Oriented matroid theory and loop quantum gravity in (2+2) and eight dimensions

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We establish a connection between oriented matroid theory and loop quantum gravity in (2+2) (two time and two space dimensions) and 8-dimensions. We start by observing that supersymmetry implies that the structure constants of the real numbers, complex numbers, quaternions and octonions can be identified with the chirotope concept. This means, among other things, that normed divisions algebras, which are only possible in 1,2, 4 or 8-dimensions, are linked to oriented matroid theory. Therefore, we argue that the possibility for developing loop quantum gravity in 8-dimensions must be taken as important alternative. Moreover, we show that in 4-dimensions, loop quantum gravity theories in the (1+3) or (0+4) signatures are not the only possibilities. In fact, we show that loop quantum gravity associated with the (2+2)-signature may also be an interesting physical structure.

Keywords: Loop quantum gravity; eight dimensions; (2+2)-dimensions.

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It is known that the self-dual (or antiself-dual) concept associated with the 2-form Riemann tensor R^{AB} plays a central role in quantum gravity *a la* Ashtekar [1-3]. Mathematically, the self-dual sector of R^{AB} is realized by introducing a dual tensor ${}^*R^{AB}$ such that the self-dual curvature,

$$ {}^+R^{AB} = \frac{1}{2}(R^{AB} + \alpha^* R^{AB}), \tag{1}$$

where $\alpha = \{1, i\}$, is again a 2-form. Using the completely antisymmetric density $\epsilon_{A_1..A_D}$ (ϵ-symbol) which takes values in the set $\{-1, 0, 1\}$ one can define ${}^*R_{A_1...A_{D-3}}$ as

$$ {}^*R_{A_1...A_{D-3}} = \frac{1}{2}\epsilon_{A_1..A_{D-3}A_{D-2}A_{D-1}} R^{A_{D-2}A_{D-1}}. \tag{2}$$

In this case, one immediately sees that the dual ${}^*R_{A_0...A_{D-3}}$ is a 2-form only in 4-dimensions. This seems to indicate that, from a quantum gravity perspective, 4-dimensions is an exceptional signature. However, in Refs. 4 to 7 it is shown that also makes sense to consider self-dual gravity in 8-dimensions, but one should define ${}^*R^{AB}$ in terms of the η-symbol.(see Refs. 8, 9, 10 and 12) rather that in terms of the ϵ-symbol. In fact, the η-symbol is very similar to the ϵ-symbol in 4-dimensions; it is a 4-index completely antisymmetric object and takes values also in the set $\{-1, 0, 1\}$. However, the η-symbol is defined in 8-dimensions rather than in 4. Moreover, while the ϵ-symbol in 4-dimensions can be connected with quaternions, the η-symbol is related to the octonion structure constants (see Ref. 13 and References therein). Thus, in 8-dimensions we can also introduce the dual tensor

$$ {}^*R_{A_1A_2} = \frac{1}{2}\eta_{A_1A_2A_3A_4} R^{A_3A_4}, \tag{3}$$

and consequently the self-dual object ${}^+R^{AB}$, given in (1), is again a 2-form. Since in 2-dimensions one can always write $R^{AB} = R\epsilon^{AB}$, ${}^*R^{AB} ={}^* R\epsilon^{AB}$ and ${}^+R^{AB} ={}^+ R\epsilon^{AB}$, with $R = (1/2)R^{AB}\epsilon_{AB}$,, one sees that self-duality requirement (1) can also be achieved in 2-dimensions. The case of 1-dimension corresponds to an ϵ without indices and may be identified with the real numbers. Therefore, this shows that the set

$$ D = \{1, 2, 4, 8\}, \tag{4}$$

describes the dimensionality of the "spacetime" where self-duality can be accomplished. One may recognize in (4) the only possible dimensions for a real division algebras [14-15] (see also Ref. 16 and references therein). Moreover, the set (4) corresponds to the dimensions associated with the normed division algebras; real numbers, complex numbers, quaternions and octonions. From the point of view of string theory and massless vector field, the dimensions in the set D can be understood as the true physical degrees of freedom, corresponding to dimensions $3, 4, 6$ and 10 in the covariant approach, respectively. It turns out interesting that this normed division algebras are related to the objects ϵ, ϵ_{AB}, ϵ_{ABCD} and η_{ABCD}, respectively. In fact, we shall see below that this is not a coincidence but as a result of a link between supersymmetry, division algebras and oriented matroids.

The next step is to analyze the above scenario of $1, 2, 4$ or 8-dimensions from the point of view of the "spacetime"-signature. The Milnor-Bott [14] and Kervair [15] theorem for real division algebras and Hurwitz theorem [17] for normed division algebras refer to Euclidean space, but in Refs. 18, 19 and 20 it is shown that the set D may also be linked to other signatures. In 2-dimensions we have the two possible signatures $(1 + 1)$ and $(0 + 2)$ which may be identified with

2-dimensional gravity (see Ref. 21 and references therein). Traditionally, in 4-dimensions one assumes the signatures $(0+4)$ or $(1+3)$, but in this article we will show that the $(2+2)$-signature (two time and two space dimensions) may emerge also as interesting possibility (see Ref. 22 and references therein). Similarly, in 8-dimensions one can consider the signatures $(0+8)$, $(1+7)$, $(2+6)$ and $(4+4)$ which in the covariant context may correspond to $(0+10)$, $(1+9)$, $(2+8)$, $(4+6)$ and $(5+5)$ (see Ref. 23). Of course, it will be wonderful to have a theory which predicts no only the dimensionality of the "spacetime" but also its signature (See Ref. 24). At least the self-duality concept predicts the dimensionality of the "spacetime". But in the lack of a sensible theory which determines the signature of the "spacetime" we need to explore all possibilities. Eventually this may help to find, for a fixed dimensionality, a connection between the different signatures.

Let us analyze the above scenario from the point of view of gauge group theory. It is known that the algebra $so(1,3)$ can be written as $so(1,3) = su(2) \times su(2)$, or the algebra $so(4)$ as $so(4) = so(3) \times so(3)$, corresponding to the signatures $(1+3)$ and $(0+4)$ respectively. So, in both cases the curvature R^{AB} can be decomposed additively:

$$R^{AB}(\omega) =^{+} R^{AB}(^{+}\omega) + {}^{-}R^{AB}(^{-}\omega), \tag{5}$$

where $^{+}\omega$ and $^{-}\omega$ are the self-dual and antiself-dual parts of the spin connection ω. In an Euclidean context, this is equivalent to write the normed group for quaternions $O(4)$ as $O(4)$=S^3×S^3, where S^3 denotes the 3-sphere. The situation in 8-dimensions is very similar since $O(8)$=S^7×S^7×G_2, with S^7 denoting the 7-sphere, suggesting that one can also define self-duality in 8-dimensions, but modulo the exceptional group G_2 [8-9].

In $(2+2)$-dimensions we have analogue situation since $SO(2,2) = SU(1,1) \times SU(1,1)$. It is worth mentioning a number of properties of the group $SU(1,1)$. First of all, the group $SU(1,1)$ is isomorphic to the groups $SL(2,R)$ and $Sp(2)$. Secondly, just as $SU(2)$ is the double cover of $SO(3)$, we have that $SU(1,1)$, $SL(2,R)$ and $Sp(2)$ are double cover of $SO(1,2)$. Moreover, $SU(1,1)$ manifold is topologically $R^2 \times S^1$. In general, the important role played by the groups $SU(1,1)$, $SL(2,R)$ and $Sp(2)$ has been recognized, for long time, in a various physical scenarios, including 2-dimensional black-holes [25], 2t physics [26], and string theory [27-28]. In this context, $^{+}\omega$ (or $^{-}\omega$) must be understood as a connection associated with the gauge groups $SU(1,1)$, $SL(2,R)$ and $Sp(2)$. As a consequence, in (2+2)-dimensions the self-dual connection $^{+}\omega$ can be linked with the group $SO(1,2)$. We will see that in this case an interesting possibility arises at the quantum gravity level.

Let us consider now the Clifford algebra

$$\Gamma^{AC}_{\mu}\Gamma_{\upsilon CB} + \Gamma^{AC}_{\nu}\Gamma_{\mu CB} = 2\delta^{A}_{B}\eta_{\mu\nu}. \tag{6}$$

In order to have a supersymmetric Yang-Mills theories it is necessary that Γ^{AB}_{μ} satisfies the additional condition (see Ref. 29)

$$\Gamma_{\mu A(B}\Gamma^{\mu}_{CD)} = 0, \tag{7}$$

where the bracket (BCD) means completely antisymmetric. It can be shown that the two relations (6) and (7) are equivalent to the condition for normed division algebras. So, the possible dimensions of supersymmetric Yang-Mills theories are limited to only $1, 2, 4$ or 8 (see Refs. 29 and 20). The interesting thing that we would like to add is that the expression (7) can be identified with a Grassman-Plücker relation and consequently the Γ^{AB}_{μ} satisfying (7) is a chirotope [30] which takes the values ϵ, ϵ^{AB}, ϵ^{ABCD} and η^{ABCD} depending if we are considering $1, 2, 4$ or 8-dimensions, respectively. In Refs. 31 it is shown that the ϵ-symbol is a chirotope. Similarly, in connection to maximal supersymmetry in Ref. 32 it is shown the η-symbol is also a chirotope. The new ingredient is that by using the Clifford structure, expressions (6) and (7), both cases, ϵ-symbol-chirotope and η-symbol-chirotope, can also be obtained. This result suggests a link between maximal supersymmetry and Clifford structure. Now, there exist a definition of an oriented matroid in terms of chirotopes [30]. So, we have established a connection between supersymmetry, division algebras and oriented matroids. It is important to mention that the set $D = \{1, 2, 4, 8\}$, and the corresponding quantities ϵ, ϵ^{AB}, ϵ^{ABCD} and η^{ABCD}, can also be connected with the so called r-fold cross product [8].

The next step is now to bring these results at the level of canonical Diffeomorphism and Hamiltonian constraints of quantum gravity. First, suppose that the Hamiltonian operators $\hat{H}$ and $\hat{H}_l$ act on the physical sates $|\Psi\rangle$ in the form

$$\hat{H}|\Psi\rangle = 0 \tag{8}$$

and

$$\hat{H}_l|\Psi\rangle = 0, \tag{9}$$

respectively. We shall assume that $\hat{H}$ and $\hat{H}_l$ can be written in terms of the canonical variables $\hat{A}^a_i$ and $\hat{E}_{(a)}{}^{i}$. Here, $\hat{E}_{(a)}{}^{i}$ is an operator associated with the $E_i{}^{(a)}$ part of the general vielbein on a M^D-manifold (see Ref. 7 and references therein)

$$e_{\mu}{}^{(A)} = \begin{pmatrix} E_0{}^{(0)} & E_0{}^{(a)} \\ 0 & E_i{}^{(a)} \end{pmatrix}, \tag{10}$$

and $\hat{A}^a_i$ is an operator associated with the self-dual connection $^{+}\omega_i{}^{(0a)} \equiv A^a_i$.

In the case of $(2+2)$-dimensions one has the constrains

$$H = \frac{1}{4}\tilde{E}\epsilon^{ijk}\, E_i{}^{(a)}\, {}^{+}R_{jk(0a)} = 0 \tag{11}$$

and

$$H_l = \frac{1}{4}\tilde{E}\epsilon^{ijk}\, \epsilon^{a}{}_{bc} E_i{}^{(b)} E_l{}^{(c)}\, {}^{+}R_{jk(0a)} = 0. \tag{12}$$

Here, $^{+}R_{jk(0a)}$ is a reduction to seven dimensions of $^{+}R^{AB}$ and

$$\epsilon^{ijk} = \frac{1}{\tilde{E}}\varepsilon^{ijk},$$

with $\varepsilon^{123} = 1$. Furthermore, $\tilde{E}$ is the determinant of $E_i^{\ (a)}$. Although these constrains have the same form as the case of $(1+3)$-signature there are important differences. First, the symbols ϵ^{ijk} refers to $(1+2)$- "spacetime" rather than to $(0+3)$. Second, ${}^+\omega_i^{\ (0a)} \equiv A_i^a$ will be $SU(1,1)$ gauge field rather than $SU(2)$. It turns out useful to change the notation in (11) and (12) by writing ${}^+R_{jk(0a)} \equiv F_{jka}$, so that F_{jk}^a can be identified with the curvature of A_i^a, $F = dA + A \wedge A$. We also write

$$P_a^i = \tilde{E} E_{(a)}^{\ i}. \tag{13}$$

Thus, in terms of F_{jk}^a and P_i^a the constraints (11) and (12) become (see Ref. 33 and references therein)

$$H = \frac{1}{4\sqrt{\det(P_a^i)}} P_a^i P_b^j \epsilon^{ab}{}_c \, F_{ij}^c = 0 \tag{14}$$

and

$$H_l = \frac{1}{2} P_a^i F_{li}^a = 0. \tag{15}$$

Here, we have used the identities $\epsilon^{ijk} E_i^{\ (a)} = \epsilon^{abc} E_b^{\ j} E_c^{\ k}$ and $\epsilon_{abc} E_i^{\ (b)} E_j^{\ (c)} = \epsilon_{ijk} E_{(a)}^{\ k}$ which can be derived from $\epsilon^{ijk} \epsilon_{abc} E_i^{\ (a)} E_j^{\ (b)} E_k^{\ (c)} = 1$.

The only non-vanishing Poisson bracket between the pair of canonical variables $A_i^a(x)$ and $P_a^i(y)$ is

$$\{A_j^a(x), P_b^i(y)\} = \delta_j^i \delta_b^a \delta(x,y). \tag{16}$$

One may assume that the physical states $|\Psi\rangle$ can be written in terms of a Wilson loop wave function

$$\Psi_\gamma(A) = trP \exp \int_\gamma A, \tag{17}$$

which satisfies the representation conditions

$$\hat{A}_i^a \Psi(A) = A_i^a \Psi(A),$$
$$\hat{P}_a^i \Psi(A) = \frac{\delta \Psi(A)}{\delta A_i^{\ a}}. \tag{18}$$

Here, the integral (17) is over the loop γ. If we want to go further and consider interactions one first needs to make finite computations. The strategy in this case is to decompose the loop γ in a finite number of edges e, in other words, one represents γ as a graph G. This allows us to write the function $\Psi_\gamma(A)$ as [1]

$$\Psi_\gamma(A) = \psi(h_{e_1}(A), ..., h_{e_m}(A)), \tag{19}$$

where h_e is holonomy along each edge e. However, in order to implement this strategy one needs to complete the computations by considering all possible graphs G. It is worth mentioning that the program of considering Wilson loops for a gauge field A associated with a noncompact group $S(1,2)$ has already been considered in the context of $(1+2)$-dimensional gravity (see Ref. 25). Of course, $(2+2)$-dimensional gravity, with gauge group $SU(1,1)$, is different theory, but at least in both cases the gauge field A can be associated with the noncompact group $SO(1,2)$. One can even think in a connection between the two theories by assuming a compactification of one of the time dimensions in the $(2+2)$-gravitational theory.

In the case of 8-dimensions, one has that the classical constraints H and H_l are given by [7]

$$H = \frac{1}{4} \tilde{E} \eta^{ijk} \, E_i^{\ (a)} \, {}^+R_{jk(0a)} = 0 \tag{20}$$

and

$$H_l = \frac{1}{4} \tilde{E} \eta^{ijk} \, \eta^a{}_{bc} E_i^{\ (b)} E_l^{\ (c)} \, {}^+R_{jk(0a)} = 0. \tag{21}$$

It can be expected that these constraints can also be written in the form

$$H = \frac{1}{4\sqrt{\det(P_a^i)}} P_a^i P_b^j \epsilon^{ab}{}_c \, F_{ij}^c = 0 \tag{22}$$

and

$$H_l = \frac{1}{2} P_a^i F_{li}^a = 0. \tag{23}$$

However, one should be careful in this case with the meaning of the determinant $\det(P_a^i)$ because now it is defined in terms of the octonion structure constant η^{ijk} and η^{abc} rather than in terms of ϵ^{ijk} and ϵ^{abc}. In this case one can choose A_i^a as a $spin(7)$ gauge field. The formulae (16-19) for the pair of canonical variables $A_i^a(x)$ and $P_a^i(y)$ also applies to this case. This means that in 8-dimensions one can also use graph theory to make finite computations. Of course the topology of a given 7-dimensional manifold is more complicated (see Ref. 34) than in 4-dimensions. Nevertheless, one should expect that a more rich structure may emerge beyond graph theory. For instance, one may look for physical states in terms of the analogue of the Chern-Simons states in 4-dimensions [35]. The reason for this is because Chern-Simons theory is linked to instantons in 4-dimensions via the topological term

$$\int_{M^4} tr \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta},$$

while in 8-dimensions the topological term should be of the form

$$\int_{M^8} tr \eta^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

which can be related with G_2-instantons (see [36] and references therein).

Thus in both cases, in $(2+2)$-dimensions and 8-dimensions, the loop quantum gravity approach [37-41] indicates that it is necessary for computations to use directed graph formalism. But a directed graph G is a particular case of a oriented matroid $\mathcal{M}$. So one may expect that oriented matroid theory may play an important mathematical tool in new developments on this program. And, in fact, this seems to have been recently confirmed [42]. However, we believe that the importance of oriented matroid theory in loop quantum gravity should be extended beyond graph theory. The reason for this expectation comes from a number of previous connections between matroids and different scenarios [43-47], including Chern-Simons theory, superstrings, p-branes and M-theory. In the process we have even develop the idea of the gravitoid [47] which refers to any connection between matroids and gravitons. In all these cases, the main motivation is the search for a duality principle underlying M-theory. Oriented matroid theory seems to provide the mathematical tool necessary for this goal, since one of its central topics is precisely duality. In fact, we have proposed [43] the oriented matroid theory as the mathematical framework for M-theory. In the case of loop quantum gravity similar duality motivation can be considered. This idea emerges natural since we have proved that in $(2+2)$ and 8-dimensions, oriented matroid theory is linked to loop quantum gravity at both levels, namely the constraints operators (Heisenberg-like approach) and physical states (Schrödinger-like approach). Since one can associate with every oriented matroid $\mathcal{M}$ a dual matroid $\mathcal{M}^*$. One should expect that duality also plays a central role in loop quantum gravity. Let us outline how this can be accomplish. The following arguments are, in fact, true for any of the dimensions $1, 2, 3$ or 8 and any of the corresponding signatures.

We shall be brief in our comments (see the Ref. 48 for details). Consider any graph G. Let B be the incidence matrix of G. One can introduce a pair of complementary subspaces $L, L^\perp$ in R^m, where m is the number of edges in G, which can be can be associated with B by the expressions $L = \ker B$ and $L^\perp = imB$. Indeed, L and $L^\perp$ corresponds to the circuit and cocircuit space of G. It turns out that L and $L^\perp$ satisfy the so called Farkas property: For every edge e in G either

a) $\exists X \in L, e \in$ supp $X, X \geqslant 0$ or

b) $\exists Y \in L^\perp, e \in$ supp $Y, Y \geqslant 0$ but not both.

Here, X and Y are the incidence vectors associated with a circuit and cocircuit respectively. Note that this property is self-dual in the sense that both alternatives a) and b) can be interchanging by replacing L by $L^\perp$. The central idea in oriented matroid theory is to generalize this property to any pair of signed sets (S, S'), with S' properly defined, such that (S, S') satisfies the analogue of the Farkas property. In fact, an oriented matroid can be defined in terms of the pair (S, S') and such a generalized Farkas property. One interesting thing is that given this definition of an oriented matroid one finds that there are oriented matroids which can not be realized as graphs. So the oriented matroid notion is a more general structure than the graph concept. Another interesting aspect of this construction of oriented matroids is that the two spaces L and $L^\perp$ (or S and S') are equally important. This is one of the reasons why every oriented matroid $\mathcal{M}$ has always a dual $\mathcal{M}^*$.

How this definition of an oriented matroid in terms of the Farkas property can be linked to loop quantum gravity? Let us assume that a physical state has the form

$$\Psi_C(A, L) = trP \exp \int_C A, \tag{24}$$

where C is a circuit of a given graph G. We write $\Psi_C(A, L)$ to emphasis that C is contained in the circuit space $L = \ker B$, with B the incidence matrix of G. But according to the Farkas property it must be equally important to consider the physical state,

$$\Psi_{C^*}(A^*, L^\perp) = trP \exp \int_{C^*} A^*. \tag{25}$$

Here, C^* is a cocircuit in $L^\perp$ and A^* is a dual gauge field. Observe that (25) completely dualize (24). This Schrödinger-like schema for $\Psi_C(A, L)$ and $\Psi_{C^*}(A^*, L^\perp)$ must have Heisenberg-like counterpart in terms of dual Hamiltonian operators constraints. In principle these dual Hamiltonian constraints can be $\hat{H}$ and $\hat{H}_l$ themselves. However, in a more general scenario one must consider dual Hamiltonian operators constrains $\hat{H}^*$ and $\hat{H}_l^*$ acting on the physical states $|\Psi^*\rangle$ associated with $\Psi_{C^*}(A^*, L^\perp)$. In other words one must have the symbolic formulae

$$\hat{H}^*|\Psi^*\rangle = 0 \tag{26}$$

and

$$\hat{H}_l^*|\Psi^*\rangle = 0. \tag{27}$$

Going backwards the constraints operators $\hat{H}^*$ and $\hat{H}_l^*$ must come from classical constrains H^* and H_l^* which in turn should be possible to derive from a dual gravitational field E^* and dual connection ω^* via the corresponding self-dual curvature ${}^+R^{*AB}$. Note that we have distinguish between two different dualities in ${}^+R^{*AB}$. This is because we are considering the most general dual theory but at some level one should expect that both kind of dualities are related. In S-duality for linearized gravity [49], for instance, one starts with a curvature ${}^+R^{AB}$ and finds the dual curvature ${}^+W^{AB}$ which can be identified with ${}^+R^{*AB}$. Some of these ideas are under intensive research and we expect to report our results elsewhere.

It is worth mentioning that a possible connection between oriented matroid theory and Ashtekar formalism is mentioned in the Refs. 4 to 7. Further, in the literature (see [50-51] and references therein) exist a canonical approach of the $(2+2)$-imbedding, but this should be called $((1+1)+(0+2))$-imbedding since refers to the $(1+3)$-signature rather than to the case of 2-time and 2-space dimensions $((2+2)$-dimensions$)$ which we have been considered in this work. Nevertheless, it may be interesting for further research to see if there is a link between $((1+1)+(0+2))$-imbedding and $(2+2)$-loop quantum gravity.

Perhaps, the link between oriented matroid theory and loop quantum gravity may provide new fascinating insights into other contexts in which $(2+2)$-signature makes its appearance, including qubit-strings [52] and N=2 strings [53].

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