Recursive parametrization of quark flavour mixing matrices

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We examine quark flavour mixing matrices for three and four generations using the recursive parametrization of $U(n)$ and $SU(n)$ matrices developed earlier. After a brief summary of the recursive parametrization, we obtain expressions for the independent rephasing invariants and also the constraints on them that arise from the requirement of mod symmetry of the flavour mixing matrix.

Keywords: Quark mixing matrix; recursive parametrization.

Las matrices de mezcla de sabor de quarks para tres y cuatro generaciones son examinadas usando la parametrización recursiva de las matrices $U(n)$ y $SU(n)$ desarrolladas con anterioridad. Despues de un breve resumen de la parametrización recursiva, se obtienen las expresiones para los invariantes independientes de un cambio de fase y las restricciones sobre estos que se producen del requerimiento de que la matriz de mezcla de sabor sea modulo simétrica.

Descriptores: Matriz de mezcla de quarks; parametrización recursiva.

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1. Introduction

It has been over thirty-five years since the first explicit parametrization for the six-quark case was given in Ref. 1 for the so-called Cabibbo-Kobayashi-Maskawa (CKM) matrix. Since then many different parametrization have been suggested [2-14][i]. Still there is no deep understanding of the observed quark mixing.

Study of flavor mixing in weak interactions provides a low energy window for new physics. Currently, experiments are underway at Belle and BaBar to check the "unitarity triangle" for the 3×3 flavor mixing matrix, as accurately as possible. If there is a significant deviation then it would be a signal for the existence of more than three generations. Furthermore, the 3×3 CKM mixing matrix contains only one CP-violating phase thus implying that CP-violations in different processes are related. Again, the violation of any one of these relations would be a signal for more generations. Consequently, in this paper we study some general properties of a 4×4 flavor mixing matrix. Such a matrix in general has six angles and three phases. However, a moduli symmetric 4×4 unitary matrix has fewer parameters. We study such a matrix in detail and present parametrizations which would be useful for confrontation with experiments in the future.

In Sec. 2, rephasing invariants for a $n \times n$ unitary matrices are defined. In addition, relations between plaquettes for the particular cases $n = 3$ and 4 are given. In Sec. 3, recursive parametrization [15,16] for the $n \times n$ case is given together with that for $n = 2$, 3, and 4. Rephasing invariants in the recursive parametrization are presented in Sec. 4. A moduli symmetric unitary matrix has fewer parameters and the results for $n = 3$ and 4 are given in Sec. 5. In Sec. 6 the standard PDG parametrization [17] is obtained using the recursive approach. The conclusions are presented in Sec. 7.

2. Rephasing invariants of $U(n)$ matrices

We begin by recalling some known facts about the group $U(n)$ of $n \times n$ unitary matrices $V = (V)_{jk}$, $(j, k = 1, \ldots, n)$. This group is of real dimension n^2, *i.e.*, we need n^2 real independent parameters or coordinates to label the elements of V. Of these, some may be taken to be of the modulus type and the rest to be of the phase type. The number of independent modulus type parameters, chosen from out of all the n^2 moduli $|V_{jk}|$ is $n(n-1)/2$. The number of independent phase type parameters chosen from out of all the n^2 phases $\arg(V_{jk})$, is $n(n+1)/2$. The reduction from n^2 to $n(n-1)/2$ and $n(n+1)/2$ in the two cases respectively is the result of the unitarity conditions on the matrix elements V_{jk}. The recursive parametrisation of $U(n)$ taken from [15] and which we describe in Sec. 3 makes it particularly easy to arrive at these numbers of independent parameters of each type.

The rephasing transformation is the multiplication of a general $V \in U(n)$ by independent diagonal matrices of $U(n)$ on the left and on the right is expressed as:

$$V \to V' = D(\theta')VD(\theta), \tag{1}$$

with

$$D(\theta') = \mathrm{diag}\{e^{i\theta'_1}, \ldots, e^{i\theta'_n}\}$$
$$\text{and} \quad D(\theta) = \mathrm{diag}\{e^{i\theta_1}, \ldots, e^{i\theta_n}\}. \tag{2}$$

Of course all the n^2 (non independent) moduli $|V_{jk}|$ are invariant under this transformation and so also the $n(n-1)/2$ independent modulus parameters formed out of them (these are essentially the Cabibbo angles). On the other hand, out of the $n(n+1)/2$ independent phase type parameters of $U(n)$, only $(n-1)(n-2)/2$ suitably formed combinations are rephasing invariants (these are essentially the Kobayashi-Maskawa angles). These can be obtained in more than one way, one of which is to use "plaquettes" or (the phase of) Bargmann invariants (BI's), as outlined below.

To avoid possible ambiguities, it is worth repeating that all the $n(n-1)/2$ independent modulus type coordinates of $U(n)$ remain rephasing invariant, while out of the $n(n+1)/2$ independent phase type coordinates only $(n-1)(n-2)/2$ survive as rephasing invariant.

$$\Delta_{j\ell km} \equiv V_{jk}V^*_{\ell k}V_{\ell m}V^*_{jm} \quad (j<\ell,\ k<m). \tag{3}$$

The number of such BI's is clearly $n^2(n^2-1)/4$ and each of them is rephasing invariant. It is from the phases arg$(\Delta_{j\ell km})$ that we essentially pick up the expected number $(n-1)(n-2)/2$ of independent ones.

Simple algebra shows that all $n^2(n-1)^2/4$ BI's in Eq. (3) can be expressed in terms of just $(n-1)^2$ "elementary" BI's of the form

$$\Delta_{jk} \equiv \Delta_{j,j+1,k,k+1} \quad (j,k = 1,2,\ldots n-1), \tag{4}$$

apart from modulus type factors. Therefore in searching for independent rephasing invariants phases, as a first step we can limit ourselves to arg(Δ_{jk}), $(n-1)^2$ in number.

To proceed further, the restrictions coming from the unitarity of V have to be analysed. As shown in Ref. 15 this is a somewhat intricate analysis and leads to the result that the independent rephasing invariant phases are the phases of the $(n-1)(n-2)/2$ primitive BI's Δ_{jk} for $j<k\le n-1$. Thus one has the recursive reduction in numbers:

$$\begin{array}{rl} n^2(n-1)^2/4 & \mathrm{BI's}: \Delta_{j\ell km}. \quad \xrightarrow{\text{algebraic simplification}} \\ (n-1)^2 & \mathrm{Elementary\ BI's}: \Delta_{jk}. \quad \xrightarrow{\text{unitarity simplification}} \\ (n-1)(n-2)/2 & \mathrm{Primitive\ BI's}: \Delta_{jk} \ \ j<k\le n-1. \end{array} \tag{5}$$

At the conclusion of this analysis then, the independent $(n-1)(n-2)/2$ rephasing invariant phases are arg(Δ_{jk}) for $j<k\le n-1$.

For $n=3$ and $n=4$, the reductions and relations are explicitly given below.

2.1. Relations between plaquettes for $n=3$

The number of independent modulus type invariant parameters is 3 and of phase type is 1. Here there is only one primitive, *viz.*, Δ_{12}. Orthogonality of the rows of V gives the relations,

$$\Delta_{i1} = -|V_{i2}|^2|V_{i+12}|^2 - \Delta_{i2}. \tag{6}$$

Likewise the orthogonality of the columns gives,

$$\Delta_{1i} = -|V_{2i}|^2|V_{2i+1}|^2 - \Delta_{2i}. \tag{7}$$

These are four inhomogeneous equations for four quantities Δ_{11}, Δ_{21}, Δ_{22}, and Δ_{12}. One of them is derivable from the other three leaving us with three equations which allow us to solve for Δ_{11}, Δ_{21}, and Δ_{22} in terms of Δ_{12}:

$$\begin{aligned} \Delta_{11} &= -|V_{12}|^2|V_{22}|^2 - \Delta^*_{12}, \\ \Delta_{22} &= -|V_{22}|^2|V_{23}|^2 - \Delta^*_{12}, \\ \Delta_{21} &= |V_{22}|^2(|V_{23}|^2 - |V_{32}|^2) + \Delta_{12} \end{aligned} \tag{8}$$

(for ease in counting, we have used 4 modulus type parameters here; there is no conflict with the fact that there are only three independent parameters of this type, as by unitarity $|V_{32}|$ can be expresed in terms of $|V_{12}|$ and $|V_{22}|$).

Any other plaquette, *e.g.*, Δ_{1213}, can be expressed as $\Delta_{11}\Delta_{12}/|V_{12}|^2|V_{22}|^2$ and, using the relations above, as $-|V_{13}|^2|V_{23}|^2 - \Delta_{12}$.

As is well known, these relations have the consequence that the imaginary parts of all the plaquettes are the same, up to a sign. Furthermore, if even one V_{ij}, say V_{11}, vanishes, then all the plaquettes become real. It is also evident that, imposing mod symmetry on V, *i.e.*, requiring $|V_{ij}| = |V_{ji}|$, while reducing the number of independent modulus type invariants from three to two, has no effect on the number of independent phase type invariants.

2.2. Relations between plaquettes for $n = 4$

In this case row and column orthogonality of V respectively yield:

$$|V_{i3}|^2|V_{i+13}|^2(\Delta_{i1}+\Delta^*_{i2}) = -\Delta^*_{i2}(\Delta_{i2}+\Delta^*_{i3}) \quad (i=1,2,3), \tag{9}$$

$$|V_{3i}|^2|V_{3i+1}|^2(\Delta_{1i}+\Delta^*_{2i}) = -\Delta^*_{2i}(\Delta_{2i}+\Delta^*_{3i}) \quad (i=1,2,3), \tag{10}$$

which may alternatively be written as

$$|V_{i2}|^2|V_{i+12}|^2(\Delta_{i3}+\Delta^*_{i2}) = -\Delta^*_{i2}(\Delta_{i2}+\Delta^*_{i1}) \quad (i=1,2,3), \tag{11}$$

$$|V_{2i}|^2|V_{2i+1}|^2(\Delta_{3i}+\Delta^*_{2i}) = -\Delta^*_{2i}(\Delta_{2i}+\Delta^*_{1i}) \quad (i=1,2,3). \tag{12}$$

Choosing Eqs. (9) and (12) we have:

$$\begin{aligned}
|V_{13}|^2|V_{23}|^2(\Delta_{11}+\Delta^*_{12}) &= -\Delta^*_{12}(\Delta_{12}+\Delta^*_{13}),\\
|V_{23}|^2|V_{33}|^2(\Delta_{21}+\Delta^*_{22}) &= -\Delta^*_{22}(\Delta_{22}+\Delta^*_{23}),\\
|V_{33}|^2|V_{43}|^2(\Delta_{31}+\Delta^*_{32}) &= -\Delta^*_{32}(\Delta_{32}+\Delta^*_{33}),\\
|V_{21}|^2|V_{22}|^2(\Delta_{31}+\Delta^*_{21}) &= -\Delta^*_{21}(\Delta_{21}+\Delta^*_{11}),\\
|V_{22}|^2|V_{23}|^2(\Delta_{32}+\Delta^*_{22}) &= -\Delta^*_{22}(\Delta_{22}+\Delta^*_{12}),\\
|V_{23}|^2|V_{24}|^2(\Delta_{33}+\Delta^*_{23}) &= -\Delta^*_{23}(\Delta_{23}+\Delta^*_{13})
\end{aligned} \tag{13}$$

(for ease in writing, here too we have used 8 modulus type parameters, but using unitarity both $|V_{43}|$ and $|V_{24}|$ can be expressed in terms of the other 6).

These six equations for the nine plaquettes allow us to solve all of them in terms of the primitives which we choose to be Δ_{12}, Δ_{13}, and Δ_{23}. The relevant equations are:

$$\Delta_{11} = -|V_{12}|^2|V_{22}|^2 - \Delta^*_{12}\left(1+\frac{\Delta^*_{13}}{|V_{13}|^2|V_{23}|^2}\right) \tag{14}$$

$$\Delta_{33} = -|V_{33}|^2|V_{34}|^2 - \Delta^*_{23}\left(1+\frac{\Delta^*_{13}}{|V_{23}|^2|V_{24}|^2}\right) \tag{15}$$

$$\left(1+\frac{\Delta_{11}}{|V_{21}|^2|V_{22}|^2}\right)\Delta_{21} - \left(1+\frac{\Delta_{33}}{|V_{33}|^2|V_{43}|^2}\right)\Delta_{32} = |V_{32}|^2(|V_{42}|^2-|V_{31}|^2) \tag{16}$$

$$\begin{aligned}
&\left(1+\frac{\Delta^*_{12}}{|V_{22}|^2|V_{23}|^2}\right)\Delta_{21} - \left(1+\frac{\Delta^*_{23}}{|V_{23}|^2|V_{33}|^2}\right)\Delta_{32}\\
&= |V_{32}|^2|V_{33}|^2\left(1+\frac{\Delta^*_{23}}{|V_{23}|^2|V_{33}|^2}\right)\\
&- |V_{22}|^2|V_{32}|^2\left(1+\frac{\Delta^*_{12}}{|V_{22}|^2|V_{23}|^2}\right)
\end{aligned} \tag{17}$$

$$\Delta_{31} = -|V_{31}|^2|V_{32}|^2 - \Delta^*_{21}\left(1+\frac{\Delta^*_{11}}{|V_{21}|^2|V_{22}|^2}\right) \tag{18}$$

$$\Delta_{22} = \frac{\Delta^*_{21}+|V_{22}|^2|V_{32}|^2}{\left(1+\dfrac{\Delta_{23}}{|V_{23}|^2|V_{33}|^2}\right)}. \tag{19}$$

3. Recursive parametrization of $U(n)$ ($SU(n)$) matrices

Let $U(n)$ denote the group of unitary matrices acting on all n dimensions. For $m = 1, 2, \ldots, n-1$, we will denote by $U(m)$ the unitary group acting on the first m dimensions, leaving the dimensions $m+1, \ldots, n$, unaffected. Then we have the canonical subgroup chain

$$U(1) \subset U(2) \subset \cdots \subset U(n-1) \subset U(n). \tag{20}$$

General matrices of $U(n), U(n-1), \ldots$ will be written as $\mathcal{A}_n, \mathcal{A}_{n-1}, \ldots$, respectively. In a matrix $\mathcal{A}_m \in U(m)$ the last rows and columns are trivial, with ones along the diagonals and zeros elsewhere (when no confusion is likely to arise, $\mathcal{A}_m$ will also denote an unbordered $m \times m$ unitary matrix).

It was shown in Ref. 15 that (except for a set of measure zero) any matrix $\mathcal{A}_n \in U(n)$ can be expressed uniquely as an n-fold product

$$\begin{aligned}
\mathcal{A}_n &= \mathcal{A}_n(\boldsymbol{\zeta})\mathcal{A}_{n-1}(\boldsymbol{\eta})\\
&\times \mathcal{A}_{n-2}(\boldsymbol{\xi})\cdots\mathcal{A}_4(\boldsymbol{\gamma})\mathcal{A}_3(\boldsymbol{\beta})\mathcal{A}_2(\boldsymbol{\alpha})\mathcal{A}_1(\chi),
\end{aligned} \tag{21}$$

where $\mathcal{A}_n(\boldsymbol{\zeta})$ is a special $U(n)$ element determined by an n-component complex unit vector $\boldsymbol{\zeta}$, $\mathcal{A}_{n-1}(\boldsymbol{\eta})$ is a special $U(n-1)$ element determined by an $n-1$-component complex unit vector $\boldsymbol{\eta}$, and so on down to $\mathcal{A}_2(\boldsymbol{\alpha})$ that is a special $U(2)$ element determined by a two-component complex unit vector $\boldsymbol{\alpha}$, and $\mathcal{A}_1(\chi)$ is a phase factor belonging to $U(1)$. The complex unit vectors $\{\boldsymbol{\zeta}, \boldsymbol{\eta}, \ldots\}$, appear as the last columns of the (unbordered) matrices $\{\mathcal{A}_n(\boldsymbol{\zeta}), \mathcal{A}_{n-1}(\boldsymbol{\eta}), \ldots\}$ and can be identified with the labels of the cosets $\{U(n)/U(n-1), U(n-1)/U(n-2), \ldots\}$. Remembering that $\{\boldsymbol{\zeta}, \boldsymbol{\eta}, \ldots\}$ are complex unit vectors of dimensions $\{n, n-1, \ldots\}$, it is easily seen that the number of real independent parameters add up to n^2 as they should.

The same considerations as above apply to $SU(n)$ matrices as well. Denoting by $A_n(\boldsymbol{\zeta})$ the corresponding matrices in $SU(n)$, except for a set of measure zero, any $A_n \in SU(n)$ can be decomposed as

$$A_n = A_n(\boldsymbol{\zeta})A_{n-1}(\boldsymbol{\eta})A_{n-2}(\boldsymbol{\xi})\cdots A_4(\boldsymbol{\gamma})A_3(\boldsymbol{\beta})A_2(\boldsymbol{\alpha}) \tag{22}$$

The above construction fixes only the last column of the unitary matrix $\mathcal{A}_n(\boldsymbol{\zeta})$ as $\boldsymbol{\zeta}$, and one has a great deal of freedom in arranging the remaining $n-1$ columns leading to many explicit forms for these matrices. In this work we consider two explicit forms which correspond to those discussed in Refs. 15 and 16 respectively.

The explicit expressions for the nonzero matrix elements of $\mathcal{A}_n(\boldsymbol{\zeta})$ considered in Ref. 15 are

$$\begin{aligned}
\mathcal{A}_n(\boldsymbol{\zeta}) &= (a_{jk}(\boldsymbol{\zeta})) \in U(n); \quad j,k=1,2,\ldots,n.\\
a_{jn}(\boldsymbol{\zeta}) &= \zeta_j; \quad j=1,2,\ldots,n.\\
a_{jj-1}(\boldsymbol{\zeta}) &= \rho_{j-1}/\rho_j; \quad j=2,3,\ldots,n;\\
&\qquad \rho_j = \sqrt{1-|\zeta_{j+1}|^2-|\zeta_{j+2}|^2-\cdots-|\zeta_n|^2} = \sqrt{|\zeta_1|^2+\cdots+|\zeta_j|^2}.\\
a_{jk}(\boldsymbol{\zeta}) &= -\zeta_j\zeta_{k+1}^*/\rho_k\rho_{k+1}; \quad j\le k\le n-1.
\end{aligned} \tag{23}$$

Thus, for instance, for $n=2,3,4$ we have:

$$\mathcal{A}_2(\boldsymbol{\alpha})=\begin{pmatrix} -\alpha_2^*\alpha_1/\mu_1 & \alpha_1\\ \mu_1 & \alpha_2\end{pmatrix}, \qquad \mu_1=|\alpha_1|. \tag{24}$$

$$\mathcal{A}_3(\boldsymbol{\beta})=\begin{pmatrix} -\beta_2^*\beta_1/\sigma_1\sigma_2 & -\beta_3^*\beta_1/\sigma_2 & \beta_1\\ \sigma_1/\sigma_2 & -\beta_3^*\beta_2/\sigma_2 & \beta_2\\ 0 & \sigma_2 & \beta_3\end{pmatrix},$$

$$\sigma_1=|\beta_1|, \quad \sigma_2=\sqrt{|\beta_1|^2+|\beta_2|^2}. \tag{25}$$

$$\mathcal{A}_4(\boldsymbol{\gamma})=\begin{pmatrix} -\gamma_2^*\gamma_1/\rho_1\rho_2 & -\gamma_3^*\gamma_1/\rho_2\rho_3 & -\gamma_4^*\gamma_1/\rho_3 & \gamma_1\\ \rho_1/\rho_2 & -\gamma_3^*\gamma_2/\rho_2\rho_3 & -\gamma_4^*\gamma_2/\rho_3 & \gamma_2\\ 0 & \rho_2/\rho_3 & -\gamma_4^*\gamma_3/\rho_3 & \gamma_3\\ 0 & 0 & \rho_3 & \gamma_4\end{pmatrix},$$

$$\rho_1=|\gamma_1|, \quad \rho_2=\sqrt{|\gamma_1|^2+\gamma_2|^2}, \quad \rho_3=\sqrt{|\gamma_1|^2+|\gamma_2|^2+|\gamma_3|^2}. \tag{26}$$

The determinant of the matrices $\mathcal{A}_n(\boldsymbol{\zeta})$ turns out to be $(-1)^{n-1}\zeta_1/|\zeta_1|$ and hence the corresponding $SU(n)$ matrices can be obtained by multiplying, for instance, the first column by $(-1)^{n-1}\zeta_1^*/|\zeta_1|$. Thus for $n=2,3,4$ we have

$$A_2(\boldsymbol{\alpha})=\begin{pmatrix} \alpha_2^* & \alpha_1\\ -\alpha_1^* & \alpha_2\end{pmatrix}, \tag{27}$$

$$A_3(\boldsymbol{\beta})=\begin{pmatrix} -\beta_2^*/\sigma_2 & -\beta_3^*\beta_1/\sigma_2 & \beta_1\\ \beta_1^*/\sigma_2 & -\beta_3^*\beta_2/\sigma_2 & \beta_2\\ 0 & \sigma_2 & \beta_3\end{pmatrix}, \tag{28}$$

$$A_4(\boldsymbol{\gamma})=\begin{pmatrix} \gamma_2^*/\rho_2 & -\gamma_3^*\gamma_1/\rho_2\rho_3 & -\gamma_4^*\gamma_1/\rho_3 & \gamma_1\\ -\gamma_1^*/\rho_2 & -\gamma_3^*\gamma_2/\rho_2\rho_3 & -\gamma_4^*\gamma_2/\rho_3 & \gamma_2\\ 0 & \rho_2/\rho_3 & -\gamma_4^*\gamma_3/\rho_3 & \gamma_3\\ 0 & 0 & \rho_3 & \gamma_4\end{pmatrix}. \tag{29}$$

This parametrization assumes that ζ_1 is nonzero. As a result, in the extreme case when $\boldsymbol{\zeta}=(1,0,\ldots,0)$, the matrix $\mathcal{A}_n(\boldsymbol{\zeta})$ does not reduce to the identity matrix. A parametrization where this does happen and which corresponds to that given in Ref. 16 is given below:

$$\begin{aligned}
\mathcal{A}_n(\boldsymbol{\zeta}) &= (a_{jk}(\boldsymbol{\zeta})) \in U(n); \quad j,k=1,2,\ldots,n.\\
a_{jn}(\boldsymbol{\zeta}) &= \zeta_j; \quad j=1,2,\ldots,n.\\
a_{jj}(\boldsymbol{\zeta}) &= \rho_j/\rho_{j-1}; \quad j=1,2,\ldots,n-1; \quad \rho_0=1;\\
&\qquad \rho_j = \sqrt{1-|\zeta_1|^2-|\zeta_2|^2-\cdots-|\zeta_j|^2} = \sqrt{|\zeta_{j+1}|^2+\cdots|\zeta_n|^2}.\\
a_{jk}(\boldsymbol{\zeta}) &= -\zeta_j\zeta_k^*/(\rho_{k-1}\rho_k); \quad j>k, \quad k=1,2,\ldots,n-1.
\end{aligned} \tag{30}$$

Note that we are using the same symbols as in the parametrization earlier though with different meanings. For $n=2,3,4$, we explicitly have:

$$\mathcal{A}_2(\boldsymbol{\alpha}) = \begin{pmatrix} \mu_1 & \alpha_1 \\ -\alpha_1^*\alpha_2/\mu_1 & \alpha_2 \end{pmatrix}; \quad \mu_1 = |\alpha_2|. \tag{31}$$

$$\mathcal{A}_3(\boldsymbol{\beta}) = \begin{pmatrix} \sigma_1 & 0 & \beta_1 \\ -\beta_1^*\beta_2/\sigma_1 & \sigma_2/\sigma_1 & \beta_2 \\ -\beta_1^*\beta_3/\sigma_1 & -\beta_2^*\beta_3/\sigma_1\sigma_2 & \beta_3 \end{pmatrix}; \quad \sigma_1 = \sqrt{|\beta_2|^2+|\beta_3|^2}, \quad \sigma_2 = |\beta_3|. \tag{32}$$

$$\mathcal{A}_4(\boldsymbol{\gamma}) = \begin{pmatrix} \rho_1 & 0 & 0 & \gamma_1 \\ -\gamma_1^*\gamma_2/\rho_1 & \rho_2/\rho_1 & 0 & \gamma_2 \\ -\gamma_1^*\gamma_3/\rho_1 & -\gamma_2^*\gamma_3/\rho_1\rho_2 & \rho_3/\rho_2 & \gamma_3 \\ -\gamma_1^*\gamma_4/\rho_1 & -\gamma_2^*\gamma_4/\rho_1\rho_2 & -\gamma_3^*\gamma_4/\rho_2\rho_3 & \gamma_4 \end{pmatrix};$$

$$\rho_1 = \sqrt{|\gamma_2|^2+|\gamma_3|^2+|\gamma_4|^2}, \quad \rho_2 = \sqrt{|\gamma_3|^2+|\gamma_4|^2}, \quad \rho_3 = |\gamma_4|. \tag{33}$$

The determinant of the matrices $\mathcal{A}_n(\boldsymbol{\zeta})$ is $\zeta_n/|\zeta_n|$. We can convert the above matrices to $SU(n)$ matrices by multiplying, say the $(n-1)$-th column by $\zeta_n^*/|\zeta_n|$. Thus, for $n = 2, 3, 4$, we have:

$$A_2(\boldsymbol{\alpha}) = \begin{pmatrix} \alpha_2^* & \alpha_1 \\ -\alpha_1^* & \alpha_2 \end{pmatrix}, \tag{34}$$

$$A_3(\boldsymbol{\beta}) = \begin{pmatrix} \sigma_1 & 0 & \beta_1 \\ -\beta_1^*\beta_2/\sigma_1 & \beta_3^*/\sigma_1 & \beta_2 \\ -\beta_1^*\beta_3/\sigma_1 & -\beta_2^*/\sigma_1 & \beta_3 \end{pmatrix}, \tag{35}$$

$$A_4(\boldsymbol{\gamma}) = \begin{pmatrix} \rho_1 & 0 & 0 & \gamma_1 \\ -\gamma_1^*\gamma_2/\rho_1 & \rho_2/\rho_1 & 0 & \gamma_2 \\ -\gamma_1^*\gamma_3/\rho_1 & -\gamma_2^*\gamma_3/\rho_1\rho_2 & \gamma_4^*/\rho_2 & \gamma_3 \\ -\gamma_1^*\gamma_4/\rho_1 & -\gamma_2^*\gamma_4/\rho_1\rho_2 & -\gamma_3^*/\rho_2 & \gamma_4 \end{pmatrix}. \tag{36}$$

Given a matrix $\mathcal{A}_n \in U(n)$, we can determine the parameters, the complex unit vectors, $\{\boldsymbol{\zeta}, \boldsymbol{\eta}, \ldots\}$ in a recursive fashion through the following steps.

- Write $\mathcal{A}_n = (a_{jk}) \in U(n)$ as

$$\mathcal{A}_n = \mathcal{A}_n(\boldsymbol{\zeta})\mathcal{B}_{n-1}, \tag{37}$$

where $\boldsymbol{\zeta}$ is the last column of $\mathcal{A}_n$

$$\zeta_j = a_{jn}. \tag{38}$$

- With $\mathcal{A}_n(\boldsymbol{\zeta})$ thus determined, we have

$$\mathcal{B}_{n-1} = \mathcal{A}_n^\dagger(\boldsymbol{\zeta})\mathcal{A}_n. \tag{39}$$

The matrix elements (b_{ij}), $n-1 \geq i, j \geq 1$, of $\mathcal{B}_{n-1}$ in the first form [15] are given by

$$b_{ij} = \sum_{k=1}^{n} a_{ki}^*(\boldsymbol{\zeta})a_{kj} = -\frac{1}{\rho_i\rho_{i+1}} \times \sum_{k=i+1}^{n} a_{kn}^*(a_{kj}a_{i+1n} - a_{kn}a_{i+1j}) \tag{40}$$

and in the second form [16] by

$$b_{ij} = \sum_{k=1}^{n} a_{ki}^*(\boldsymbol{\zeta})a_{kj} = \frac{1}{\rho_i\rho_{i-1}} \sum_{k=i+1}^{n} a_{kn}^*(a_{ij}a_{kn} - a_{in}a_{kj}). \tag{41}$$

- Write $\mathcal{B}_{n-1}$ as

$$\mathcal{B}_{n-1} = \mathcal{A}_{n-1}(\boldsymbol{\eta})\mathcal{C}_{n-2} \tag{42}$$

with

$$\eta_j = b_{jn-1}. \tag{43}$$

- Repeat the same procedure as above with $\mathcal{C}_{n-2}$.

The same procedure applies to the decomposition of an $SU(n)$ matrix. Thus, for instance, using the second form [16], a matrix $V \in SU(3)$ can be decomposed as

$$V = A_3(\boldsymbol{\beta})A_2(\boldsymbol{\alpha}), \tag{44}$$

where

$$\beta_1 = V_{13}, \quad \beta_2 = V_{23}, \quad \beta_3 = V_{33}, \tag{45}$$

and

$$\alpha_1 = \frac{[V_{23}^*(V_{12}V_{23} - V_{13}V_{22}) + V_{33}^*(V_{12}V_{33} - V_{13}V_{32})]}{\sqrt{|V_{23}|^2+|V_{33}|^2}},$$

$$\alpha_2 = \frac{(V_{33}V_{22} - V_{32}V_{23})}{\sqrt{|V_{23}|^2+|V_{33}|^2}}. \tag{46}$$

4. Rephasing invariants in the recursive parametrization

Having shown how to parametrize a given $U(n)$ ($SU(n)$) matrix in terms of a sequence of complex unit vectors $\{\boldsymbol{\zeta}, \boldsymbol{\eta}, \ldots\}$ of dimensions $\{n, n-1, \ldots\}$, we now examine how these parameters transform under rephasing with the purpose of constructing rephasing invariants out of them. For simplicity and without any loss of generality we will assume that the given matrix V belongs to $SU(n)$ and will consider the cases $n = 3, 4$ and discuss the transformation properties of the parameters in both the forms [15,16] given above.

In the first form [15], any $SU(3)$ can be written as

$$V = A_3(\boldsymbol{\beta})A_2(\boldsymbol{\alpha}) = \begin{pmatrix} -\beta_2^*\alpha_2^*/\sigma_2 + \beta_3^*\beta_1\alpha_1^*/\sigma_2 & -\beta_2^*\alpha_1 - \beta_3^*\beta_1\alpha_2\sigma_2 & \beta_1 \\ \beta_1^*\alpha_2^*/\sigma_2 + \beta_3^*\beta_2\alpha_1^*/\sigma_2 & \beta_1^*\alpha_1/\sigma_2 - \beta_3^*\beta_2\alpha_2/\sigma_2 & \beta_2 \\ -\sigma_2\alpha_1^* & \sigma_2\alpha_2 & \beta_3 \end{pmatrix}. \tag{47}$$

Under rephasing by independent diagonal $SU(3)$ matrices $D(\theta)$ and $D(\theta')$ where

$$D(\theta) = \mathrm{diag}(e^{i(\theta_1+\theta_2)}, e^{i(-\theta_1+\theta_2)}, e^{-2i\theta_2})$$

and $D(\theta')$ is similarly defined, we have

$$V \to V' = D(\theta')VD(\theta) = A_3(\boldsymbol{\beta}')A_2(\boldsymbol{\alpha}') \tag{48}$$

From the locations of α_1, α_2, β_1, β_3, and β_3 in Eq. (47) one can easily deduce the transformation properties of $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$:

$$\alpha_1' = \alpha_1 e^{i(2\theta_2'-\theta_1-\theta_2)}, \quad \alpha_2' = \alpha_2 e^{i(-2\theta_2'-\theta_1+\theta_2)}, \tag{49}$$

$$\beta_1' = \beta_1 e^{i(\theta_1'+\theta_2'-2\theta_2)}, \quad \beta_2' = \beta_2 e^{i(-\theta_1'+\theta_2'-2\theta_2)},$$
$$\beta_3' = \beta_3 e^{i(-2\theta_2'-2\theta_2)}. \tag{50}$$

From these transformation properties it is evident that $(\alpha_1\alpha_2^*\beta_1^*\beta_2^*\beta_3)$ and hence $\arg(\alpha_1\alpha_2^*\beta_1^*\beta_2^*\beta_3)$ is invariant under rephasing.

For $n = 4$, parametrizing $D(\theta)$ as

$$D(\theta)=\mathrm{diag}(e^{i(\theta_1+\theta_2+\theta_3)}, e^{i(-\theta_1+\theta_2+\theta_3)}, e^{-2i\theta_2+i\theta_3}, e^{-3i\theta_3}) \tag{51}$$

and similarly for $D(\theta')$, one finds that

$$\begin{aligned} V&=A_4(\boldsymbol{\gamma})A_3(\boldsymbol{\beta})A_2(\boldsymbol{\alpha}) \to \\ V'&=D(\theta')VD(\theta) \\ &=A_4(\boldsymbol{\gamma'})A_3(\boldsymbol{\beta'})A_2(\boldsymbol{\alpha'}) \\ &=D(\theta')A_4(\boldsymbol{\gamma})\mathrm{diag}(e^{i\theta_3}, e^{i\theta_3}, e^{i\theta_3}, e^{-3i\theta_3}) \\ &\times A_3(\boldsymbol{\beta})A_2(\boldsymbol{\alpha})\mathrm{diag}(e^{i\theta_1+i\theta_2}, e^{-i\theta_1+i\theta_2}, e^{-2i\theta_2}, 1) \end{aligned} \tag{52}$$

The expressions for $\boldsymbol{\gamma'}$ can easily be read off:

$$\gamma_1' =\gamma_1 e^{i(\theta_1'+\theta_2'+\theta_3'-3\theta_3)}, \quad \gamma_2' = \gamma_1 e^{i(-\theta_1'+\theta_2'+\theta_3'-3\theta_3)},$$
$$\gamma_3' =\gamma_1 e^{i(-2\theta_2'+\theta_3'-3\theta_3)}, \quad \gamma_4' = \gamma_1 e^{-3i(\theta_3'+3\theta_3)}. \tag{53}$$

A little algebra shows that

$$\begin{aligned} &D(\theta')A_4(\boldsymbol{\gamma})\mathrm{diag}(e^{i\theta_3}, e^{i\theta_3}, e^{i\theta_3}, e^{-3i\theta_3}) \\ &\quad = A_4(\boldsymbol{\gamma'})\mathrm{diag}(e^{i(2\theta_2'+2\theta_3'-2\theta_3)}, \\ &\qquad e^{i(-2\theta_2'+\theta_3'+\theta_3)}, e^{i(-3\theta_3'+\theta_3)}, 1), \end{aligned} \tag{54}$$

so that the rest reduces to an $SU(3)$ problem in a 3×3 matrix form

$$\begin{aligned} A_3(\beta')A_2(\alpha') &= \mathrm{diag}\big(e^{i(2\theta_2'+2\theta_3'-2\theta_3)}, \\ &\qquad e^{i(-2\theta_2'+\theta_3'+\theta_3)}, e^{i(-3\theta_3'+\theta_3)}, 1\big) \\ &\times A_3(\beta)A_2(\alpha)\mathrm{diag}\big(e^{i\theta_1+i\theta_2}, e^{-i\theta_1+i\theta_2}, e^{-2i\theta_2}\big). \end{aligned} \tag{55}$$

We see that for the $SU(4)$ problem to accompany Eqs. (53) we have,

$$\alpha_1' = \alpha_1 e^{i(3\theta_3'-\theta_1-\theta_2-\theta_3)},$$
$$\alpha_2' = \alpha_2 e^{i(-3\theta_3'-\theta_1+\theta_2+\theta_3)}, \tag{56}$$

$$\beta_1' = \beta_1 e^{i(2\theta_2'+2\theta_3'-2\theta_2-2\theta_3)},$$
$$\beta_2' = \beta_2 e^{i(-2\theta_2'+\theta_3'-2\theta_2+\theta_3)},$$
$$\beta_3' = \beta_3 e^{i(-3\theta_3'-2\theta_2+\theta_3)}. \tag{57}$$

With the transformation properties of $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, and $\boldsymbol{\alpha}$ at hand, we can now systematically construct rephasing invariant quantities out of them as shown in Ref. 15. The three independent invariants turn out to be $(\alpha_1\alpha_2^*\beta_1^*\beta_2^*\beta_3)$, $(\beta_2\beta_3^*\gamma_3^*\gamma_4)$, and $(\beta_1\beta_2^*\gamma_1^*\gamma_2^*\gamma_3)$. The arguments of these quantities furnish the three independent phase type invariants for the $SU(3)$ problem. Notice that the first of these is the rephasing invariant for the $SU(3)$ problem and this is indeed a rather desirable feature of the recursive parametrization outlined here as one goes from n to $n+1$ one retains the parameters at the n^{th} level.

In the second form [16], for $n = 3$, the analogues of Eqs. (47), (49), and (50) are

$$V = A_3(\boldsymbol{\beta})A_2(\boldsymbol{\alpha}) = \begin{pmatrix} \sigma_1\alpha_2^* & \sigma_1\alpha_1 & \beta_1 \\ -\beta_1^*\beta_2\alpha_2^*/\sigma_1 - \beta_3^*\alpha_1^*/\sigma_1 & -\beta_1^*\beta_2\alpha_1/\sigma_1 + \beta_3^*\alpha_2/\sigma_1 & \beta_2 \\ -\beta_1^*\beta_3\alpha_2^*/\sigma_1 + \beta_2^*\alpha_1/\sigma_1 & -\beta_1^*\beta_3\alpha_1/\sigma_1 - \beta_2^*\alpha_2/\sigma_1 & \beta_3 \end{pmatrix}. \tag{58}$$

$$\alpha'_1 = \alpha_1 e^{i(\theta'_1+\theta'_2-\theta_1+\theta_2)},$$
$$\alpha'_2 = \alpha_2 e^{-i(\theta'_1+\theta'_2+\theta_1+\theta_2)}, \quad (59)$$
$$\beta'_1 = \beta_1 e^{i(\theta'_1+\theta'_2-2\theta_2)}, \quad \beta'_2 = \beta_2 e^{i(-\theta'_1+\theta'_2-2\theta_2)},$$
$$\beta'_3 = \beta_3 e^{-2i(\theta'_2+\theta_2)}, \quad (60)$$

and the rephasing invariant is $(\alpha_1\alpha_2^*\beta_1^*\beta_2^*\beta_3)$.

For $n = 4$ the corresponding equations to (53), (56), and (57) are

$$\gamma'_1 = \gamma_1 e^{i(\theta'_1+\theta'_2+\theta'_3-3\theta_3)}, \quad \gamma'_2 = \gamma_1 e^{i(-\theta'_1+\theta'_2+\theta'_3-3\theta_3)},$$
$$\gamma'_3 = \gamma_1 e^{i(-2\theta'_2+\theta'_3-3\theta_3)}, \quad \gamma'_4 = \gamma_1 e^{-3i(\theta'_3+3\theta_3)}. \quad (61)$$
$$\alpha'_1 = \alpha_1 e^{i(\theta'_1+\theta'_2+\theta'_3-\theta_1+\theta_2+\theta_3)},$$
$$\alpha'_2 = \alpha_2 e^{-i(\theta'_1+\theta'_2+\theta'_3+\theta_1+\theta_2+\theta_3)}. \quad (62)$$
$$\beta'_1 = \beta_1 e^{i(\theta'_1+\theta'_2+\theta'_3-2\theta_2+\theta_3)},$$
$$\beta'_2 = \beta_2 e^{i(-\theta'_1+\theta'_2+\theta'_3-2\theta_2+\theta_3)},$$
$$\beta'_3 = \beta_3 e^{-2i(\theta'_2+\theta'_3+\theta_3)}. \quad (63)$$

The three independent invariants turn out to be $(\alpha_1\alpha_2^*\beta_1^*\beta_2^*\beta_3)$, $(\beta_1\beta_2^*\gamma_1^*\gamma_2)$, and $(\beta_2\beta_3^*\gamma_2^*\gamma_3\gamma_4)$.

5. Constraints due to mod symmetry

In this section we examine the constraints on the parameters that arise from demanding that the given $SU(n)$ matrix be mod symmetric, *i.e.*, $|V_{ij}| = |V_{ji}|$. For convenience we shall use the first form [15] for this discussion.

For $n = 3$, mod symmetry requires that $|\alpha_2| = |\beta_2|/\sigma_2$. The number of independent angle type invariants comes down from three to two leaving the phase type invariant unchanged.

For $n = 4$, with $V = A_4(\boldsymbol{\gamma})A_3(\boldsymbol{\beta})A_2(\boldsymbol{\alpha})$, after some algebra one finds,

$$|V_{14}| = |V_{41}| \Rightarrow |\alpha_2| = |\gamma_2|/\rho_2, \quad (64)$$
$$|V_{34}| = |V_{43}| \Rightarrow |\beta_3| = |\gamma_3|/\rho_3, \quad (65)$$
$$|V_{23}| = |V_{32}| \Rightarrow \cos((\delta_1+\delta_2+\delta_3)/2)\times \left[\frac{|\beta_2|}{\rho_2}\cos((\delta_1-\delta_2-\delta_3)/2)+\frac{|\gamma_4|}{\rho_3}\right] = 0. \quad (66)$$

Here δ_1, δ_2, and δ_3 denote the three independent invariant phases $\arg(\alpha_1\alpha_2^*\beta_1^*\beta_2^*\beta_3)$, $\arg(\beta_2\beta_3^*\gamma_3^*\gamma_4)$, and $\arg(\beta_1\beta_2^*\gamma_1^*\gamma_2^*\gamma_3)$, respectively. The equalities $|V_{24}|=|V_{42}|$, $|V_{12}| = |V_{21}|$, and $|V_{13}| = |V_{31}|$ give no new conditions. It can be seen from the above equations that one can obtain mod symmetry by requiring

$$|\alpha_2| = |\gamma_2|/\rho_2,\ |\beta_3| = |\gamma_3|/\rho_3, \quad \delta_1+\delta_2+\delta_3 = \pi, \quad (67)$$

and in this situation the mod symmetric matrix mixing is parametrized by four angles and two phases.

A simpler moduli symmetric parametrization can be obtained if some of the eigenvalues E_i $(i = 1, 2, \ldots, n)$, of the $n \times n$ unitary matrix V are equal. For the case when $n-1$ eigenvalues are equal, *viz.*, $E_2 = E_3 = \cdots = E_n$, V can be expressed in terms of $n-1$ real parameters and only one phase [18].

6. Comparison with the "standard" (PDG) parametrization

For the case of three generations, the standard or PDG [17] parametrization of the mixing matrix is obtained by putting

$$\alpha_1 = c_{12}, \quad \alpha_2 = s_{12}, \quad \beta_1 = s_{13}e^{-i\delta_{13}},$$
$$\beta_2 = s_{23}c_{13}, \quad \beta_3 = c_{23}c_{13}, \quad (68)$$

($c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$) in $V = A_3(\boldsymbol{\beta})A_2(\boldsymbol{\alpha})$ with $A_3(\boldsymbol{\beta})$ and $A_2(\boldsymbol{\alpha})$ given by Eqs. (35) and (34), respectively.

The extension of the mixing matrix to four generations is given by $V = A_4(\boldsymbol{\gamma})A_3(\boldsymbol{\beta})A_2(\boldsymbol{\alpha})$, where $A_4(\boldsymbol{\gamma})$ is given by Eq. (36) with $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ as before and

$$\gamma_1 = s_{14}e^{-\delta_{14}}, \quad \gamma_2 = c_{14}s_{24}e^{-i\delta_{24}},$$
$$\gamma_3 = s_{34}c_{24}c_{14}, \quad \gamma_4 = c_{34}c_{24}c_{14}, \quad (69)$$

which conveniently reduces to the case of three generations when θ_{14}, θ_{24}, and θ_{34} are all set equal to zero.

We note here that the parametrization given above is closely related to the Harari-Leurer parametrization [5] where the mixing matrix is expressed as an ordered product of essentially 2×2 "rotation" matrices. Our parametrization results when one suitably combines the factors appearing in that form. For instance, in the 4×4 case, the Harari-Leurer form for the mixing matrix has the structure $\Omega_{34}\Omega_{24}\Omega_{14}\Omega_{23}\Omega_{13}\Omega_{12}$ and reduces to our form by the identifications $A_4(\boldsymbol{\gamma}) \equiv \Omega_{34}\Omega_{24}\Omega_{14}$, $A_3(\boldsymbol{\beta}) = \Omega_{23}\Omega_{13}$, and $A_2(\boldsymbol{\alpha}) = \Omega_{12}$, provided we choose γ_4, β_3, and α_2 to be real.

7. Conclusions

In this work we have examined in detail the question of parametrizing quark flavor mixing matrices for three and four flavors within the framework of the recursive parametrization. In particular we have shown, given the matrix, how to determine the corresponding parameters. We have also studied in detail aspects of rephasing invariants in this parametrization scheme and have derived conditions for the mixing matrix to be moduli symmetric.

In the near future we plan to confront the 3×3 and 4×4 recursive parametrization with available data [17].

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i. Parametrizations of P. Kielanowski [9] and V. Gupta [14] are interesting as they give CP-violation but have only three real parameters.

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