

Complex dynamics and chaos in commutable pendulum

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This paper deals with a commutable pendulum which has two different natural frequencies in two space regions. The first region corresponds to the first three quadrants of the phase space; while the second region coincides with the fourth quadrant. This apparently trivial system shows a very complex behavior, regardless of the fact that it is based on the simple pendulum model. In the case where the two natural frequencies coincide, the flow is stable. However, just by varying the natural frequency in the fourth quadrant, the system may be asymptotically stable or unstable and also may have simple limit cycle, complex limit cycles and even chaotic behavior. In all these cases the trajectories have simple analytical descriptions.

Keywords: Chaos; commutable pendulum; complex dynamical system.

En este trabajo se estudia el péndulo con conmutaciones de frecuencia en dos regiones del plano. La primera región coincide con los tres primeros cuadrantes del plano, mientras que la segunda región coincide con el cuarto cuadrante. Este sistema, aparentemente trivial, presenta un comportamiento realmente complejo a pesar de estar basado en el modelo del péndulo simple. En situaciones cuando las dos frecuencias naturales coinciden, el comportamiento es estable. Sin embargo, cuando se cambia la frecuencia natural en el cuarto cuadrante, el sistema puede ser asintóticamente estable o inestable, puede tener un ciclo límite sencillo, ciclos límites complejos y más aún, puede presentar comportamiento caótico. Otra característica importante es que en todos estos casos las trayectorias pueden ser descritas analíticamente de forma sencilla.

Descriptores: Caos; péndulo conmutable; sistema dinámico complejo.

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1. Introduction

The complex and chaotic behavior of trajectories of dynamical systems is of great interest in Physics and Mathematics and it has been extensively studied by different authors [1-3]. For instance, first chaotic model appears in paper of E. Lorenz [8] which is considered as foundation of modern chaos theory. Lorenz description of the butterfly effect is commonly used to describe the atmospheric behavior. In telecommunications, chaos can be used to encode private electronic information. In the industry, chaotic mechanisms are used to mix liquids. In medicine, chaos may be used to stimulate the brain and to develop more efficient pacemakers [7-15].

There are two different cases of chaotic systems: discrete-time and continuous-time dynamic systems. In discrete-time dynamic systems, chaos may appear in all dimensions. The sufficient conditions of chaos in one-dimension for discrete system are given by Sharkovskii and Li-Yorke theorem [4,5].

In case of autonomous continuous-time systems described by ordinary differential equations with continuous right-hand side the chaos may appear only if the system dimension is higher or equal to three. Chaos cannot appear in autonomous continuous-time two-dimensional systems due to Poincaré-Bendixson theorem [1,2]. All well known examples of continuous chaos (Lorenz [8], Rössler [9], Sprott [10] and other attractors, Chua *et al.* [11], Pikovskii-Rabinovich [12] and Anisichenko-Astakhov [13] generators)

have the dimension three and chaos may be verified numerically only.

Chaos appears in two-dimensional nonlinear and non autonomous oscillators with sufficiently great external or parametric excitation, for example in Ueda oscillator [14]. In the nonlinear Zaslavsky oscillator [15], chaos appears if there exists an impulsive excitation. Other authors study the generation of chaos on the basis of commutable linear systems; however the complexity of their models avoid the obtaining of analytical solutions [16,17].

In this paper a simple piece-wise linear two-dimensional dynamical system with constant piece-wise coefficients and a nonlinear commutation law for these coefficients is considered. More exactly, the trajectories of commutable ideal pendulum with two regions in phase space having different frequencies are studied. One of these regions coincides with the fourth quadrant of plane while the other region coincides with the rest of the quadrants. The trajectories of such commutable pendulum have a simple analytical description for each region of the plane; the trajectories are continuously differentiable and formed by two sections: three quarters of a circle and one quarter of an ellipse. The unique changing parameter is the frequency on ellipsoidal region of the trajectory. Nevertheless, the global behavior of the trajectories may be very complicated. This paper is a continuation of [19] having new results specially in proofs of chaos existence. The main result is the proof that the commutable two-dimensional pendulum may have chaotic behavior for a large set of commu-

tation laws and different values of parameter in commutation law. This result does not contradict to Poincaré-Bendixson theorem because the right-hand part of equation describing commutable pendulum is piece-wise constant.

The rest of the paper is organized as follows. In Sec. 2, the commutable pendulum is defined. The appearance of chaos in discrete one-dimensional equations is briefly analyzed in Sec. 3. Section 4 is devoted to the study of complex trajectories described by the commutable pendulum despite its simple equations. The chaotic behavior of the commutable pendulum is analyzed in Sec. 5. Finally, in Sec. 6 some conclusions are drawn.

2. Commutable ideal pendulum

Consider the special type of commutable pendulum S defined as a continuous-time dynamical system in two-dimensional phase space. Two ideal pendulums $S1$ and $S2$ are acting in two regions of state space $Q1$ and $Q2$. The region $Q1$ coincides with the fourth quadrant, $Q1 = \{x < 0, \dot{x} > 0\}$ and $Q2 = \mathbb{R}^2 - Q1$. In that sense, the commutable pendulum S is defined as:

$$S = \begin{cases} S1 : \ddot{x} + \omega^2 x = 0, & x \in Q1, \\ S2 : \ddot{x} + x = 0, & x \in Q2. \end{cases} \quad (1)$$

The commutable pendulum S has two commutation lines: $L_{1,2}$ for the transition from region $Q1$ to region $Q2$ and $L_{2,1}$ from region $Q2$ to region $Q1$:

$$\begin{aligned} L_{1,2} &= \{x = 0, \dot{x} > 0\}, \\ L_{2,1} &= \{x < 0, \dot{x} = 0\}. \end{aligned} \quad (2)$$

The graphical representation of commutable system is given in Fig. 1. If the trajectories of pendulum S pass through line $L_{2,1}$, it changes its natural frequency and passes from $S2$ to $S1$. The frequency ω has constant value in IV quadrant, but may depend on the initial conditions in the point $P2 : \omega = \text{const} = \omega(x(2))$, where $x(2)$ is the angular position at the moment of passing from III quadrant to IV quadrant, clearly at this instant the angular velocity is equal to zero. The trajectories and their first derivatives are continuous on the lines $L_{1,2}, L_{2,1}$. Figure 1 shows typical trajectory of commutable pendulum S defined by Eq. (1) and commutation lines (2).

Solving Eq. $S2$ is just trivial. This solution represents three quarters of a circle. Suppose the solution starts at a point

$$P1 = \{x(0) = A, \dot{x}(0) = 0\}, \quad (3)$$

then the solution trajectory in quadrants II and III is $x(t) = A \cos(t), 0 \leq t \leq \pi$. After half a turn, at commutation point $P2$ from $S2$ to $S1$, the coordinates are

$$P2 = \{x(2) = x(\pi) = -A, \dot{x}(2) = \dot{x}(\pi) = 0\}. \quad (4)$$

Equation $S1$ defines commutable pendulum dynamics in IV quadrant and its trajectory is given by

$$x(t) = -A \cos(\omega(t - \pi)), \pi \leq t \leq \pi + \frac{\pi}{2\omega}. \quad (5)$$

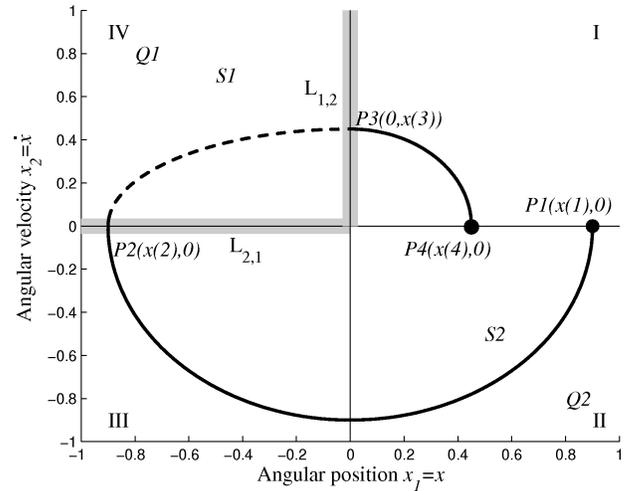


FIGURE 1. Commutation regions and commutation lines for a commutable pendulum.

Thus, the trajectory reaches the point $P3$ of commutation from $S1$ to $S2$ at instant $t_3 = \pi + (\pi/2\omega)$ and

$$P3 = \{x(3) = x(t_3) = 0, \dot{x}(3) = \dot{x}(t_3) = A\omega\}. \quad (6)$$

The dynamics of the commutable pendulum in first quadrant is defined by Eq. $S2$ and, considering its initial conditions at $P3$, its trajectory is

$$\begin{aligned} x(t) &= A\omega \sin\left(t - \pi - \frac{\pi}{2\omega}\right), \\ \pi + \frac{\pi}{2\omega} &\leq t \leq \frac{3\pi}{2} + \frac{\pi}{2\omega}. \end{aligned} \quad (7)$$

Thus, the first recurrent point of commutable pendulum trajectory on the transversal axis $\{x > 0, \dot{x} = 0\}$ is $P4$ with coordinates

$$P4 = \left\{ x\left(\frac{3\pi}{2} + \frac{\pi}{2\omega}\right) = A\omega, \dot{x}\left(\frac{3\pi}{2} + \frac{\pi}{2\omega}\right) = 0 \right\}. \quad (8)$$

By comparing $x(0) = A$ and

$$x\left(\frac{3\pi}{2} + \frac{\pi}{2\omega}\right) = A\omega,$$

it can be readily noticed that if $\omega = \text{const} < 1$, the trajectory of this commutable pendulum is asymptotically stable. But, if $\omega = \text{const} > 1$ the commutable pendulum is unstable [18].

It is clear that the dynamics of the commutable pendulum is defined by a law of frequency change which may be described by a simple one-dimensional discrete equation. In the following section, it is studied the appearance of chaos in one-dimensional discrete equations. This is because the chaotic behavior of the commutable pendulum is directly related with the chaos of the discrete equation describing the change of frequency ω in IV quadrant.

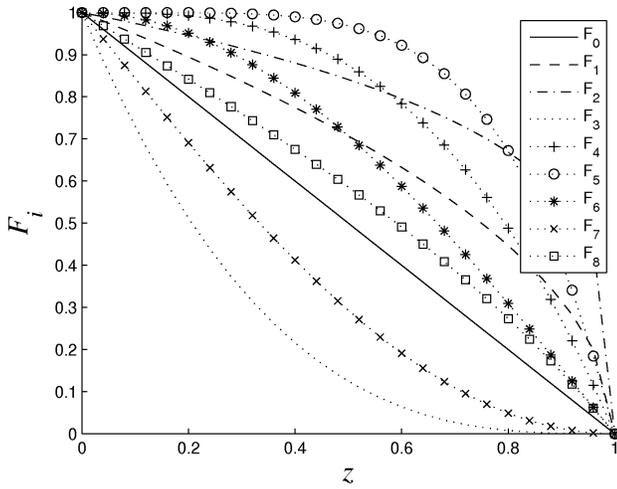


FIGURE 2. Graphs of commutation laws $F_i(z)$, $i = 0, \dots, 8$.

3. Chaos in discrete one-dimensional equations

Consider the discrete one dimensional equations of the form

$$z(n + 1) = \alpha z(n)F_i(z(n)), \quad n = 0, 1, 2, \dots \quad (9)$$

Thus, in this section, the following functions are studied as illustrative examples:

$$\begin{aligned} F_0(z) &= (1 - z), & F_1(z) &= \sqrt{1 - z}, \\ F_2(z) &= \sqrt[4]{1 - z}, & F_3(z) &= (1 - z)^3, \\ F_4(z) &= (1 - z^3), & F_5(z) &= (1 - z^5), \\ F_6(z) &= \cos\left(\frac{\pi}{2}z\right), & F_7(z) &= \left(1 - \sin\left(\frac{\pi}{2}z\right)\right), \\ F_8(z) &= \left(1 - \tan\left(\frac{\pi}{4}z\right)\right). \end{aligned} \quad (10)$$

Plots of functions (10), with $\alpha = 1$, are given in Fig. 2. Notice that $F_i(z)$ must to fulfill condition: $F_i : [0, 1] \rightarrow [0, 1]$ for $i = 0, \dots, 8$. Therefore the upper limit for parameter α can be obtained as follows:

$$\max_{z \in [0,1]} (zF_i(z)) \leq \beta_i, \quad \beta_i < 1, \quad i = 1, \dots, 8, \quad (11)$$

and

$$\alpha_{\max}^i = \frac{1}{\beta_i} > 1. \quad (12)$$

Chaotic trajectories may appear for any law in (10) with adequate values of parameter $\alpha > 1$. In order to compute the interval for parameter α which ensures chaotic trajectories, it is sufficient to solve Li-Yorke equation for existence of solutions with period 3 [5].

For functions $F_0(z), \dots, F_8(z)$ in (10), Li-Yorke condition of existence of 3-periodic solutions is

$$z(3) = z(0). \quad (13)$$

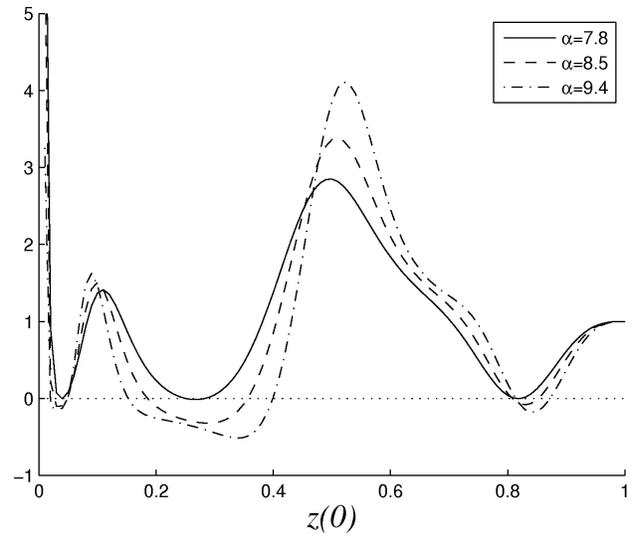


FIGURE 3. Graph of Eq. (14) for function $F_3(z)$ with $\alpha = 7.8, 8.5, 9.4$.

It is clear that any 1-periodic point $z(0)$ such that $z(1)=z(0)$ of discrete Eq. (9) is also 3-periodic solution of this equation. Therefore it is necessary to prove existence of 3-periodic solutions of (9) which are not 1-periodic solutions of this equation, $z(1) = z(0)$. In other words, it is necessary to find at least one solution $z(0)$ for a given value α of the following equation:

$$\varphi(\alpha, z(0)) = \frac{z(3) - z(0)}{z(1) - z(0)} = 0. \quad (14)$$

Here $z(1), z(2)$ and $z(3)$ are presented in terms of $z(0)$ by following recurrent formulas

$$\begin{aligned} z(1) &= \alpha z(0)F_i(z(0)), \\ z(2) &= \alpha z(1)F_i(z(1)) \\ &= \alpha(\alpha z(0)F_i(z(0)))F_i(\alpha z(0)F_i(z(0))), \\ z(3) &= \alpha z(2)F_i(z(2)) \\ &= \alpha(\alpha z(1)F_i(z(1)))F_i(\alpha z(1)F_i(z(1))) \\ &= \alpha(\alpha(\alpha z(0)F_i(z(0)))F_i((\alpha z(0)F_i(z(0)))) \\ &\times F_i(\alpha(\alpha z(0)F_i(z(0)))F_i(\alpha(\alpha z(0) \\ &\times F_i(z(0)))F_i(\alpha z(0)F_i(z(0)))))) \end{aligned} \quad (15)$$

By substituting (15) in (14) the final form of Li-Yorke Eq. (14) is obtained. This equation is a nonlinear equation both in z and α . It is possible to solve it only numerically by fixing α and seeking for a root $z(0)$ of (14) such that $0 < z(0) < 1$.

Observe that if for parameter α_{\max}^i Eq. (14) has a solution, then by continuity there exists α_{\min}^i such that for all $\alpha \in [\alpha_{\min}^i, \alpha_{\max}^i]$ Eq. (14) has a solution. Here $\alpha_{\max}^i = (1/\beta_i)$, but α_{\min}^i is obtained graphically without extreme precision.

TABLE I. Intervals of chaos existence in commutable pendulum

Function F_i	Value of β_i	Interval of $\alpha \in [\alpha_{\min}^i, \alpha_{\max}^i]$ for chaos existence
F_0	$\frac{2}{(3\sqrt{3})} = 0.25$	$3.83 < \alpha < 4$
F_1	$\frac{2}{(3\sqrt{3})} = 0.3849$	$2.56 < \alpha < 2.59$
F_2	$\frac{4}{5} \sqrt[4]{\frac{1}{5}} = 0.5349$	$1.858 < \alpha < 1.869$
F_3	$\frac{27}{256} = 0.1055$	$7.8 < \alpha < 9.4814$
F_4	$\frac{3}{4\sqrt[3]{4}} = 0.47247$	$1.98 < \alpha < 2.116$
F_5	$\frac{5}{6(6)^{1/5}} = 0.5823$	$1.6 < \alpha < 1.7171$
F_6	0.357	$2.65 < \alpha < 2.80$
F_7	0.1670	$5.3 < \alpha < 5.98$
F_8	0.296	$3.23 < \alpha < 3.37$

Figure 3 illustrate this process for function F_3 and $\alpha = 7.8, 8.5, 9.4$. For $\alpha = 7.8$ the graph of function $\varphi(7.8, z(0))$ has zeros. For $\alpha < 7.8$ the graph of function $\varphi(7.8, z(0))$ has no zeros.

All initial conditions $z(0)$ which generate κ -periodic satisfy the equation

$$z(\kappa) - z(0) = 0. \tag{16}$$

It is clear that for functions $F_0(z), F_3(z), F_4(z), F_5(z)$ and for all κ Eq. (16) is an algebraic equation with real coefficients. For this reason Eq. (16) has a finite number of zeros for any κ . Therefore, the set $U(0)$ of all initial values $z(0)$ which generate periodic solution of any period is a denumerable set. It is well-known that the Lebesgue measure of $U(0)$ is equal to zero. Therefore, the set M which is the complement of a set $U(0)$ in the interval $[0, 1]$, i.e. $M = [0, 1]/U(0)$, has a Lebesgue measure equal to 1.

As consequence, for $z(0) \in M$ the sequence $z(0), z(1), \dots, z(\kappa), \dots$ generated by initial condition $z(0)$ and Eq. (9) is not periodic but it is bounded. Li and Yorke have shown that this sequence is chaotic if there exists a solution of period 3.

From Fig. 3 it is clear that Li-Yorke Eq. (14) for $F_3(z)$ and $\alpha \in [7.8, 9.48]$ has solutions, and as consequence 3-periodic solutions can be obtained for these values of α . Therefore, for $\alpha \in [7.8, 9.48]$ and $F_i(z) = F_3(z)$ Eq. (9) has chaotic behavior.

The results of numerical investigation for functions $F_0(z), \dots, F_8(z)$ in (10) are summarized in Table I. Notice that these calculations only give sufficient conditions for appearance of chaotic trajectories. For other values of parameter α chaotic solutions may appear or not. In similar way, different laws which are not included in Table I may be also studied. However, it is sufficiently sure that the chaotic trajectories will appear. for any monotonic function $F(z)$ similar to those considered in (10) with an appropriated value of parameter α .

4. Complexity in commutable pendulum

Consider now the commutable pendulum (1)-(2) with the frequency for region $S1$ defined as

$$\omega = \alpha(1 + x(2)) = \alpha(1 - x(1)) = \alpha F_0(x(1)). \tag{17}$$

According to Fig. 1 and equations (3)-(8), the first recurrence point $x(4)$ of the trajectory is equal to

$$\begin{aligned} x(4) &= \omega x(1) \\ &= \alpha((1 + x(2)))x(1) = \alpha(1 - x(1))x(1). \end{aligned} \tag{18}$$

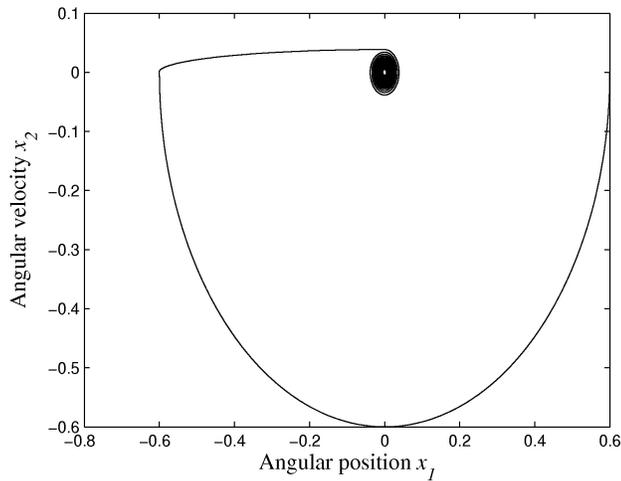
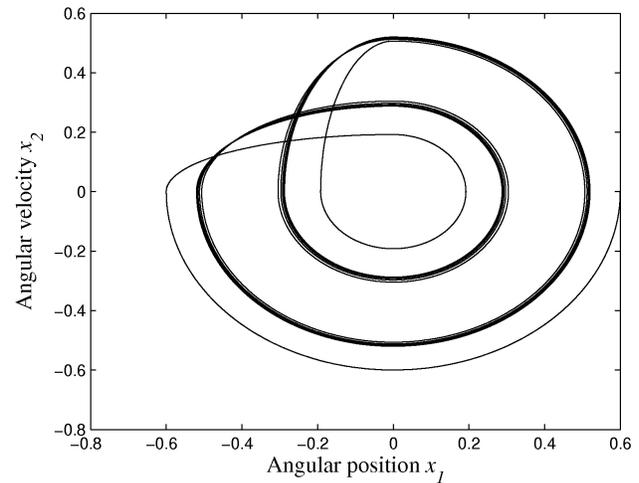
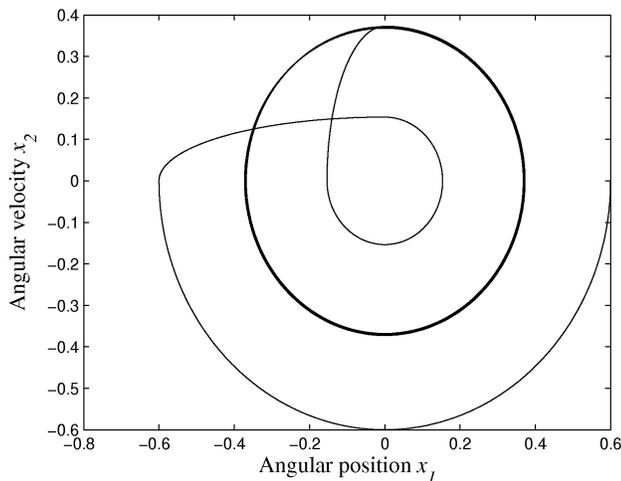
Equation (18) coincides with classical discrete logistic equation [5]. The trajectories of commutable pendulum (1)-(2) with the frequency defined by (17) have a wide range of behavior as the parameter α is modified. The behavior of the commutable pendulum with function $F_0(z)$ has been thoroughly analyzed in Ref. 18 and 19. For that reason in the present work function $F_3(z)$ is used to validate the existence of many discrete equations, which are capable of generating complex trajectories in the pendulum when they are used to define the frequency in region $S1$.

Thus, by taking into account definition of function $F_3(z)$

$$\omega = \alpha(1 + x(2))^3 = \alpha(1 - x(1))^3 = \alpha F_3(x(1)) \tag{19}$$

as the frequency for the commutable pendulum in region $S1$, the results given in Figs. 4-7 can be readily obtained when initial angular position is $x(1) = x_1(0) = 0.6$ and initial angular velocity is $\dot{x}(1) = x_2(0) = 0$.

Firstly, it is considered the case when $\alpha = 1$. The simulation result given in Fig. 4 shows that the commutable pendulum evolving under this condition is asymptotically stable.

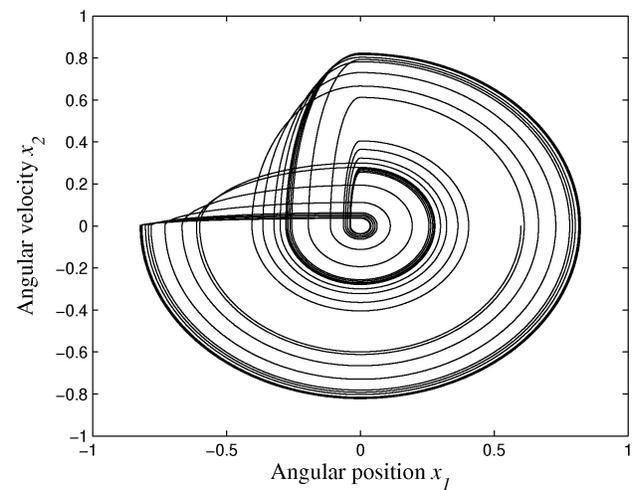
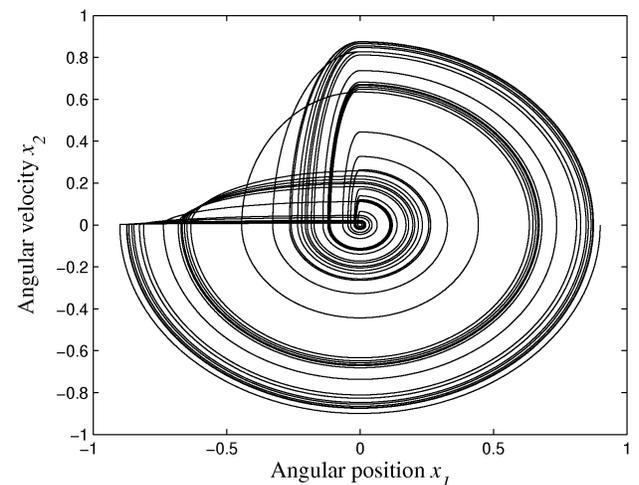
FIGURE 4. Asymptotically stable behavior for $\alpha = 1$.FIGURE 6. Triple limit cycle for $\alpha = 5$.FIGURE 5. Simple limit cycle for $\alpha = 4$.

However, the behavior of the commutable pendulum can be changed by selecting a different value for parameter α . Consider now $\alpha = 4$ and the previous initial conditions. In this case there exists a simple limit cycle as the one presented in Fig. 5.

In the same way, a double limit cycle can be obtained by the suitable choice of parameter α . The commutable pendulum has a double limit cycle when $\alpha = 5$ (see Fig. 6).

A triple limit cycle can be obtained when $\alpha = 7.8$ and the commutable pendulum starts from $x(1) = x_1(0) = 0.6$ and $\dot{x}(1) = x_2(0) = 0$, as shown in Fig. 7. Li-Yorke theorem states that in this case for other initial condition the commutable pendulum has periodic solutions of any period and also chaotic solutions similar to those shown in Figs. 8-10. This conclusion will be verified in the following section.

It is important to remark that the trajectories depicted in Figs. 4 to 7 have been generated by considering (19) as the frequency for the commutable pendulum in region $S1$. Therefore, the same equation is applied in the following section to obtain chaos in the commutable pendulum.

FIGURE 7. Triple limit cycle for $\alpha = 7.8$.FIGURE 8. Chaotic trajectory for $\alpha = 8.3$.

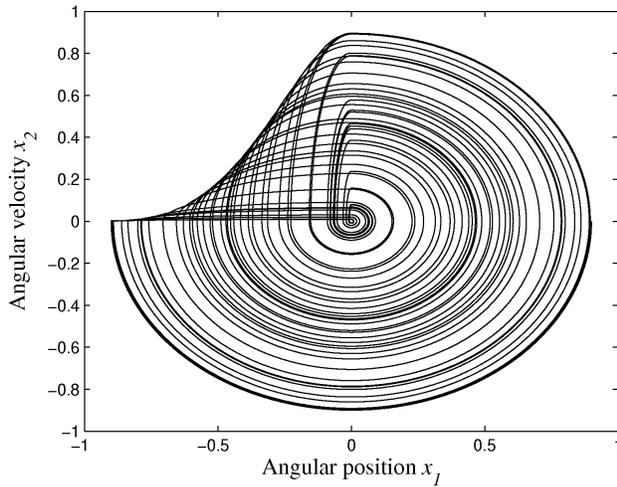


FIGURE 9. Chaotic trajectory for $\alpha = 8.5$.

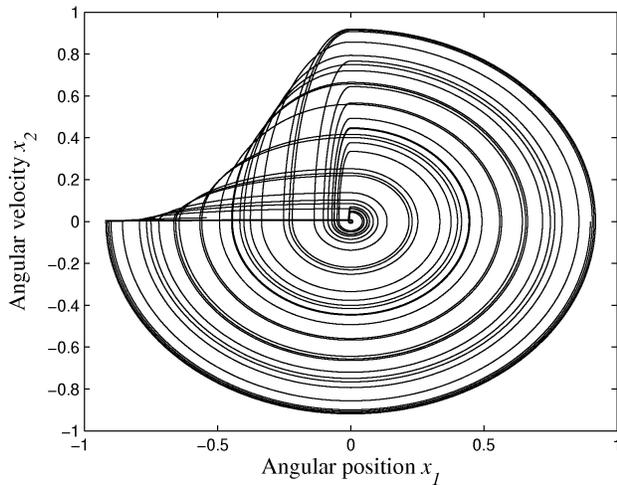


FIGURE 10. Chaotic trajectory for $\alpha = 8.7$.

5. Chaos in a commutable pendulum

According to Table I for $7.8 < \alpha < 9.4814$ the function $F_3(z)$ have chaotic solutions [4,5]. For this reason the behavior of commutable pendulum may be also chaotic when α is defined within such an interval.

Figures 8, 9 and 10 show the evolution in time of chaotic solutions for $\alpha = 8.3$, $\alpha = 8.5$ and $\alpha = 8.7$, respectively, and initial $x(1) = x_1(0) = 0.9$, and $\dot{x}(1) = x_2(0) = 0$.

The simplicity of this model relies on the fact that each one of Figs. 8, 9 and 10 is formed by a $3/4$ of a circle (S_2) and an $1/4$ of an ellipse (S_1).

Remark 1 *It is important to mention that each trajectory is deterministic and it is possible to compute analytically any point at any instant t .*

On the other hand, Figs. 11, 12 and 13 show position, velocity and averages values as function of time for $\alpha = 8.5$, respectively, and $x(1) = x_1(0) = 0.9$ $\dot{x}(1) = x_2(0) = 0$.

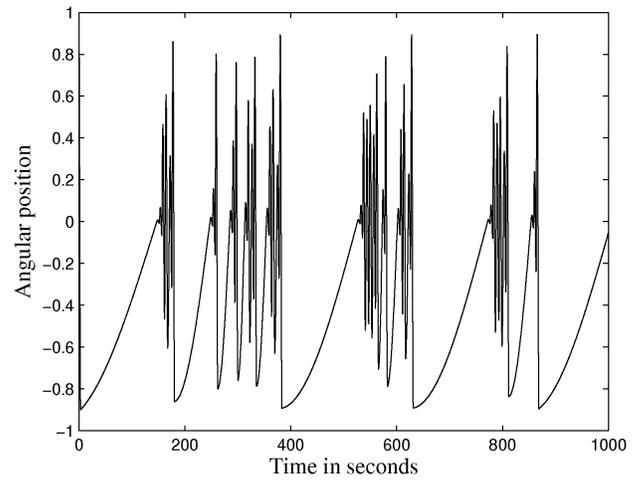


FIGURE 11. Angular position of the commutable pendulum when $\alpha = 8.5$.

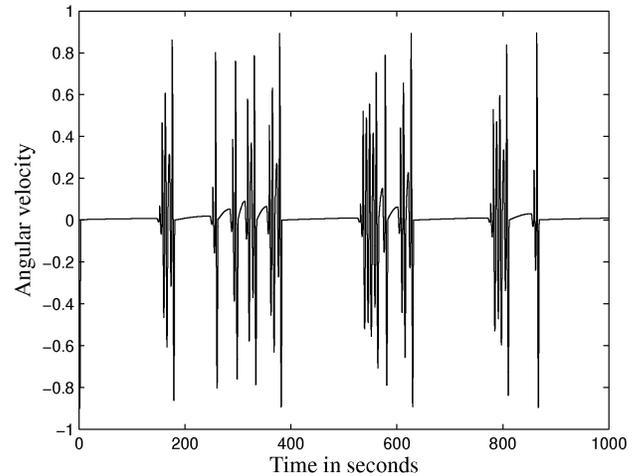


FIGURE 12. Angular velocity of the commutable pendulum when $\alpha = 4$.

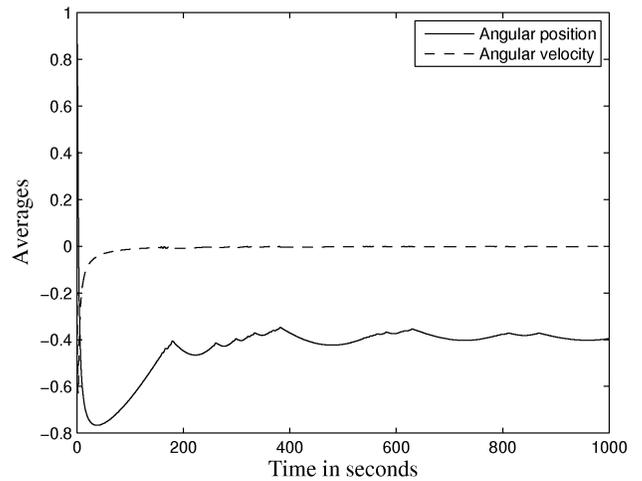


FIGURE 13. Averages for position and velocity of the commutable pendulum when $\alpha = 8.5$.

It is well known that in case of ideal pendulum these mean values tend to zero as time tends to infinity. However, for the

chaotic commutable pendulum the mean velocity tends to a small positive value as time tends to infinity, while the mean value of position is negative. This feature will be analyzed next.

The time of one revolution T_{rev} for commutable pendulum is equal to

$$T_{\text{rev}} = \frac{3\pi}{2} + \frac{\pi}{2\omega}. \quad (20)$$

In Eq. (20), the first term represents the sum of transition time from position $P1$ to $P2$ and from position $P3$ to $P4$. The second term is the transition time from position $P2$ to $P3$. For $P2$ close to -1 the frequency ω in IV quadrant is close to zero and the transition time from $P2$ to $P3$ is large. During this time the value of $x(t)$ is negative and the velocity has a small positive value. As consequence, the mean value of position is negative while the mean velocity is equal to a small positive number.

6. Conclusions

Generating continuous chaos can be useful in many systems, for instance: liquid mixing, brain simulation, financial studies, etc. In physics, pendulum models are often used to de-

scribed the behavior of atoms in crystal structures [7,15]. The emission of light quantum may be considered as instantaneous commutation of some atom characteristics, which are modeled, in this paper, as pendulum frequency commutation.

In that sense, it has been shown that complex dynamics and chaos may appear in relatively simple dynamical systems such as the commutable pendulum with two different frequencies.

In this paper, the nonlinear commutations are based on discrete mappings which allows the system to generate very complex trajectories including chaos. However, the existence of other simple commutation laws is not denied and they must to be analyzed in the future.

The proposed approach can be easily extended to other dynamical systems of different areas of science and engineering as mechanical and electronic systems, robotics, etc.

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