WKB quantization for completely bound quadratic dissipative systems

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We study the energy quantization for completely bound dissipative systems over a full cycle of motion. We approach the problem by means of an effective phenomenological Hamiltonian and the WKB quantization rule to obtain the energy levels in the system. An example of this approach is given for the quantum bouncer with quadratic dissipation.

Keywords: WKB quantization rule; dissipative systems; semiclassical quantization.

Se estudia la cuantización de energía para sistemas completamente ligados con disipación cuadrática utilizando la teoría WKB. Se propone un nuevo Hamiltoniano efectivo que restaura la continuidad de la trayectoria en el espacio fase y permite obtener la cuantización de energía a través del área encerrada. Ilustramos nuestro método para el caso del rebotador cuántico con disipación.

Descriptores: Cuantización WKB; sistemas disipativos; cuantización semiclásica.

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1. Introduction

The study of quantum dissipative systems has been a topic of great interest because of its fundamental importance in real world applications [1]. In classical mechanics, the equations of motion for conservative systems, *i.e.* systems in which the sum of the kinetic energy K and potential energy U is constant, can be derived from a Hamiltonian function which represents the energy of the system in terms of generalized coordinates q and momenta p, and is used as a basis for the so-called canonical quantization to obtain the corresponding Hamiltonian operator and, thus, the Schrödinger equation for the system [2–5]. Unfortunately, the Schrödinger equation has exact solutions for a few simple systems and its therefore essential to develop special techniques for attacking more complex problems. The WKB method is a technique for obtaining approximate solutions to the time-independent Schrödinger equation in one dimension and it is particularly useful in calculating bound-state energies [6–8]. The WKB theory sheds light on how quantization occurs and it gives the right answer for the order of magnitude for the quantum involved [9].

For dissipative systems difficulties arise in the quantum formalism and even the WKB theory cannot be applied due to the non periodic orbits for this particular systems. Although the Hamiltonian function yielding the correct equations of motion can be quantized by solving the Schrödinger equation problems arise in the interpretation of the wave function [10, 11]. Alternative methods to study dissipation in quantum systems have been proposed, the main approach to address dissipation at the quantum level consists of the coupling between the quantum dissipative system and the environment, as a result, one brings about a master equation with the dissipation parameter included in the solution [1].

In this article a new approach to tackle quadratic dissipative systems is proposed. We will study quantum systems under a quadratic velocity dissipative field where only bound states are allowed, *i.e.* the particle is completely spatially confined for any energy. For this type of quantum systems an effective Hamiltonian will be proposed which not only yields the correct equations of motion for position and momentum but also restores the periodicity of the system and hence the WKB quantization rule can be applied. The article is organized as follows. First we will start by giving the general formulation of the theory and the effective Hamiltonian. Then we will apply our theory to the quantum bouncer with quadratic dissipation using the WKB approach to obtain the energies of the system. The conclusions are summarized in the last section.

2. Effective Hamiltonian

Let us consider the motion of a particle of mass m in one dimension subject both to a field U depending only on the coordinates and a frictional field proportional to the square of the particle's velocity. The equation of motion for the system is given by

$$m\frac{dv}{dt} = -\frac{dU}{dx} - \gamma(x)v|v| = \begin{cases} -\frac{dU}{dx} - \gamma(x)v^2 & \text{if} \quad v > 0 \\ -\frac{dU}{dx} + \gamma(x)v^2 & \text{if} \quad v < 0, \end{cases} \tag{1}$$

where $\gamma(x) > 0$ is the dissipation parameter and might depend on the particle's coordinate. Equation (1) can be rewritten as

$$\frac{m}{2}\frac{d}{dx}(v_\pm^2) = -\frac{dU}{dx} \mp \tilde{\gamma}(x)\frac{m}{2}v_\pm^2, \tag{2}$$

where $\tilde{\gamma}(x) = 2\gamma(x)/m$ and $v_\pm$ represents the velocity of the particle when $v > 0$ and $v < 0$, respectively. The general solution to Eq. (2) is given by

$$\frac{m}{2}v_\pm^2 + U = e^{\mp\int\tilde{\gamma}(x)dx}\left(C + \int U\tilde{\gamma}(x)e^{\pm\int\tilde{\gamma}(x)dx}dx\right), \tag{3}$$

where C is an integration constant that depends on the initial conditions. Note that when $\gamma = 0$ the sum of the kinetic energy and potential energy is constant. Solving Eq. (3) for $v_\pm^2$ and inserting the result into Eq. (1) we obtain the following equations of motion of the system

$$m\frac{dv_+}{dt} = -\frac{dU}{dx} - \tilde{\gamma}(x)e^{-\int\tilde{\gamma}(x)dx}\left(C + \int U\tilde{\gamma}(x)e^{\int\tilde{\gamma}(x)dx}dx\right), \tag{4}$$

$$m\frac{dv_-}{dt} = -\frac{dU}{dx} + \tilde{\gamma}(x)e^{\int\tilde{\gamma}(x)dx}\left(C + \int U\tilde{\gamma}(x)e^{-\int\tilde{\gamma}(x)dx}dx\right). \tag{5}$$

From Eq. (4) and Eq. (5) we can construct an effective Hamiltonian in the usual way since the force is now only a function of the coordinate. Let $H_\pm$ represent the Hamiltonian for the case when $v > 0$ and $v < 0$, respectively, then the effective Hamiltonian of the system can be written as

$$H_{eff} = \frac{H_+ + H_-}{2} + \frac{p}{|p|}\left(\frac{H_+ - H_-}{2}\right). \tag{6}$$

For the case of weak dissipation, *i.e.* $H_+ \approx H_-$, and for a full cycle of motion we can drop the last term in Eq. (6).

We end this section by considering the problem studied in Ref. 12, which corresponds to the motion of a particle of mass m dropped a distance d above the surface of the Earth and we consider that during its motion there is a friction force which is proportional to the square of the particle's velocity. The equation of motion which describes the dynamics of the particle is given by

$$m\frac{dv}{dt} = -mg - \gamma v|v| = \begin{cases} -mg - \gamma v^2 & \text{if} \quad v > 0 \\ -mg + \gamma v^2 & \text{if} \quad v < 0, \end{cases} \tag{7}$$

where $\gamma > 0$ is the dissipation parameter. The system given in Eq. (7) is a particular case of Eq. (1) where $U = mgx$ and $\gamma(x) = \gamma > 0$. Using Eq. (3) we have the square of the velocity in terms of the particle's position

$$v_-^2(x) = \frac{mg(1 - e^{-2\gamma(d-x)/m})}{\gamma}, \tag{8}$$

$$v_+^2(x) = \frac{mg(2e^{-2\gamma x/m} - e^{-2\gamma(d-x)/m} - 1)}{\gamma}, \tag{9}$$

where we have taken into account that the particle undergoes a perfectly elastic collision when it bounces on the surface of the Earth. Plugging Eq. (8) and Eq. (9) into Eq. (7) we have

$$m\frac{dv_+}{dt} = -mg(2e^{-2\gamma x/m} - e^{-2\gamma(d+x)/m}), \tag{10}$$

$$m\frac{dv_-}{dt} = -mge^{-2\gamma(d-x)/m}. \tag{11}$$

The Hamiltonian for Eq. (10) and Eq. (11) is given by

$$H_- = \frac{p^2}{2m} + \frac{m^2 g}{2\gamma}e^{-2\gamma d/m}[e^{2\gamma x/m} - 1], \tag{12}$$

$$H_+ = \frac{p^2}{2m} + \frac{m^2 g}{2\gamma}[e^{-2\gamma d/m} - 2][e^{-2\gamma x/m} - 1], \tag{13}$$

which is identical to the Hamiltonian given in Ref. 12. The phase space diagram described by the Hamiltonian given in Eq. (12) and Eq. (13) is depicted in Fig. 1 for different energies. Note how the trajectories in the phase space diagram are discontinuous at $p = 0$. On the other hand, the effective Hamiltonian which is given by

$$H_{eff} = \frac{H_+ + H_-}{2} = \frac{p^2}{2m} + \frac{A(e^{2\gamma x/m} - 1) + B(e^{-2\gamma x/m} - 1)}{2}, \tag{14}$$

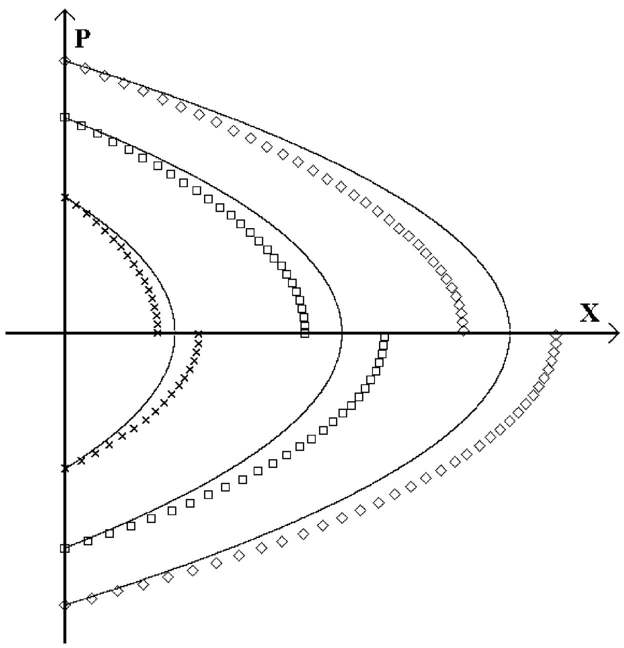

FIGURE 1. Phase space diagram for the effective Hamiltonian (solid line) describing a particle bouncing on the surface of the Earth and subjected to a frictional force proportional to the square of the particle's velocity. The pointers correspond to the Hamiltonian $H_\pm$ of the system when the particle is going up and down, respectively.

where

$$A = m^2 g e^{-2\gamma d/m}/2\gamma$$

and

$$B = m^2 g(e^{-2\gamma d/m} - 2)/2\gamma,$$

is a continuous function. Note that $A > 0$ and $B < 0$. The phase space diagram described by the effective Hamiltonian given in Eq. (14) is depicted in Fig. 1 by the solid line. Note how the effective Hamiltonian encloses the same area in phase space as the other Hamiltonians. These features of the effective Hamiltonian will allow us to use the WKB theory to obtain the energy levels of the quantized dissipative system.

3. WKB Quantization

To see the effects of dissipation in the eigenvalues of the quantum bouncer we are going to use the WKB quantization rule which may be written as

$$\oint p(x, E)dx = 2\pi\hbar(n + \beta), \tag{15}$$

where β is a phase angle which is determined by the connection procedure of the wave function at the turning points.
Eq. (15) means that the area enclosed by the curve representing the motion in phase space determines the possible quantized values of E. Making the assumption $mgd \ll m^2g/2\gamma$, we can take $A = m^2g/2\gamma = -B$, and using Eq. (14) we have for the effective energy

$$E = \frac{p^2}{2m} + A\sinh(2\gamma x/m). \tag{16}$$

The area enclosed by the curve representing the motion in phase space for the energy given in Eq. (16) is obtained by solving the following integral

$$\frac{m}{2\gamma}\int_{-\sqrt{2mE}}^{\sqrt{2mE}} \sinh^{-1}\left(\frac{E}{A} - \frac{p^2}{2mA}\right)dp = 2\pi\hbar\left(n + \frac{3}{4}\right), \tag{17}$$

where the phase angle for a half-space potential is given by $\beta = 3/4$ and $n = 0, 1, 2\ldots$ [6]. The WKB method is expected to be valid when $|dk/dx| \ll k^2$, where k is the wavenumber [7–9]. For our case

$$\hbar^2k^2/2m = E - A\sinh(2\gamma x/m),$$

hence

$$\left|\frac{dk}{dx}\right| = \left|\frac{1}{2k}\frac{dk^2}{dx}\right| = \frac{1}{kx_0^3}\sqrt{1 + \left(\frac{E}{A} - \frac{\hbar^2k^2}{2mA}\right)^2}, \tag{18}$$

where $x_0^3 = \hbar^2/m^2g$. Therefore, the WKB method is applicable if the following inequality is satisfied

$$k^3x_0^3 \gg \sqrt{1 + \left(\frac{E}{A} - \frac{\hbar^2k^2}{2mA}\right)^2}. \tag{19}$$

TABLE I. Numerical values that show the energy loss of a neutron dropped in a uniform gravitational field and subjected to a frictional force that is proportional to the square of the particle's velocity. The table shows the WKB solutions for $\gamma = 0$ and for $\gamma = 1.07 \times 10^{-22}$.

n	WKB with no dissipation	WKB with dissipation
0	2.23517×10^{-31}	2.20095×10^{-31}
1	3.93214×10^{-31}	2.94089×10^{-31}
2	5.31486×10^{-31}	3.29341×10^{-31}
3	6.53569×10^{-31}	3.53142×10^{-31}
4	7.65126×10^{-31}	3.71361×10^{-31}

The inequality given in Eq. (19) is satisfied for all energies when $\gamma = 0$ and $k \gg 1/x_0$, hence the WKB approximation is legitimate only if the term in parenthesis inside the square root of Eq. (19) is small, *i.e.* we must have weak dissipation.

Expanding the integrand in Eq. (17) in a power series and keeping only the first three terms we end up with the following trascendental equation

$$\frac{(Em)^{3/2}\left(1540m^8g^4 - 704m^4g^2\gamma^2E^2 + 1024\gamma^4E^4\right)}{\sqrt{2}1155m^{10}g^5} = \pi\hbar\left(n + \frac{3}{4}\right), \tag{20}$$

note that if the dissipation parameter is zero, *i.e.* $\gamma = 0$, we get the WKB eigenvalue solution for the quantum bouncer. In 2002 an experiment to discern this gravitational effect at the quantum level was performed by Nesvizhevsky [13]. This experiment consisted of dropping ultracold neutrons from a height of 14μm unto a neutron mirror and watching them bounce [14, 15]. They found a neutron ground state at 2.24305×10^{-31}J, this energy value is well approximated by the WKB theory.

In Table I we have the numerical values for the quantized energy of a neutron dropped in a uniform gravitational field subjected to a frictional force proportional to the square of the particle's velocity. Note how the particle loses energy due to dissipation which is consistent with the classical picture of a purely dissipative system.

4. Conclusions

We have shown that for any completely bound quantum system with quadratic friction we can obtain an effective Hamiltonian which restores the continuity of the phase space trajectory and thus enables us to apply the WKB quantization rule to obtain the quantized energies of the system. We must however recognize that the WKB solution is not exact but gives us an accurate value for the eigenvalues of the quantum system. An important remark is that the problem explored through the WKB quantization rule shows that the system loses energy due to dissipation which is perfectly consistent with the classical picture of a purely dissipative system.

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