnetic field are given by [6]

$$T^{\alpha}{}_{\beta} = -\frac{1}{4\pi}(F^{\alpha}{}_{\gamma}F^{\gamma}{}_{\beta} - \tfrac{1}{4}F^{\rho}{}_{\gamma}F^{\gamma}{}_{\rho}\delta^{\alpha}_{\beta}), \qquad (25)$$

which amounts to saying that, in terms of the matrix F, the 4×4 matrix $T = (T^{\alpha}{}_{\beta})$ is expressed as [see Eqs. (5) and (8)]

$$\begin{aligned} T &= -\frac{1}{4\pi}\left[F^2 - \tfrac{1}{4}(\mathrm{tr}\,F^2)I\right] \\ &= -\frac{1}{4\pi}\left[(S+A)^2 - \tfrac{1}{4}(\mathrm{tr}\,F^2)I\right] \\ &= -\frac{1}{4\pi}\left[2SA + (\mathbf{F}^2 + \mathbf{F}^{*2} - \tfrac{1}{4}\mathrm{tr}\,F^2)I\right], \end{aligned} \qquad (26)$$

where tr denotes the trace.

As is well known, the trace of the energy-momentum tensor (25) vanishes, $T^{\alpha}{}_{\alpha} = 0$; on the other hand,

$$\mathrm{tr}\,SA = \mathrm{tr}\,(M^{-1}KM)(M^{-1}LM) = \mathrm{tr}\,KL = 0,$$

since KL is the direct product of two traceless 2×2 matrices. Hence, from Eq. (26) we see that

$$\mathrm{tr}\,F^2 = 4(\mathbf{F}^2 + \mathbf{F}^{*2}) = 8(\mathbf{E}^2 - \mathbf{B}^2) \qquad (27)$$

[which also follows directly from Eq. (3)] and

$$T = -\frac{1}{2\pi}SA. \qquad (28)$$

An important, nontrivial consequence of this last equation is that, by virtue of Eq. (8),

$$T^2 = \frac{1}{4\pi^2}S^2A^2 = \frac{1}{4\pi^2}[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2]\,I$$

or, equivalently,

$$T^{\alpha}{}_{\gamma}T^{\gamma}{}_{\beta} = \frac{1}{4\pi^2}[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2]\,\delta^{\alpha}_{\beta}. \qquad (29)$$

With the aid of Eq. (29) one can prove that the electromagnetic field satisfies the so-called *dominant energy condition*. For an observer whose four-velocity is V^{α}, the energy flux four-vector of the electromagnetic field is $T^{\alpha}{}_{\beta}V^{\beta}$. This four-vector is nonspacelike since, using the symmetry of $T_{\alpha\beta}$ and Eq. (29),

$$\begin{aligned} (T^{\alpha}{}_{\beta}V^{\beta})(T_{\alpha\gamma}V^{\gamma}) &= T^{\beta}{}_{\alpha}T^{\alpha}{}_{\gamma}V_{\beta}V^{\gamma} \\ &= \frac{1}{4\pi^2}[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2]V_{\beta}V^{\beta} \leqslant 0. \end{aligned}$$

As another application of Eq. (29) we can prove that the energy-momentum tensor of the electromagnetic field is of the form $T_{\alpha\beta} = k_{\alpha}k_{\beta}$, for some four-vector k_{α} only if both invariants of the electromagnetic field vanish. In effect, the fact that the trace of $T_{\alpha\beta}$ is equal to zero amounts to $k^{\alpha}k_{\alpha} = 0$; hence, $T^{\alpha}{}_{\gamma}T^{\gamma}{}_{\beta} = k^{\alpha}k_{\gamma}k^{\gamma}k_{\beta} = 0$ and from Eq. (29) it follows that $\mathbf{E}^2 - \mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$ must vanish simultaneously.

3.2. Orthogonal transformations

As pointed out in Refs. 1 to 5, the exponential appearing in Eq. (4) is analogous to the expression of a Lorentz transformation. As we show explicitly below, the exponential in Eq. (4) is a Lorentz transformation. In fact, the four-velocity U^{α} of a particle must satisfy the condition $U^{\alpha}U_{\alpha} = -c^2$ and the solution $U^{\alpha}(\tau)$ to the system of linear equations (1) must depend linearly of the initial condition $U^{\alpha}(0)$ (even if the $F_{\alpha\beta}$ are not constant). Since the Lorentz transformations are the only linear transformations of the Minkowski space into itself that leave invariant the products $A^{\alpha}A_{\alpha}$, $U^{\alpha}(\tau)$ must be related to $U^{\alpha}(0)$ by means of a Lorentz transformation.

Let V be a real four-dimensional vector space, with a non-singular metric tensor, $g_{\alpha\beta}$ (that is, $\det(g_{\alpha\beta}) \neq 0$). In the case of the Minkowski space-time, with respect to an appropriate basis,

$$(g_{\alpha\beta}) = \mathrm{diag}\,(-1,1,1,1) \qquad (30)$$

(or its negative, depending on the conventions adopted), but we can also consider, *e.g.*, $(g_{\alpha\beta}) = \mathrm{diag}(1,1,1,1)$, corresponding to the Euclidean four-dimensional space, and $(g_{\alpha\beta}) = \mathrm{diag}\,(-1,-1,1,1)$.

The orthogonal transformations are those linear transformations from V onto itself that leave the metric tensor invariant; in the case of the metric tensor (30) the orthogonal transformations are just the homogeneous Lorentz transformations. If the 4×4 matrix $(a^{\alpha}{}_{\beta})$ is an orthogonal transformation, then

$$a^{\alpha}{}_{\mu}a^{\beta}{}_{\nu}g_{\alpha\beta} = g_{\mu\nu}, \qquad (31)$$

from which one can prove that the orthogonal transformations form a group under composition.

If $A(s)$ is an orthogonal matrix depending on a real parameter, s, in such a way that

$$A(t+s) = A(t)A(s), \qquad (32)$$

then, differentiating with respect to t at $t = 0$ we obtain $A'(s) = A'(0)A(s)$, or, defining $F \equiv A'(0)$

$$A'(s) = FA(s). \qquad (33)$$

From Eq. (32) it follows that $A(0) = I$, and therefore the solution to Eq. (33) is given by

$$A(s) = (\exp sF)A(0) = \exp sF. \qquad (34)$$

On the other hand, denoting by $a^{\alpha}{}_{\beta}(s)$ and $F^{\alpha}{}_{\beta}$ the entries of the matrices $A(s)$ and F, respectively, from Eq. (31) we have $a^{\alpha}{}_{\mu}(s)a^{\beta}{}_{\nu}(s)g_{\alpha\beta} = g_{\mu\nu}$. Differentiating with respect to s at $s=0$, taking into account that $A(0) = I$ and $A'(0) = F$, we get

$$F^{\alpha}{}_{\mu}g_{\alpha\nu} + F^{\beta}{}_{\nu}g_{\mu\beta} = 0$$

or, equivalently, with the standard definition $F_{\alpha\beta} \equiv g_{\alpha\gamma}F^{\gamma}{}_{\beta}$,

$$F_{\nu\mu} + F_{\mu\nu} = 0, \tag{35}$$

i.e., $F_{\mu\nu}$ is antisymmetric, as in the case of the electromagnetic field tensor. Thus, the exponential appearing in Eq. (4) is an orthogonal transformation (*i.e.*, a Lorentz transformation) and from Eq. (34), by suitably choosing F and the value of s, one can obtain Lorentz transformations. In fact, any proper orthochronous Lorentz transformation is of this form (see, *e.g.*, Ref. 9).

3.2.1. *Orthogonal transformations of the Euclidean space*

In the case where the metric tensor has the form

$$(g_{\alpha\beta}) = \operatorname{diag}(1,1,1,1),$$

we have $F^{\alpha}{}_{\beta} = F_{\alpha\beta}$ and by virtue of Eq. (35) the matrix $F = (F^{\alpha}{}_{\beta})$ is of the form

$$\begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & b_3 & -b_2 \\ -e_2 & -b_3 & 0 & b_1 \\ -e_3 & b_2 & -b_1 & 0 \end{pmatrix},$$

where e_1, e_2, e_3 and b_1, b_2, b_3 are six arbitrary real numbers [*cf.* Eq. (3)]. Now ${}^*({}^*t_{\alpha\beta}) = t_{\alpha\beta}$, therefore the eigenvalues of the duality operator are ± 1, and the self-dual and anti-self-dual parts of $F_{\alpha\beta}$ are defined by $S_{\alpha\beta} \equiv F_{\alpha\beta} + {}^*F_{\alpha\beta}$, and $A_{\alpha\beta} \equiv F_{\alpha\beta} - {}^*F_{\alpha\beta}$, respectively, so that ${}^*S_{\alpha\beta} = S_{\alpha\beta}$, and ${}^*A_{\alpha\beta} = -A_{\alpha\beta}$.

Thus

$$F = \frac{1}{2}(S + A), \tag{36}$$

where,

$$S = \begin{pmatrix} 0 & g_1 & g_2 & g_3 \\ -g_1 & 0 & g_3 & -g_2 \\ -g_2 & -g_3 & 0 & g_1 \\ -g_3 & g_2 & -g_1 & 0 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 0 & \tilde{g}_1 & \tilde{g}_2 & \tilde{g}_3 \\ -\tilde{g}_1 & 0 & -\tilde{g}_3 & \tilde{g}_2 \\ -\tilde{g}_2 & \tilde{g}_3 & 0 & -\tilde{g}_1 \\ -\tilde{g}_3 & -\tilde{g}_2 & \tilde{g}_1 & 0 \end{pmatrix},$$

with the definitions

$$\mathbf{g} \equiv \mathbf{e} + \mathbf{b}, \quad \tilde{\mathbf{g}} \equiv \mathbf{e} - \mathbf{b}.$$

By means of a straightforward computation one finds that, as in the previous case, S commutes with A, and the squares of S and A are proportional to the identity matrix

$$SA = AS, \quad S^2 = -\mathbf{g}^2 I, \quad A^2 = -\tilde{\mathbf{g}}^2 I \tag{37}$$

[*cf.* Eqs. (8)], but now S and A are two independent real antisymmetric matrices, $\mathbf{g}^2$ and $\tilde{\mathbf{g}}^2$ are always real and non-negative (see the discussion below).

From Eqs. (37) we have, *e.g.*,

$$S^{2n} = (-1)^n |\mathbf{g}|^{2n} I, \quad S^{2n+1} = (-1)^n |\mathbf{g}|^{2n} S,$$

for $n = 0, 1, 2, \ldots$. Thus,

$$\exp tF = \left(\exp \frac{1}{2}tS\right)\left(\exp \frac{1}{2}tA\right)$$

and

$$\exp \frac{1}{2}tS = \left(\cos \frac{t}{2}|\mathbf{g}|\right) I + \frac{1}{|\mathbf{g}|}\left(\sin \frac{t}{2}|\mathbf{g}|\right) S$$

[*cf.* Eq. (11)], with an analogous expression for $\exp(1/2)tA$, replacing $|\mathbf{g}|$ by $|\tilde{\mathbf{g}}|$. Hence

$$\begin{aligned} \exp tF = {} & \left(\cos \frac{t}{2}|\mathbf{g}|\right)\left(\cos \frac{t}{2}|\tilde{\mathbf{g}}|\right) I \\ & + \frac{1}{|\mathbf{g}|}\left(\sin \frac{t}{2}|\mathbf{g}|\right)\left(\cos \frac{t}{2}|\tilde{\mathbf{g}}|\right) S \\ & + \frac{1}{|\tilde{\mathbf{g}}|}\left(\sin \frac{t}{2}|\tilde{\mathbf{g}}|\right)\left(\cos \frac{t}{2}|\mathbf{g}|\right) A \\ & + \frac{1}{|\mathbf{g}||\tilde{\mathbf{g}}|}\left(\sin \frac{t}{2}|\mathbf{g}|\right)\left(\sin \frac{t}{2}|\tilde{\mathbf{g}}|\right) SA. \end{aligned}$$

Making use of Eqs. (36) and (37), this last expression can be written in terms of I, S, A, and F^2.

As in the case where the metric tensor is given by Eq. (30), one can readily see that relations (37) trivially follow from the expression of S and A in an appropriate basis. In fact, with the aid of the unitary 4×4 matrix

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} -\mathrm{i} & 0 & 0 & 1 \\ 0 & 1 & -\mathrm{i} & 0 \\ 0 & 1 & \mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 & -1 \end{pmatrix}$$

one obtains

$$\begin{aligned} MFM^{-1} = {} & \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -\mathrm{i}g_3 & -\mathrm{i}g_1 + g_2 \\ -\mathrm{i}g_1 - g_2 & \mathrm{i}g_3 \end{pmatrix} \\ & + \frac{1}{2}\begin{pmatrix} -\mathrm{i}\tilde{g}_3 & -\mathrm{i}\tilde{g}_1 - \tilde{g}_2 \\ -\mathrm{i}\tilde{g}_1 + \tilde{g}_2 & \mathrm{i}\tilde{g}_3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The first term corresponds to S and the second one to A.

4. Conclusions

By contrast with the approaches followed in other works, the procedure employed in this paper allows us to solve the equations of motion (2) in a straightforward manner and to derive certain algebraic relations [notably Eq. (29)] that do not seem obtainable with other elementary methods in such a simple way. As we have shown, the fundamental relations established in order to solve Eq. (2) have a wider applicability and analogs that enable us to readily find the exponentials of the generators of all the orthogonal groups in four dimensions.

Acknowledgment

One of the authors (C.S.S.) wishes to thank the Vicerrectoría de Investigación y Estudios de Posgrado of the Universidad Autónoma de Puebla for financial support through the programme “Jóvenes Investigadores.” The authors would like to thank the referee and José Luis López Bonilla for helpful comments and for pointing out Ref. 1 to them.

1. A.H. Taub, *Phys. Rev.* **73** (1948) 786.
2. A.T. Hyman, *Am. J. Phys.* **65** (1997) 195.
3. G. Muñoz, *Am. J. Phys.* **65** (1997) 429.
4. J. Fredsted, *J. Math. Phys.* **42** (2001) 4497.
5. S.A. Chin, *J. Math. Phys.* **50** (2009) 012904.
6. J.D. Jackson, *Classical Electrodymanics*, 2nd ed., (Wiley, New York, 1995).
7. M. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, (Academic Press, New York, 1974).
8. K.B. Wolf, *Rev. Mex. Fís.* **49** (2003) 465.
9. G.F. Torres del Castillo, *Spinors in Four-Dimensional Spaces*, (Birkhäuser, Boston, 2010).
10. G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists*, 6th ed., (Elsevier Academic Press, Amsterdam, 2005), Chap. 3.