Applications and extensions of the Liouville theorem on constants of motion

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We give an elementary proof of the Liouville theorem, which allows us to obtain n constants of motion in addition to n given constants of motion in involution, for a mechanical system with n degrees of freedom, and we give some examples of its application. For a given set of n constants of motion that are not in involution with respect to the standard symplectic structure, there exist symplectic structures with respect to which these constants will be in involution and the Liouville theorem can then be applied. Using the fact that any second-order ordinary differential equation (not necessarily related to a mechanical problem) can be expressed in the form of the Hamilton equations, the knowledge of a first integral of the equation allows us to find its general solution.

Keywords: Hamilton–Jacobi equation; constants of motion; symplectic structures.

Se da una prueba elemental del teorema de Liouville, el cual permite obtener n constantes de movimiento adicionales a n constantes de movimiento en involución dadas, para un sistema mecánico con n grados de libertad, y se dan algunos ejemplos de su aplicación. Para un conjunto dado de n constantes de movimiento que no están en involución con respecto a la estructura simpléctica estándar, existen estructuras simplécticas con respecto a las cuales estas constantes estarán en involución y puede aplicarse entonces el teorema de Liouville. Usando el hecho de que cualquier ecuación diferencial ordinaria de segundo orden (no necesariamente relacionada con un problema mecánico) puede expresarse en la forma de las ecuaciones de Hamilton, el conocer una primera integral de la ecuación permite hallar su solución general.

Descriptores: Ecuación de Hamilton–Jacobi; constantes de movimiento; estructuras simplécticas.

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1. Introduction

The Hamilton–Jacobi (HJ) equation provides a powerful method to solve various problems of classical mechanics (see, *e.g.*, Refs. 1 and 2) as well as to relate different problems [3]. As is well known, given a Hamiltonian for a mechanical system with n degrees of freedom, the knowledge of a complete solution (that is, a solution containing n non-additive arbitrary constants) of the corresponding HJ equation allows one to obtain the solution of the equations of motion. The n arbitrary parameters contained in a complete solution of the HJ equation are then constants of motion, that are identified with half of a new set of canonical coordinates.

Usually, the complete solutions of the HJ equation are obtained by means of separation of variables, which requires expressing this equation in a suitable coordinate system (see, *e.g.*, Refs. 1 and 2). Actually, in most textbooks on classical mechanics the method of separation of variables is the only one employed to solve the HJ equation. (Similarly, in most textbooks on quantum mechanics, the only method employed in the solution of the Schrödinger equation is that of separation of variables.) Nevertheless, there exist some other methods for solving *first-order* partial differential equations (see, *e.g.*, Ref. 4) such as the HJ equation. In one of these less-known methods, when applied to the HJ equation, one has to express the canonical momenta in terms of the coordinates and n constants of motion; a complete solution, S, of the HJ equation can then be obtained from $\mathrm{d}S = p_i \mathrm{d}q^i - H\mathrm{d}t$. However, it turns out that the expression on the right-hand side is an exact differential if and only if the constants of motion employed in this process are in involution, that is, their Poisson brackets are all equal to zero, as Liouville found by 1855 [2]. In the case where there is only one degree of freedom, the Liouville theorem can be applied making use of an arbitrary constant of motion, since the Poisson bracket of a function with itself is trivially equal to zero.

The aim of this paper is to give an elementary proof of Liouville's theorem, with some illustrative examples of its application finding complete solutions of the HJ equation, without relying on an specific coordinate system (by contrast with the method of separation of variables). When one has n constants of motion that are not in involution, it is still possible to find a different symplectic structure (that is, another definition of the Poisson bracket) so that these constants of motion be in involution. Since any second-order ordinary differential equation (ODE), or any pair of first-order ODEs, can be expressed in the form of the Hamilton equations (in an infinite number of different ways) [5], with the aid of Liouville's theorem, making use of a first integral one can find the complete solution.

In Sec. 2 we state the Liouville theorem stressing its analogy with the procedure followed in the use of a complete solution of the HJ equation in the solution of the equations of motion. Section 3 contains four examples of the application of the Liouville theorem and in Sec. 4 an elementary proof of the Theorem is given.

2. The Liouville theorem

In this section we begin by recalling some basic facts related with the application of the complete solutions of the HJ equation and we show that, in a certain sense, the same steps appear in the Liouville theorem but in the opposite order.

For a given Hamiltonian $H(q^i, p_i, t)$ of a system with n degrees of freedom, the corresponding HJ equation, in its standard form, is the first-order partial differential equation

$$H\left(q^i, \frac{\partial S}{\partial q^i}, t\right) + \frac{\partial S}{\partial t} = 0. \tag{1}$$

A complete solution of this equation is a function $S(q^i, t, Q^i)$ containing n non-additive arbitrary parameters $Q^1, \ldots, Q^n$ that satisfies Eq. (1). Under the appropriate regularity conditions, such a function generates a canonical transformation relating the original canonical coordinates q^i, p_i with a new set of canonical coordinates Q^i, P_i, which are constants of motion (since the new Hamiltonian is equal to zero), according to

$$\mathrm{d}S = p_i \mathrm{d}q^i - H\mathrm{d}t - P_i \mathrm{d}Q^i, \tag{2}$$

that is,

$$p_i = \frac{\partial S}{\partial q^i}, \qquad P_i = -\frac{\partial S}{\partial Q^i}, \qquad i = 1, 2, \ldots, n \tag{3}$$

(see, *e.g.*, Refs. 1 and 2).

Since the Q^i, P_i are canonical coordinates, the Poisson brackets among the Q^i are all equal to zero

$$\{Q^i, Q^j\} = 0, \qquad i, j = 1, 2, \ldots, n. \tag{4}$$

As pointed out in the Introduction, the complete solutions of the HJ equation are usually obtained by separation of variables, in which case the separation constants can be taken as the Q^i, but the application of this method requires an appropriate choice of the coordinates q^i, p_i. As we shall show, given a set of n functionally independent constants of motion $Q^1, \ldots, Q^n$, satisfying Eq. (4) (that is, the Q^i are in *involution*) one can find a complete solution of the HJ equation (and, therefore, the solution of the equations of motion) without having to use some special coordinate system. Indeed, if we have n constants of motion

$$Q^i = Q^i(q^j, p_j, t), \qquad i = 1, 2, \ldots, n \tag{5}$$

(which may depend explicitly on the time), assuming that these relations can be inverted to express the p_i in terms of Q^j, q^j, and t, we obtain n functions F_i such that

$$p_i = F_i(q^j, t, Q^j). \tag{6}$$

Substituting these expressions into the Hamiltonian we obtain a function

$$\widetilde{H}(q^i, t, Q^i) \equiv H(q^i, F_i(q^j, t, Q^j), t). \tag{7}$$

Then, treating the Q^i as constants, the linear differential form $F_i \mathrm{d}q^i - \widetilde{H}\mathrm{d}t$ is exact, that is, $F_i \mathrm{d}q^i - \widetilde{H}\mathrm{d}t$ is the differential of some function S (which depends parametrically on the Q^i)

$$\mathrm{d}S = F_i \mathrm{d}q^i - \widetilde{H}\mathrm{d}t \tag{8}$$

[*cf.* Eq. (2)] and $S(q^i, t, Q^i)$ is therefore a complete solution of the HJ equation. In the next section we give some examples, deferring the proof of the exactness of $F_i \mathrm{d}q^i - \widetilde{H}\mathrm{d}t$ to Sec. 4.

3. Examples

In this section we give some illustrative examples of the procedure outlined above to find complete solutions of the HJ equation.

3.1. The Kepler problem in two dimensions

As is well known, the HJ equation for the Kepler problem in two dimensions, which corresponds to the Hamiltonian

$$H = \frac{1}{2m}({p_x}^2 + {p_y}^2) - \frac{k}{\sqrt{x^2 + y^2}},$$

expressed in Cartesian coordinates x, y, where m is the mass of the particle and k is a positive constant, is separable in polar and parabolic coordinates (see, *e.g.*, [1,6]) but it *is not* separable in Cartesian coordinates.

Since H is time-independent and invariant under rotations about the origin,

$$Q^1 \equiv H, \qquad Q^2 \equiv x p_y - y p_x$$

(the total energy and the angular momentum about the origin) are constants of motion, which are in involution (as can be seen from the fact that the angular momentum is a constant of motion). Inverting these expressions one finds

$$p_x = \frac{-Q^2 y \pm x\sqrt{2mQ^1 r^2 + 2mkr - (Q^2)^2}}{r^2},$$
$$p_y = \frac{Q^2 x \pm y\sqrt{2mQ^1 r^2 + 2mkr - (Q^2)^2}}{r^2},$$

where $r^2 \equiv x^2 + y^2$, which gives the functions F defined by Eq. (6). Thus, the right-hand side of Eq. (8) becomes

$$Q^2 \frac{(-y\mathrm{d}x + x\mathrm{d}y)}{r^2} \pm \frac{\sqrt{2mQ^1 r^2 + 2mkr - (Q^2)^2}}{r^2}(x\mathrm{d}x + y\mathrm{d}y)$$

or, equivalently,

$$Q^2 \mathrm{d}\left(\arctan\frac{y}{x}\right) \pm \sqrt{2mQ^1 + \frac{2mk}{r} - \frac{(Q^2)^2}{r^2}}\,\mathrm{d}r.$$

This last expression is indeed the differential of a function, which must be a complete solution of the HJ equation. It may be noticed that this function is the sum of separate functions of the polar coordinates θ, r (which is a consequence of using the angular momentum as one of the constants of motion Q^i).

3.2. A time-dependent Hamiltonian

Now we shall consider the time-dependent Hamiltonian

$$H = \frac{p^2}{2m} - ktx, \tag{9}$$

where k is a constant. The corresponding HJ equation is not separable, but one can readily verify that

$$Q \equiv p - kt^2/2$$

is a constant of motion (note that Q depends explicitly on the time). Hence

$$F(x,t,Q) = Q + kt^2/2$$

and

$$\widetilde{H}(x,t,Q) = (1/2m)(Q + kt^2/2)^2 - ktx.$$

Then, it can readily be verified that the differential form

$$F\mathrm{d}x - \widetilde{H}\mathrm{d}t = \left(Q + \frac{kt^2}{2}\right)\mathrm{d}x - \left[\frac{1}{2m}\left(Q + \frac{kt^2}{2}\right)^2 - ktx\right]\mathrm{d}t$$

is exact and that it is the differential of the function

$$S = Qx + \frac{1}{2}kt^2x - \frac{1}{2m}\left(Q^2t + Qk\frac{t^3}{3} + k^2\frac{t^5}{20}\right), \tag{10}$$

which is, therefore, a complete solution of the HJ equation. It may be noticed that this function is not the sum of separate functions of x and t.

From Eqs. (3) and (10) one finds a second constant of motion

$$-P = \frac{\partial S}{\partial Q} = x - \frac{Qt}{m} - \frac{kt^3}{6m}.$$

As usual, the values of Q and P are determined, *e.g.*, by the initial conditions, and the formulas above give the solution of the equations of motion.

This example is also interesting because the Schrödinger equation corresponding to the Hamiltonian (9) cannot be solved by separation of variables, but it turns out that $\psi = \exp(\mathrm{i}S/\hbar)$, with S given by Eq. (10), is a solution of this equation.

3.3. Two constants of motion that are not in involution

The Hamiltonian

$$H = \frac{{p_x}^2 + {p_y}^2}{2m} + mgy \tag{11}$$

corresponds to a particle of mass m in a uniform gravitational field in Cartesian coordinates and one can readily verify that the two functions

$$Q^1 \equiv p_x, \qquad Q^2 \equiv \frac{p_x p_y}{m} + mgx \tag{12}$$

are constants of motion (as well as the Hamiltonian itself). Making use of the standard definition of the Poisson bracket ($\{x, p_x\} = 1 = \{y, p_y\}$) one finds that $\{Q^1, Q^2\}$ is different from zero and therefore these two constants of motion do not seem suitable to find a complete solution of the HJ equation.

However, by *defining* the Poisson bracket in such a way that

$$\{x, p_y\} = 1 = \{y, p_x\}, \tag{13}$$

Q^1 and Q^2 are in involution. But, then, taking into account that p_x is the momentum conjugate to y and p_y is the momentum conjugate to x, in order to reproduce the equations of motion

$$\dot{x} = p_x/m, \quad \dot{y} = p_y/m, \quad \dot{p_x} = 0, \quad \dot{p_y} = -mg,$$

in place of (11), we have to use Q^2 as the Hamiltonian; in fact,

$$\dot{x} = \frac{\partial Q^2}{\partial p_y}, \quad \dot{y} = \frac{\partial Q^2}{\partial p_x}, \quad \dot{p_x} = -\frac{\partial Q^2}{\partial y}, \quad \dot{p_y} = -\frac{\partial Q^2}{\partial x}$$

(see Ref. 7 for details).

Thus

$$p_x = Q^1, \qquad p_y = \frac{m(Q^2 - mgx)}{Q^1}$$

and [performing the appropriate changes on the right-hand side of Eq. (8)]

$$\mathrm{d}S = \frac{m(Q^2 - mgx)}{Q^1}\mathrm{d}x + Q^1\mathrm{d}y - Q^2\mathrm{d}t$$

must yield a complete solution of the HJ equation (corresponding to the new Hamiltonian Q^2).

3.4. Application to second-order ODEs

Following the algorithm given in Ref. 5, any second-order ODE, or any system of two first-order ODEs, can be expressed in the form of the Hamilton equations in an infinite number of different ways. Then, any constant of motion is useful to apply Liouville's Theorem, and one can make use of the complete solution of the HJ equation thus obtained to find a second constant of motion and, therefore, the general solution of the original ODE or ODEs.

For instance, in Ref. 5 the Emden–Fowler equation (which arises in the study of a self-gravitating gas)

$$\ddot{x} + \frac{2\dot{x}}{t} + x^k = 0,$$

where k is a constant, has been considered, showing that it is equivalent to the Hamilton equations with

$$H = \frac{p^2}{2t^2} + \frac{t^2 q^{k+1}}{k+1}$$

and $q = x$, $p = t^2 \dot{x}$, as can be readily verified. It was also shown that in the particular case where $k = 5$,

$$Q \equiv \frac{3p^2}{t} + 3pq + t^3 q^6$$

is a constant of motion. Inverting this last expression, one finds

$$p = F(q, t, Q) = \frac{t}{6}\big(-3x \pm \sqrt{9q^2 - 12t^2 q^6 + 12Q/t}\big)$$

and the right-hand side of Eq. (8) takes the form

$$\mathrm{d}\left(-\frac{u}{4} - \frac{Q}{6}\ln|t|\right) \pm \frac{1}{12}\sqrt{9 - 12u^2 + 12Q/u}\,\mathrm{d}u,$$

with $u \equiv q^2 t$. Clearly, this differential form is the differential of a function, which must be a complete solution of the HJ equation (it may be noticed that this function is the sum of separate functions of $q^2 t$ and t).

A second constant of motion, which together with Q, yields the complete solution of the original equation, is given by

$$-P = \frac{\partial S}{\partial Q} = -\frac{1}{6}\ln|t| \pm \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{9u^2 - 12u^4 + 12Qu}}.$$

4. Proof of the Liouville theorem

We shall prove that, if $Q^1, \ldots, Q^n$ are n functionally independent constants of motion, and the original momenta p_i can be expressed in the form $p_i = F_i(q^j, t, Q^j)$, the differential form $F_i \mathrm{d}q^i - \widetilde{H}\mathrm{d}t$ is exact if and only if the Q^i are in involution. According to the standard criterion, this differential form is exact if and only if

$$\frac{\partial F_i}{\partial q^j} = \frac{\partial F_j}{\partial q^i}, \qquad i, j = 1, 2, \ldots, n \tag{14}$$

and

$$\frac{\partial F_i}{\partial t} = -\frac{\partial \widetilde{H}}{\partial q^i}, \qquad i = 1, 2, \ldots, n \tag{15}$$

(these last equations *look like* half of the Hamilton equations, but, as we shall see, they hold as a consequence of the constancy of the Q^i).

Substitution of the relations $p_i = F_i(q^j, t, Q^j)$ into the expressions for the constants of motion Q^i give the equations

$$Q^i = Q^i(q^j, F_j(q^k, t, Q^k), t)$$

which have to hold identically; hence, making use of the chain rule, we obtain

$$0 = \frac{\partial Q^i}{\partial q^m} + \frac{\partial Q^i}{\partial p_k}\frac{\partial F_k}{\partial q^m}, \tag{16}$$

and

$$0 = \frac{\partial Q^i}{\partial t} + \frac{\partial Q^i}{\partial p_k}\frac{\partial F_k}{\partial t}. \tag{17}$$

Making use of Eq. (16) we find that

$$\begin{aligned}
\{Q^i, Q^j\} &= \frac{\partial Q^i}{\partial q^m}\frac{\partial Q^j}{\partial p_m} - \frac{\partial Q^j}{\partial q^m}\frac{\partial Q^i}{\partial p_m} \\
&= -\frac{\partial Q^i}{\partial p_k}\frac{\partial F_k}{\partial q^m}\frac{\partial Q^j}{\partial p_m} + \frac{\partial Q^j}{\partial p_k}\frac{\partial F_k}{\partial q^m}\frac{\partial Q^i}{\partial p_m} \\
&= -\frac{\partial Q^i}{\partial p_k}\frac{\partial F_k}{\partial q^m}\frac{\partial Q^j}{\partial p_m} + \frac{\partial Q^j}{\partial p_m}\frac{\partial F_m}{\partial q^k}\frac{\partial Q^i}{\partial p_k} \\
&= \frac{\partial Q^i}{\partial p_k}\frac{\partial Q^j}{\partial p_m}\left(\frac{\partial F_m}{\partial q^k} - \frac{\partial F_k}{\partial q^m}\right),
\end{aligned}$$

so that $\{Q^i, Q^j\} = 0$ if and only if Eqs. (14) hold.

Similarly, from the definition of $\widetilde{H}$ we have

$$\frac{\partial \widetilde{H}}{\partial q^i} = \frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial p_j}\frac{\partial F_j}{\partial q^i}, \tag{18}$$

and the fact that Q^i is a constant of motion amounts to

$$0 = \frac{\partial Q^i}{\partial t} + \frac{\partial Q^i}{\partial q^m}\frac{\partial H}{\partial p_m} - \frac{\partial Q^i}{\partial p_m}\frac{\partial H}{\partial q^m}.$$

Substituting Eqs. (16), (17), and (18) into this last equation we find that

$$\begin{aligned}
0 = &-\frac{\partial Q^i}{\partial p_k}\frac{\partial F_k}{\partial t} - \frac{\partial Q^i}{\partial p_k}\frac{\partial F_k}{\partial q^m}\frac{\partial H}{\partial p_m} \\
&- \frac{\partial Q^i}{\partial p_m}\left(\frac{\partial \widetilde{H}}{\partial q^m} - \frac{\partial H}{\partial p_k}\frac{\partial F_k}{\partial q^m}\right) \\
= &-\frac{\partial Q^i}{\partial p_k}\left(\frac{\partial F_k}{\partial t} + \frac{\partial \widetilde{H}}{\partial q^k}\right) \\
&- \frac{\partial Q^i}{\partial p_k}\frac{\partial H}{\partial p_m}\left(\frac{\partial F_k}{\partial q^m} - \frac{\partial F_m}{\partial q^k}\right),
\end{aligned}$$

thus completing the proof.

5. Conclusions

The Liouville theorem allows one to get the n complementary constants of motion to a given set of n constants of motion of a system with n degrees of freedom, without having to use some particular coordinate system. Among other things, the results presented here show the usefulness of having various symplectic structures (and Hamiltonians) for a given mechanical system and of expressing an arbitrary second-order ODE in the form of the Hamilton equations.

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