The Jones vector as a spinor and its representation on the Poincaré sphere

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It is shown that the two complex Cartesian components of the electric field of a monochromatic electromagnetic plane wave, with a temporal and spatial dependence of the form $e^{i(kz-\omega t)}$, form a SU(2) spinor that corresponds to a tangent vector to the Poincaré sphere representing the state of polarization and phase of the wave. The geometrical representation on the Poincaré sphere of the effect of some optical filters is reviewed. It is also shown that in the case of a partially polarized beam, the coherency matrix defines two diametrically opposite points of the Poincaré sphere.

Keywords: Jones vector; Poincaré sphere; polarization; spinors.

Se muestra que las dos componentes Cartesianas complejas del campo eléctrico de una onda plana electromagnética monocromática, con dependencia temporal y espacial de la forma $e^{i(kz-\omega t)}$, forma un espinor SU(2) que corresponde al vector tangente a la esfera de Poincaré que representa el estado de polarización y fase de la onda. Se revisa la representación geométrica en la esfera de Poincaré del efecto de algunos filtros ópticos. Se muestra también que en el caso de un haz parcialmente polarizado, la matriz de coherencia define dos puntos diametralmente opuestos de la esfera de Poincaré.

Descriptores: Vector de Jones; esfera de Poincaré; polarización; espinores.

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1. Introduction

In a recent paper [1] it has been shown that the (real) Cartesian components of the electric field of a monochromatic electromagnetic plane wave can be expressed in terms of a two-component SU(2) spinor, which specifies the amplitude, state of polarization, and phase of the wave in such a way that two real mutually orthogonal vectors made out of this spinor define the point of the Poincaré sphere corresponding to the state of polarization and a tangent vector to the Poincaré sphere that determines the phase of the wave. Furthermore, the inner product of the spinors corresponding to two of these waves with the same wavevector (which is related to the parallel transport of tangent vectors to the Poincaré sphere along a great circle arc [1, 2]), determines if the waves are in phase according to Pancharatnam's definition [3]. (The relationship between the inner product of spinors and the parallel transport along geodesics of the sphere was already recognized in Payne's 1952 paper [2], without developing, however, its relationship with the interference of electromagnetic waves. See also Ref. 4.)

The fact that the amplitude, state of polarization, and phase of a monochromatic electromagnetic plane wave can be represented by a two-component spinor allows us to derive many useful relations employing the same formalism as in Quantum Mechanics [1], instead of the not so widely known results of spherical trigonometry [3] (see also Ref. 5).

The state of polarization of a wave is usually specified making use of the Stokes parameters or the Jones vector (see, *e.g.*, Refs. 6–11). The Stokes parameters can be expressed in terms of the two-component spinor mentioned above [1] and, as we shall show below, the Jones vector is essentially this spinor, expressed in an appropriate basis.

In Sec. 2 we give a summary of the relevant results of Ref. 1, relating them with the definition of the Jones vector. We show that, apart from the phase factor that gives the time and space dependence of the electric field, the Jones vector is a two-component spinor on which the rotations on the Poincaré sphere act through the spin-1/2 representation. In Sec. 3 we review the effect of some optical filters and its geometrical representation on the Poincaré sphere. We show that the effect of a phase shifter corresponds to a rotation of the Poincaré sphere, while that of an attenuator corresponds to a conformal transformation of this sphere (see also Refs. 10 and 11). In Sec. 4 we consider partially polarized beams, showing that the Stokes parameters can be arranged into a 2×2 matrix that, except in the case of unpolarized light, defines two diametrically opposite points of the Poincaré sphere.

Although some of the results obtained in this paper, such as the matrix form for phase shifters and attenuators, are found in the literature using other approaches (see, *e.g.*, Ref. 12 and the references cited therein), one remarkable feature of the spinor formalism is that, besides the state of polarization represented by a point of the Poincaré sphere, we also have the phase of the wave through the direction of a tangent vector to the sphere at that point, which is not included in other approaches. Thus the action of the optical filters are transformations not only on the points of the Poincaré sphere, but also on the tangent vectors to this sphere.

2. The Poincaré sphere

The Cartesian components of the electric field of a monochromatic electromagnetic plane wave propagating in the z-direction in a dielectric medium are usually expressed in the form

$$
\begin{aligned}
E_x &= \mathrm{Re}\{A_1 \exp[\mathrm{i}(kz - \omega t + \phi_1)]\}, \\
E_y &= \mathrm{Re}\{A_2 \exp[\mathrm{i}(kz - \omega t + \phi_2)]\},
\end{aligned} \tag{1}
$$

where A_1, A_2 are real, positive constants, ω and k are the angular frequency and wave number of the wave, respectively. At each point of space, the resulting electric field describes an ellipse centered at the origin and, therefore, the (real) electric field can be conveniently written as

$$
\begin{aligned}
\mathbf{E} = {} & \left[a \cos \tfrac{1}{2}\phi \, \cos(\omega t - kz + \tfrac{1}{2}\chi)\right. \\
& \left. - b \sin \tfrac{1}{2}\phi \sin(\omega t - kz + \tfrac{1}{2}\chi)\right] \hat{x} \\
& + \left[a \sin \tfrac{1}{2}\phi \cos(\omega t - kz + \tfrac{1}{2}\chi)\right. \\
& \left. + b \cos \tfrac{1}{2}\phi \sin(\omega t - kz + \tfrac{1}{2}\chi)\right] \hat{y},
\end{aligned} \tag{2}
$$

where a, b are real constants, with $|a| \geqslant |b|$, $|a|$ is the major semiaxis of the ellipse, $|b|$ is the minor semiaxis, and $\phi/2$ is the angle made by the major axis of the ellipse with the x-axis, so that it suffices to consider values of ϕ between 0 and 2π. The phase $\chi/2$ is necessary when one considers the superposition of two or more waves [1].

Since $|b/a| \leqslant 1$, for each value of the ellipticity, b/a, there is a unique $\theta \in [0, \pi]$ such that

$$
\frac{b}{a} = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right).
$$

Hence,

$$
\begin{aligned}
a &= \sqrt{2}\, A \cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = A\left(\cos\frac{1}{2}\,\theta + \sin\frac{1}{2}\,\theta\right), \\
b &= \sqrt{2}\, A \sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = A\left(\cos\frac{1}{2}\,\theta - \sin\frac{1}{2}\,\theta\right),
\end{aligned} \tag{3}
$$

for some constant A, which, with no loss of generality, we can assume positive. In this way, $a \geqslant 0$, while b is positive for $0 \leqslant \theta < \pi/2$ (in which case the wave has right-hand polarization) and b is negative for $\pi/2 < \theta \leqslant \pi$ (then the wave has left-hand polarization). The values $\theta = 0$ and $\theta = \pi$ correspond to circular polarization, while $\theta = \pi/2$ in the case of linear polarization. Making use of Eq. (3), Eq. (2) can be rewritten in the form

$$
\begin{aligned}
\mathbf{E} = A\Bigg\{ & \left[\cos\frac{1}{2}\,\theta \cos\left(\omega t - kz + \frac{1}{2}\chi + \frac{1}{2}\phi\right)\right. \\
& \left. + \sin\frac{1}{2}\,\theta \cos\left(\omega t - kz + \frac{1}{2}\chi - \frac{1}{2}\phi\right)\right] \hat{x} \\
& + \left[\cos\frac{1}{2}\,\theta \sin\left(\omega t - kz + \frac{1}{2}\chi + \frac{1}{2}\phi\right)\right. \\
& \left. - \sin\frac{1}{2}\,\theta \sin\left(\omega t - kz + \frac{1}{2}\chi - \frac{1}{2}\phi\right)\right] \hat{y} \Bigg\}.
\end{aligned} \tag{4}
$$

The parametrization of the electric field given by Eq. (4) contains the same number of independent parameters as expressions (1) (four real parameters). However, by contrast with (1), the parameters appearing in Eq. (4) specify more directly the polarization state of the wave [via Eqs. (3)]. Furthermore, by considering the angles θ and ϕ as spherical coordinates in the usual manner (*i.e.*, θ as the polar angle and ϕ as the azimuthal angle), each pair of values (θ, ϕ) defines a point of the Poincaré sphere [6–8].

Another set of parameters commonly employed to specify the polarization of a wave is given by the Stokes parameters, s_0, s_1, s_2, s_3, which are related to the angles θ and ϕ by means of [6, 7] (see also Ref. 3 and the references cited therein)

$$
(s_1, s_2, s_3) = s_0(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{5}
$$

where s_0 is the total flux density. Hence, $(s_1, s_2, s_3)/s_0$ is the point of the Poincaré sphere that corresponds to the polarization of the wave.

In the Jones formalism, the *complex* Cartesian components of the electric field form a column matrix (see, *e.g.*, Ref. 11 and the references cited therein),

$$
\begin{pmatrix} E_x^{\mathrm{c}} \\ E_y^{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} A \exp[\mathrm{i}(kz - \omega t + \phi_1)] \\ B \exp[\mathrm{i}(kz - \omega t + \phi_2)] \end{pmatrix}, \tag{6}
$$

where A and B are real constants. (We employ the superscript c in the components of the electric field to emphasize the fact that they are complex.)

2.1. Two-component spinors

From Eq. (4) we see that the components of the electric field are given by the compact expression

$$
\begin{aligned}
E_x + \mathrm{i}E_y = A\Bigg[& \cos\frac{1}{2}\,\theta\, \mathrm{e}^{\mathrm{i}(\omega t - kz + \chi/2 + \phi/2)} \\
& + \sin\frac{1}{2}\,\theta\, \mathrm{e}^{-\mathrm{i}(\omega t - kz + \chi/2 - \phi/2)}\Bigg],
\end{aligned} \tag{7}
$$

or, in terms of the unit two-component spinor

$$
o = \begin{pmatrix} o^1 \\ o^2 \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\chi/2} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} \cos\frac{1}{2}\theta \\ \mathrm{e}^{\mathrm{i}\phi/2} \sin\frac{1}{2}\theta \end{pmatrix}, \tag{8}
$$

we have

$$E_x + \mathrm{i}E_y = A(\mathrm{e}^{\mathrm{i}(\omega t - kz)}\overline{o^1} + \mathrm{e}^{-\mathrm{i}(\omega t - kz)}o^2), \tag{9}$$

where the bar denotes complex conjugation.

The two-component spinor (8) may be familiar from Quantum Mechanics; it is the normalized eigenspinor with eigenvalue $+\hbar/2$ of the spin projection along the direction with angles θ, ϕ. The unit spinor o defines two mutually orthogonal vectors with Cartesian components

$$R_i \equiv o^\dagger \sigma_i o, \qquad M_i \equiv o^{\mathrm{t}} \varepsilon \sigma_i o, \tag{10}$$

where $o^\dagger$ is the transpose conjugate of o, o^{t} denotes the transpose of o, the σ_i are the standard Pauli matrices, and

$$\varepsilon \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

[2,13]. The vector R_i is real and is the point of the Poincaré sphere that represents the polarization state of the wave, *i.e.*, $(R_1, R_2, R_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Hence, the Stokes parameters are directly related to the unit spinor o by

$$\frac{s_i}{s_0} = o^\dagger \sigma_i o. \tag{11}$$

The direction of $\operatorname{Re} M_i$ does depend on the phase χ and, therefore, R_i together with $\operatorname{Re} M_i$ represent the state of polarization and the phase of the wave [1]. Since $\operatorname{Re} M_i$ is orthogonal to R_i, $\operatorname{Re} M_i$ is a tangent vector to the Poincaré sphere ($\operatorname{Re} M_i$ forms an angle χ with the meridian passing through the point R_i). In this manner, the vector R_i, gives the point of the Poincaré sphere corresponding to the polarization state of the wave, and $\operatorname{Re} M_i$ can be viewed as a tangent vector to the Poincaré sphere, whose direction gives the phase of the wave.

If $o' = Qo$, with $Q \in \mathrm{SU}(2)$, then o' is also a unit spinor and the vectors R'_i and M'_i, defined by o', are related to R_i and M_i, respectively, by means of the SO(3) transformation, (a_{ij}), given by

$$Q^\dagger \sigma_i Q = \sum_{j=1}^{3} a_{ij}\sigma_j;$$

that is

$$R'_i = \sum_{j=1}^{3} a_{ij} R_j,$$

and

$$M'_i = \sum_{j=1}^{3} a_{ij} M_j.$$

Hence, each $Q \in \mathrm{SU}(2)$ gives rise to a rotation on the Poincaré sphere. Conversely, given a rotation on the Poincaré sphere, there exists a $Q \in \mathrm{SU}(2)$, defined up to sign, corresponding to the rotation.

2.2. Two spinor bases

Since the components E_x and E_y appearing in Eq. (9) are real, Eq. (9) is equivalent to

$$E_x - \mathrm{i}E_y = A\big(\mathrm{e}^{\mathrm{i}(\omega t - kz)}\overline{o^2} + \mathrm{e}^{-\mathrm{i}(\omega t - kz)}o^1\big). \tag{12}$$

Hence, from Eqs. (9) and (12) we see that

$$\begin{aligned} E_x &= \operatorname{Re}\{A[\mathrm{e}^{\mathrm{i}(kz - \omega t)}(o^1 + o^2)]\}, \\ E_y &= \operatorname{Re}\{A[\mathrm{e}^{\mathrm{i}(kz - \omega t)}(\mathrm{i}o^1 - \mathrm{i}o^2)]\} \end{aligned} \tag{13}$$

and, therefore, the components of the electric field are the real part of the *complex* functions E_x^{c}, E_y^{c}, given by the Jones vector

$$\begin{pmatrix} E_x^{\mathrm{c}} \\ E_y^{\mathrm{c}} \end{pmatrix} = \sqrt{2}\mathrm{e}^{-\mathrm{i}\pi/4} A \mathrm{e}^{\mathrm{i}(kz - \omega t)} \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} o^1 \\ o^2 \end{pmatrix} \tag{14}$$

[*cf.* Eq. (6)].

One can readily verify that the 2×2 matrix

$$\mathcal{U} \equiv \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+\mathrm{i} & 1+\mathrm{i} \\ -1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix}, \tag{15}$$

appearing in Eq. (14), belongs to SU(2) and that

$$\mathcal{U}\sigma_1\mathcal{U}^{-1} = \sigma_3, \quad \mathcal{U}\sigma_2\mathcal{U}^{-1} = \sigma_1, \quad \mathcal{U}\sigma_3\mathcal{U}^{-1} = \sigma_2. \tag{16}$$

This means that $\mathcal{U}$ corresponds to a SO(3) transformation that permutes the coordinate axes, X, Y, Z, of the Poincaré sphere and that, apart from the factor

$$\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\pi/4} A \mathrm{e}^{\mathrm{i}(kz - \omega t)},$$

the Jones vector

$$\begin{pmatrix} E_x^{\mathrm{c}} \\ E_y^{\mathrm{c}} \end{pmatrix}$$

is essentially the two-component spinor

$$\begin{pmatrix} o^1 \\ o^2 \end{pmatrix}$$

in a different basis. That is, letting

$$\begin{pmatrix} \tilde{o}^1 \\ \tilde{o}^2 \end{pmatrix} \equiv \mathcal{U} \begin{pmatrix} o^1 \\ o^2 \end{pmatrix}, \tag{17}$$

from Eq. (14) we have

$$\begin{pmatrix} E_x^{\mathrm{c}} \\ E_y^{\mathrm{c}} \end{pmatrix} = \sqrt{2}\,\mathrm{e}^{-\mathrm{i}\pi/4} A \mathrm{e}^{\mathrm{i}(kz - \omega t)} \begin{pmatrix} \tilde{o}^1 \\ \tilde{o}^2 \end{pmatrix}. \tag{18}$$

While the basis spinors

$$\begin{pmatrix} o^1 \\ o^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} o^1 \\ o^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{19}$$

(which correspond to $\theta = 0$ and $\theta = \pi$, respectively) represent circularly polarized waves, the basis spinors

$$\begin{pmatrix} \tilde{o}^1 \\ \tilde{o}^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{o}^1 \\ \tilde{o}^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{20}$$

represent linearly polarized waves [see Eq. (18)] and correspond to the points of the Poincaré sphere on the X-axis (see Eq. (21) below). Thus, the SU(2) matrix $\mathcal{U}$, given by Eq. (15), represents the connection between these two frequently employed bases of polarization states (see also Sec. 3.1, below).

Equations (9) and (12) constitute a decomposition of a wave as a superposition of circularly polarized waves, with the components o^1 and o^2 being the relative amplitudes of this decomposition. Similarly, $\tilde{o}^1$ and $\tilde{o}^2$ are the relative amplitudes of the decomposition of the wave as a superposition of two linearly polarized waves. (In fact, *any* pair of different points of the Poincaré sphere represent a basis; the pairs of points diametrically opposite are the orthogonal bases [1, 3].)

According to Eq. (17), the vectors R_i and M_i are given in terms of the spinor $\tilde{o}$ by [see Eqs. (10)]

$$R_i = \tilde{o}^\dagger \mathcal{U}\sigma_i\mathcal{U}^{-1}\tilde{o}, \qquad M_i = \tilde{o}^{\mathrm{t}}\varepsilon\mathcal{U}\sigma_i\mathcal{U}^{-1}\tilde{o}, \tag{21}$$

where we have made use of the relation $(\mathcal{U}^{-1})^{\mathrm{t}}\varepsilon = \varepsilon\mathcal{U}$, which applies to unimodular 2×2 matrices. Equations (21) are of the same form as Eqs. (10), with o replaced by $\tilde{o}$ and σ_i replaced by $\mathcal{U}\sigma_i\mathcal{U}^{-1}$. As shown in Eqs. (16), the matrices $\mathcal{U}\sigma_i\mathcal{U}^{-1}$ are a cyclic permutation of the Pauli matrices (which explains the definition of the Pauli matrices adopted, without justification, in Ref. 11, Appendix B).

Thus, apart from the factor $\mathrm{e}^{\mathrm{i}(kz-\omega t)}$, the components of the Jones vector (6) are the components of a constant SU(2) spinor (that is, independent of t and z), $\tilde{o}$, in a basis that differs from the standard one [Eq. (18)]. The unit spinor $\tilde{o}$ allows us to find the vectors R_i and $\operatorname{Re} M_i$ that represent the polarization state and phase of the wave on the Poincaré sphere [Eqs. (21) and (16)] and, since the inner product of SU(2) spinors is invariant under SU(2) transformations, the inner product of the spinors corresponding to two waves with the same wavevector determines if the waves are in phase according to Pancharatnam's definition [1, 3] (see also Ref. 14 and the references cited therein).

3. Geometrical representation of the effect of optical filters

Since the state of polarization of a monochromatic electromagnetic plane wave is represented by a point of the Poincaré sphere or, up to a phase factor, by a unit two-component spinor, *e.g.*, o or $\tilde{o}$, the effect of an optical filter on the polarization of a wave passing through the filter corresponds to some transformation of the Poincaré sphere into itself or to some spinor transformation (see also Refs. 11 and 9).

In this section, following Ref. 11, we consider some simple examples of optical filters, finding their representation on the spinor space and on the Poincaré sphere.

3.1. Phase shifters

If an optical filter produces a phase shift δ_1 for the x-component of the electric field and a, possibly different, phase shift δ_2 for the y-component, the electric field (4) is replaced by

$$\begin{aligned}\mathbf{E} = A\Big\{ &\left[\cos\frac{1}{2}\theta\cos(\omega t - kz + \frac{1}{2}\chi + \frac{1}{2}\phi + \delta_1) + \sin\frac{1}{2}\theta\,\cos(\omega t - kz + \frac{1}{2}\chi - \frac{1}{2}\phi + \delta_1)\right]\hat{x} \\ &+ \left[\cos\frac{1}{2}\theta\,\sin(\omega t - kz + \frac{1}{2}\chi + \frac{1}{2}\phi + \delta_2) - \sin\frac{1}{2}\theta\,\sin(\omega t - kz + \frac{1}{2}\chi - \frac{1}{2}\phi + \delta_2)\right]\hat{y}\Big\}.\end{aligned} \tag{22}$$

This expression is equivalent to

$$\begin{aligned}E_x + \mathrm{i}E_y = A\Big\{ &\mathrm{e}^{\mathrm{i}(\omega t - kz + (\delta_1+\delta_2)/2)}\left[\cos\frac{1}{2}\delta\cos\frac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}(\chi+\phi)/2} - \mathrm{i}\sin\frac{1}{2}\delta\sin\frac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}(\chi-\phi)/2}\right] \\ &+ \mathrm{e}^{-\mathrm{i}(\omega t - kz + (\delta_1+\delta_2)/2)}\left[\cos\frac{1}{2}\delta\sin\frac{1}{2}\theta\mathrm{e}^{-\mathrm{i}(\chi-\phi)/2} + \mathrm{i}\sin\frac{1}{2}\delta\cos\frac{1}{2}\theta\,\mathrm{e}^{-\mathrm{i}(\chi+\phi)/2}\right]\Big\},\end{aligned}$$

which is duly of the form (9), with the two-component spinor o replaced by

$$\begin{pmatrix} o'^1 \\ o'^2 \end{pmatrix} = \mathrm{e}^{-\mathrm{i}(\delta_1+\delta_2)/2}\begin{pmatrix} \cos\frac{1}{2}\delta & \mathrm{i}\sin\frac{1}{2}\delta \\ \mathrm{i}\sin\frac{1}{2}\delta & \cos\frac{1}{2}\delta \end{pmatrix}\begin{pmatrix} o^1 \\ o^2 \end{pmatrix}, \tag{23}$$

where $\delta \equiv \delta_2 - \delta_1$.

Apart from the overall phase factor $\mathrm{e}^{-\mathrm{i}(\delta_1+\delta_2)/2}$, the transformation (23) is given by the SU(2) matrix

$$\begin{pmatrix} \cos\frac{1}{2}\delta & \mathrm{i}\sin\frac{1}{2}\delta \\ \mathrm{i}\sin\frac{1}{2}\delta & \cos\frac{1}{2}\delta \end{pmatrix} = \left(\cos\frac{1}{2}\delta\right) I + \mathrm{i}\left(\sin\frac{1}{2}\delta\right)\sigma_1 = \exp\left(\mathrm{i}\frac{1}{2}\delta\sigma_1\right), \tag{24}$$

where I is the 2×2 identity matrix, which corresponds to a rotation on the Poincaré sphere through an angle $-\delta$ about the X-axis.

There exist two diametrically opposite points of the Poincaré sphere that are invariant under this rotation (the points on the intersection of the Poincaré sphere and the X-axis), which, therefore, correspond to polarization states that are not affected by this filter. These two polarization states are linearly polarized waves with the electric field along the x-axis or the y-axis [the states (20)], as one would expect. (Note that, owing to the definition of the angle ϕ given in Sec. 2, a rotation of the coordinate axes in the xy-plane through an angle α produces the substitution of $\phi/2$ by $(\phi/2) - \alpha$, which corresponds to the action of the matrix

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} = \exp(i\alpha\,\sigma_3) \tag{25}$$

on the spinor o. This SU(2) matrix, in turn, corresponds to a rotation on the Poincaré sphere through an angle -2α about the Z-axis. Thus, a rotation by $90°$ in the xy-plane, which transforms a linear polarization along the x-axis into a linear polarization along the y-axis, corresponds to a rotation by $180°$ in the Poincaré sphere.)

According to Eqs. (16), with respect to the basis (20), formed by linearly polarized states, the spinor transformation (23) is given by the unitary matrix

$$\begin{aligned} e^{-i(\delta_1+\delta_2)/2} \exp\left(i\frac{1}{2}\delta\sigma_3\right) &= e^{-i(\delta_1+\delta_2)/2} \begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\delta_1} & 0 \\ 0 & e^{-i\delta_2} \end{pmatrix}, \end{aligned} \tag{26}$$

as one would expect, owing to the definition of δ_1 and δ_2.

In order to reduce the possible confusions coming from the simultaneous use of two different bases, it is convenient to make use of Dirac's notation, denoting by $|+\rangle$ and $|-\rangle$ the states with circular polarization (19), respectively. Then,

$$\begin{aligned} |x\rangle &\equiv \frac{1}{\sqrt{2}} e^{-i\pi/4}|+\rangle + \frac{1}{\sqrt{2}} e^{-i\pi/4}|-\rangle, \\ |y\rangle &\equiv -\frac{1}{\sqrt{2}} e^{i\pi/4}|+\rangle + \frac{1}{\sqrt{2}} e^{i\pi/4}|-\rangle, \end{aligned} \tag{27}$$

correspond to states with linear polarization (the states (20), which are essentially the states $|\mathsf{v}\rangle$ and $|\mathsf{h}\rangle$ with vertical and horizontal polarization employed in Ref. 5). (See Eq. (15).) In this manner, the SU(2) transformation (24) is expressed as

$$\left(\cos\frac{1}{2}\delta\right) I + i\left(\sin\frac{1}{2}\delta\right) (|+\rangle\langle-| + |-\rangle\langle+|),$$

which, by virtue of Eqs. (27), amounts to

$$|x\rangle e^{i\delta/2}\langle x| + |y\rangle e^{-i\delta/2}\langle y| \tag{28}$$

and corresponds to the diagonal matrix $\mathrm{diag}(e^{i\delta/2}, e^{-i\delta/2})$ appearing in Eq. (26).

The effect represented by the SU(2) transformation (28) comes from the anisotropy of the medium, which produces different effects on the linearly polarized waves with electric field along the x-axis or the y-axis. In an analogous manner, a *gyrotropic* medium (see, *e.g.*, Ref. 15) produces different effects on the waves with right or left circular polarization; therefore, the effect of a gyrotropic medium is represented by

$$\begin{aligned} |+\rangle e^{-i\delta_1}\langle+| + |-\rangle e^{-i\delta_2}\langle-| &= e^{-i(\delta_1+\delta_2)/2} \\ &\quad \times \left(|+\rangle e^{i\delta/2}\langle+| + |-\rangle e^{-i\delta/2}\langle-|\right), \end{aligned}$$

where $\delta \equiv \delta_2 - \delta_1$, or by the unitary matrix

$$e^{-i(\delta_1+\delta_2)/2} \exp\left(i\frac{1}{2}\delta\,\sigma_3\right),$$

which corresponds to a rotation on the Poincaré sphere through an angle $-\delta$ about the Z-axis.

Hence, with respect to the basis (20), formed by states with linear polarization, making use of Eqs. (16) or (27), the effect of a gyrotropic medium will be represented by a matrix of the form

$$\begin{aligned} e^{-i(\delta_1+\delta_2)/2} \exp\left(i\frac{1}{2}\delta\sigma_2\right) &= e^{-i(\delta_1+\delta_2)/2} \\ &\quad \times \begin{pmatrix} \cos\frac{1}{2}\delta & -\sin\frac{1}{2}\delta \\ \sin\frac{1}{2}\delta & \cos\frac{1}{2}\delta \end{pmatrix}. \end{aligned} \tag{29}$$

A quarter-wave plate [9] is a phase shifter corresponding to a rotation on the Poincaré sphere through $\pi/2$ about an axis on the XY-plane. Hence, with respect to the basis $\{|+\rangle, |-\rangle\}$, it is represented by the SU(2) matrix

$$\begin{aligned} &(\cos\pi/4)I - i(\sin\pi/4)[(\cos 2\theta)\,\sigma_1 + (\sin 2\theta)\,\sigma_2] \\ &\qquad = \frac{1}{\sqrt{2}}[I - i(\cos 2\theta)\,\sigma_1 - i(\sin 2\theta)\,\sigma_2], \end{aligned}$$

where θ is the angle between the axis of the plate and the x-axis [see the discussion after Eq. (25)], and, according to Eqs. (16), with respect to the basis $\{|x\rangle, |y\rangle\}$, it is represented by

$$\frac{1}{\sqrt{2}}[I - i(\cos 2\theta)\,\sigma_3 - i(\sin 2\theta)\,\sigma_1].$$

A half-wave plate is a phase shifter corresponding to a rotation on the Poincaré sphere through π about an axis on the XY-plane and, therefore, is represented by the square of the matrix corresponding to a quarter-wave plate.

3.2. Attenuators

In the case of an optical filter that produces an attenuation given by a factor $e^{-\eta_1}$ for the x-component of the electric field and an attenuation given by $e^{-\eta_2}$ for the y-component, the electric field (4) is replaced by

$$\mathbf{E} = A\Big\{\mathrm{e}^{-\eta_1}\left[\cos\frac{1}{2}\theta\,\cos\left(\omega t - kz + \frac{1}{2}\chi + \frac{1}{2}\phi\right) + \sin\frac{1}{2}\theta\,\cos\left(\omega t - kz + \frac{1}{2}\chi - \frac{1}{2}\phi\right)\right]\hat{x}$$
$$+\,\mathrm{e}^{-\eta_2}\left[\cos\frac{1}{2}\theta\,\sin\left(\omega t - kz + \frac{1}{2}\chi + \frac{1}{2}\phi\right) - \sin\frac{1}{2}\theta\,\sin\left(\omega t - kz + \frac{1}{2}\chi - \frac{1}{2}\phi\right)\right]\hat{y}\Big\}. \tag{30}$$

This expression is equivalent to

$$E_x + \mathrm{i}E_y = A\mathrm{e}^{-(\eta_1+\eta_2)/2}\Big\{\mathrm{e}^{\mathrm{i}(\omega t - kz)}\Big[\cosh\frac{1}{2}\eta\cos\frac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}(\chi+\phi)/2}$$
$$+\sinh\frac{1}{2}\eta\sin\frac{1}{2}\theta\mathrm{e}^{\mathrm{i}(\chi-\phi)/2}\Big] + \mathrm{e}^{-\mathrm{i}(\omega t - kz)}\Big[\cosh\frac{1}{2}\eta\sin\frac{1}{2}\theta\,\mathrm{e}^{-\mathrm{i}(\chi-\phi)/2} + \sinh\frac{1}{2}\eta\cos\frac{1}{2}\theta\,\mathrm{e}^{-\mathrm{i}(\chi+\phi)/2}\Big]\Big\},$$

which is of the form (9), with the two-component spinor o replaced by

$$\begin{pmatrix} o'^1 \\ o'^2 \end{pmatrix} = \mathrm{e}^{-(\eta_1+\eta_2)/2}\begin{pmatrix} \cosh\frac{1}{2}\eta & \sinh\frac{1}{2}\eta \\ \sinh\frac{1}{2}\eta & \cosh\frac{1}{2}\eta \end{pmatrix}\begin{pmatrix} o^1 \\ o^2 \end{pmatrix} = \mathrm{e}^{-(\eta_1+\eta_2)/2}\exp(\tfrac{1}{2}\eta\,\sigma_1)\begin{pmatrix} o^1 \\ o^2 \end{pmatrix}, \tag{31}$$

where $\eta \equiv \eta_2 - \eta_1$. The 2×2 matrix appearing in Eq. (31) is unimodular, but does not belong to SU(2) and, therefore, it does not correspond to a rotation on the Poincaré sphere. Rather, it corresponds to a conformal transformation of the sphere (see, *e.g.*, Ref. 16). In any case, the effect of the attenuator on the polarization state of a wave is represented by a transformation on the points of the Poincaré sphere.

Clearly, if there is an attenuation given by a factor $\mathrm{e}^{-\eta_1}$ for the x-component of the electric field and an attenuation given by $\mathrm{e}^{-\eta_2}$ for the y-component, the column matrix (6) is replaced by

$$\begin{pmatrix} E'_x \\ E'_y \end{pmatrix} = \mathrm{e}^{-(\eta_1+\eta_2)/2}\begin{pmatrix} \mathrm{e}^{\eta/2} & 0 \\ 0 & \mathrm{e}^{-\eta/2} \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix}, \tag{32}$$

and the non-unitary, unimodular matrix appearing in this last equation, which can be expressed as $\exp(\frac{1}{2}\eta\,\sigma_3)$, is exactly what we should expect taking into account Eqs. (31) and (16).

4. Partially polarized beams

As is often remarked, by contrast with the Jones vector, the Stokes parameters can also be used to deal with partially polarized beams. In this section we show that the two-component spinor formalism can be easily adapted to handle partially polarized light, and, as we shall see, the resulting description is equivalent to that given by the coherency matrix (*cf.* Ref. 11, Appendix B).

The Stokes parameters allow us to distinguish a completely polarized beam from a partially polarized beam. Letting

$$S \equiv {s_0}^2 - {s_1}^2 - {s_2}^2 - {s_3}^2, \tag{33}$$

it turns out that for a completely polarized beam, $S = 0$ [*cf.* Eq. (5)], while for a partially polarized beam, $S > 0$ (see, *e.g.*, Ref. 6, Sec. 10.8.3). The four Stokes parameters can be related to a 2×2 Hermitean matrix, C, by means of

$$s_\alpha = \mathrm{tr}\,(C\sigma_\alpha), \qquad (\alpha = 0,1,2,3) \tag{34}$$

where tr denotes the trace, $\sigma_0 \equiv I$, and σ_1, σ_2, σ_3, are the Pauli matrices, as above.

The Hermitean matrix $\rho \equiv C/s_0$ has the usual properties of a density matrix (or density operator) as defined in Quantum Mechanics (see, *e.g.*, Ref. 17), namely

$$\mathrm{tr}\,\rho = 1, \qquad \mathrm{tr}\,\rho^2 \leqslant 1. \tag{35}$$

In fact, Eqs. (34) (together with the condition $C^\dagger = C$) are equivalent to

$$C = \frac{1}{2}\begin{pmatrix} s_0 + s_3 & s_1 - \mathrm{i}s_2 \\ s_1 + \mathrm{i}s_2 & s_0 - s_3 \end{pmatrix} \tag{36}$$

that is,

$$C = \frac{1}{2}\sum_{\alpha=0}^{3} s_\alpha\sigma_\alpha \tag{37}$$

and one readily verifies that $\mathrm{tr}C{=}s_0$, and $\mathrm{tr}C^2{=}{s_0}^2 - S/2$, which amount to Eqs. (35), taking into account that $S \geqslant 0$.

Furthermore, $\det C = S/4$; hence, in the case of a completely polarized wave ($S = 0$), the matrix C, having determinant equal to zero, must be of the form $\psi\psi^\dagger$, where ψ is some two-component spinor. In fact, writing $C = s_0 oo^\dagger$, where o is a normalized spinor, we recover the ("pure state") case considered in Sec. 2. Indeed,

$$\mathrm{tr}\,(C\sigma_0) = s_0\mathrm{tr}\,(oo^\dagger) = s_0 o^\dagger o = s_0,$$

and

$$\mathrm{tr}\,(C\sigma_i) = s_0\mathrm{tr}\,(oo^\dagger\sigma_i) = s_0 o^\dagger\sigma_i o = s_i, \qquad (i = 1,2,3)$$

[see Eq. (11)], reproducing Eqs. (34).

The matrix C, being Hermitean, possesses two mutually orthogonal unit eigenspinors with real eigenvalues. These unit spinors correspond to two diametrically opposite points of the Poincaré sphere [see Ref. 1, Eq. (18)]. Since C is a

2×2 matrix, its two eigenvalues coincide only when C is a multiple of the identity matrix and, only in this case, which corresponds to "unpolarized" light ($s_1 = s_2 = s_3 = 0$), the direction of the eigenspinors of C is not uniquely defined. In all cases, the unit eigenspinors of C are defined up to a phase factor, hence, there are no uniquely defined tangent vectors to the Poincaré sphere at these points, analogous to the vector $\mathrm{Re}\,\mathbf{M}$ defined in Sec. 2.

Thus, in the case of a partially polarized beam (a "mixed state"), the polarization state defines two diametrically opposite points of the Poincaré sphere (except in the case of unpolarized light). However, these two points (which correspond to the eigenspinors of C) do not fully specify the matrix C, since the eigenvalues need to be known. According to the discussion in Sec. 2, the vectors $\pm(s_1, s_2, s_3)$ point along the directions of the points of the Poincaré sphere representing the partially polarized beam.

When the beam is completely polarized, C is of the form $C = s_0 o o^\dagger$; the unit spinor o is an eigenspinor of C ($Co = s_0 o o^\dagger o = s_0 o$) and any spinor orthogonal to o (*e.g.*, the mate of o [1]) is also an eigenspinor of C (with eigenvalue equal to zero).

As with any matrix, the form and properties of C depend on the basis employed. Fortunately, making use of Eq. (37), which gives C in terms of the Pauli matrices, and Eqs. (16), we can obtain at once the expression of C in the basis formed by the unit spinors (20); the resulting expression is

$$\widetilde{C} = \frac{1}{2}(s_0 I + s_1\sigma_3 + s_2\sigma_1 + s_3\sigma_2) = \frac{1}{2}\begin{pmatrix} s_0 + s_1 & s_2 - \mathrm{i}s_3 \\ s_2 + \mathrm{i}s_3 & s_0 - s_1 \end{pmatrix}. \tag{38}$$

Taking into account the relationship between the Stokes parameters and the elements of the coherency matrix, J_{ij} (see, *e.g.*, Ref. 6, Sec. 10.8.3), we have

$$\widetilde{C} = \begin{pmatrix} J_{xx} & J_{yx} \\ J_{xy} & J_{yy} \end{pmatrix}. \tag{39}$$

When a partially polarized beam passes through a phase shifter, the matrix C, corresponding to the initial beam, is replaced by $QCQ^\dagger$, where Q is the SU(2) matrix representing the effect of the filter on the state of polarization ($\exp(\mathrm{i}\frac{1}{2}\delta\,\sigma_1)$ or $\exp(\mathrm{i}\frac{1}{2}\delta\,\sigma_3)$ in the cases considered in Sec. 3.1; note that the factors $\mathrm{e}^{-\mathrm{i}(\delta_1+\delta_2)/2}$ appearing in Eqs. (23) and (29) are not present in $QCQ^\dagger$ because they have unit modulus). The eigenspinors of $QCQ^\dagger$ are the images under Q of those of C; therefore, the diametrically opposite points on the Poincaré sphere defined by $QCQ^\dagger$ are obtained from those defined by C by means of the rotation corresponding to Q (see also Refs. 18 and 19).

Similarly, when a partially polarized beam passes through an attenuator, the initial matrix C is transformed into

$$\mathrm{e}^{-(\eta_1+\eta_2)/2}\exp\left(\frac{1}{2}\eta\sigma_1\right) C \left[\mathrm{e}^{-(\eta_1+\eta_2)/2}\exp\left(\frac{1}{2}\eta\sigma_1\right)\right]^\dagger = \mathrm{e}^{-(\eta_1+\eta_2)}\exp\left(\frac{1}{2}\eta\sigma_1\right) C \exp\left(\frac{1}{2}\eta\sigma_1\right)$$

[see Eq. (31)], which is of the form (37), with (s_0, s_1, s_2, s_3) replaced by

$$\mathrm{e}^{-(\eta_1+\eta_2)}(s_0\cosh\eta + s_1\sinh\eta, s_1\cosh\eta + s_0\sinh\eta, s_2, s_3). \tag{40}$$

Thus, apart from the overall factor $\mathrm{e}^{-(\eta_1+\eta_2)}$, the effect of an attenuator on the Stokes parameters has the form of a Lorentz boost in the x-direction (see also Refs. 10 and 11).

5. Conclusions

We have shown that the several objects and formalisms employed in the study of the polarization of electromagnetic waves are deeply related, despite their apparent differences. In particular, the identification of the Jones vector with a SU(2) spinor, allows us to represent the Jones vector by a tangent vector to the Poincaré sphere, in terms of which, among other things, the Pancharatnam phase can be visualized.

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