Solutions of the Schrödinger equation given by solutions of the Hamilton–Jacobi equation

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We find the form of the potential depending on the coordinates and the time such that a solution, S, of the Hamilton–Jacobi equation yields an exact solution, $\exp(\mathrm{i}S/\hbar)$, of the corresponding Schrödinger equation.

Keywords: Schrödinger equation; Hamilton–Jacobi equation.

Hallamos la forma del potencial dependiente de las coordenadas y el tiempo tal que una solución, S, de la ecuación de Hamilton–Jacobi produce una solución exacta, $\exp(\mathrm{i}S/\hbar)$, de la ecuación de Schrödinger correspondiente.

Descriptores: Ecuación de Schrödinger; ecuación de Hamilton–Jacobi.

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1. Introduction

As is well known, the Hamilton–Jacobi (HJ) equation can be regarded as an approximation to the Schrödinger equation. More precisely, these two equations are related in the following way. If $\psi = \exp(\mathrm{i}S/\hbar)$ is substituted into the Schrödinger equation for a particle of mass m and electric charge e, subject to a velocity-independent potential V and a magnetic field with vector potential $\mathbf{A}$,

$$-\frac{\hbar^2}{2m}\left(\nabla - \frac{\mathrm{i}e}{\hbar c}\mathbf{A}\right)^2\psi + V\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}, \tag{1}$$

imposing the condition $\nabla\cdot\mathbf{A} = 0$, one obtains

$$\frac{1}{2m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)^2 + V + \frac{\partial S}{\partial t} = \frac{\mathrm{i}\hbar}{2m}\nabla^2 S, \tag{2}$$

which is equivalent to (1). Then, if one looks for a solution of this last equation of the form

$$S = S_0 + \frac{\hbar}{\mathrm{i}}S_1 + \left(\frac{\hbar}{\mathrm{i}}\right)^2 S_2 + \cdots,$$

one finds that S_0 satisfies the HJ equation

$$\frac{1}{2m}\left(\nabla S_0 - \frac{e}{c}\mathbf{A}\right)^2 + V + \frac{\partial S_0}{\partial t} = 0 \tag{3}$$

(see, *e.g.*, Ref. [1], Sec. 2.8). However, Eq. (2) shows that if S satisfies the Laplace equation, $\nabla^2 S = 0$, and the HJ equation (3), then $\psi = \exp(\mathrm{i}S/\hbar)$ is an *exact* solution of the Schrödinger equation (1).

The aim of this paper is to give some explicit examples of this relationship between solutions of the HJ equation and solutions of the corresponding Schrödinger equation. In Sec. 2 we show that from each solution, S, of the Laplace equation, and any given magnetic field, one can define a potential, V, in such a way that S is a solution of the HJ equation and $\exp(\mathrm{i}S/\hbar)$ is a solution of the corresponding Schrödinger equation. In Secs. 3–5 we present some examples, restricting ourselves to problems without magnetic field. An example of this class was given in Ref. [2], where the attention was focused on the HJ equation.

2. Basic results

According to Eq. (2), assuming that S is a solution of the Laplace equation

$$\nabla^2 S = 0, \tag{4}$$

S satisfies the HJ equation (3) if and only if $\psi = \exp(\mathrm{i}S/\hbar)$ is a solution of the Schrödinger equation (1). Instead of specifying the potentials V and $\mathbf{A}$, and then try to find a simultaneous solution of the Laplace equation and the HJ equation, it is simpler to start from a solution, S, of the Laplace equation and *define* the potential V so that S automatically satisfies the HJ equation (3). That is, to take

$$V = -\frac{\partial S}{\partial t} - \frac{1}{2m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)^2 \tag{5}$$

or, if the magnetic field is absent, choosing $\mathbf{A} = \mathbf{0}$,

$$V = -\frac{\partial S}{\partial t} - \frac{1}{2m}(\nabla S)^2. \tag{6}$$

Of course, the resulting potential may be of little interest, but if one makes use of the general solution of the Laplace equation, one can find all the potentials for which a solution S of the HJ equation corresponds to a solution $\exp(\mathrm{i}S/\hbar)$ of the Schrödinger equation.

As pointed out after Eq. (3), if S is any solution of the HJ equation (3) and, simultaneously, of the Laplace equation, then $\exp(\mathrm{i}S/\hbar)$ is a solution of the corresponding Schrödinger equation. What would produce only a semiclassical approximation leads, in fact, to an exact solution of the Schrödinger equation. Since the modulus of $\exp(\mathrm{i}S/\hbar)$ is equal to 1, all these wavefunctions cannot be normalized; however, just like the eigenfunctions of the linear momentum, which are also non-normalizable, in some cases they can be used to express any normalizable wavefunction (see also the comment at the end of Sec. 3.1). In fact, if one finds a complete solution of the HJ equation, S, then $\exp(\mathrm{i}S/\hbar)$ possesses the appropriate number of parameters to form a complete set of functions in terms of which any wavefunction can be expanded (see the examples below). (It is interesting to note that the word "complete" appears here with two different, but related, meanings.) The solutions of the Schrödinger equation obtained here differ from the solutions usually found because the latter are commonly obtained by separation of variables (for instance, the wavefunction (11) is not separable and does not represent a state with a well-defined energy).

In the following sections we give some explicit examples, without magnetic field, starting from the highly illustrative one-dimensional case.

3. Examples in one dimension

In the one-dimensional case, the general solution of Eq. (4) is a linear function of the Cartesian coordinate x

$$S(x,t) = \alpha(t)\, x + \beta(t), \tag{7}$$

where $\alpha(t)$ and $\beta(t)$ are real-valued functions of t only. Substituting (7) into (6) we get

$$V = -\alpha'(t)\, x - \beta'(t) - \frac{1}{2m}\alpha^2(t), \tag{8}$$

which corresponds to a possibly time-dependent uniform field of force. Apart from the trivial case of a free particle, two specific examples contained in (8) are a time-independent uniform field, and a uniform field with an intensity that changes linearly with time.

3.1. Constant uniform field

From Eq. (8) we see that in order to have a potential

$$V = -Fx, \tag{9}$$

where F is a constant, corresponding to a constant uniform force of magnitude F, $\alpha'(t)$ must be equal to F and $\beta'(t) = -\alpha^2(t)/2m$, hence

$$\alpha(t) = Ft + P, \qquad \beta(t) = -\frac{(Ft+P)^3}{6mF},$$

where P is an arbitrary constant. Thus, substituting these expressions into (7), we obtain

$$S(x,t) = (Ft+P)\, x - \frac{(Ft+P)^3}{6mF}. \tag{10}$$

The solution (10) of the HJ equation corresponding to the potential (9) is complete (by virtue of the presence of the arbitrary parameter P) and it is not separable (because of the term Ftx) and, for each value of P, the wavefunction

$$\psi(x,t) = \exp\frac{\mathrm{i}}{\hbar}\left[(Ft+P)\, x - \frac{(Ft+P)^3}{6mF}\right] \tag{11}$$

is a solution of the Schrödinger equation with the potential (9), which is not (multiplicatively) separable (and is not a stationary state). The wavefunctions (11) are non-normalizable, just as the "plane waves" $\exp(\mathrm{i}Px/\hbar)$, and, as the latter, for each value of t, form a complete set, in terms of which any wavefunction can be expanded (by means of an integral on the parameter P). In fact, in the same manner as the plane waves are eigenfunctions of the momentum operator, the wavefunctions (11) are eigenfunctions of the (time-dependent) operator $p - Ft$ (which is a conserved quantity), with eigenvalue P [3].

3.2. Field with constant growth

We now consider the time-dependent potential

$$V(x) = -ktx, \tag{12}$$

where k is a constant. This potential coincides with (8) if $\alpha'(t) = kt$ and $\beta'(t) = -\alpha^2(t)/2m$, hence

$$\alpha(t) = \frac{kt^2}{2} + P, \qquad \beta(t) = -\frac{1}{2m}\left(\frac{k^2t^5}{20} + \frac{kPt^3}{3} + P^2t\right),$$

where P is an arbitrary constant. Thus,

$$S(x,t) = \left(\frac{kt^2}{2} + P\right)x - \frac{1}{2m}\left(\frac{k^2t^5}{20} + \frac{kPt^3}{3} + P^2t\right) \tag{13}$$

is a complete solution of the HJ equation with potential (12) and

$$\psi(x,t) = \exp\frac{\mathrm{i}}{\hbar}\left[\left(\frac{kt^2}{2} + P\right)x - \frac{1}{2m}\left(\frac{k^2t^5}{20} + \frac{kPt^3}{3} + P^2t\right)\right] \tag{14}$$

is a solution of the corresponding Schrödinger equation.

The (non-normalizable) wavefunctions (14) form a complete set. In fact, (14) is eigenfunction of the conserved operator $p - (1/2)kt^2$, with eigenvalue P.

4. Examples in two dimensions

The solutions of the Laplace equation in two dimensions are the form

$$S(x,y,t) = \mathrm{Re}[f(z,t)], \tag{15}$$

where $z \equiv x + \mathrm{i}y$, and $f(z,t)$ is an analytic function of z that may also depend on t and on other parameters. Substituting (15) into Eq. (6) we obtain the expression for the potential

$$V = -\frac{1}{2}\frac{\partial(f+\overline{f})}{\partial t} - \frac{1}{2m}\left|\frac{\partial f}{\partial z}\right|^2. \tag{16}$$

In the following subsections we give two explicit examples corresponding to specific choices of f.

4.1. The repulsive isotropic harmonic oscillator

The function

$$f(z,t) = \frac{m\omega}{2}z^2 + (P_1\mathrm{e}^{-\omega t} - \mathrm{i}P_2\mathrm{e}^{\omega t})\,z + \frac{1}{4m\omega}(P_1\mathrm{e}^{-\omega t} - \mathrm{i}P_2\mathrm{e}^{\omega t})^2,$$

where m, ω, P_1, and P_2 are constants, is an analytic function of z and, from Eq. (16), we obtain the potential

$$V = -\frac{m\omega^2}{2}(x^2+y^2), \tag{17}$$

which corresponds to a "repulsive harmonic oscillator" (see, *e.g.*, Ref. 4). Then, making use of Eq. (15), we have the complete solution of the HJ equation

$$S(x,y,t) = \frac{m\omega}{2}(x^2-y^2) + P_1\,\mathrm{e}^{-\omega t}x + P_2\,\mathrm{e}^{\omega t}y + \frac{1}{4m\omega}({P_1}^2\,\mathrm{e}^{-2\omega t} - {P_2}^2\,\mathrm{e}^{2\omega t}) \tag{18}$$

and the solutions of the Schrödinger equation

$$\psi(x,y,t) = \exp\frac{\mathrm{i}}{\hbar}\left[\frac{m\omega}{2}(x^2-y^2) + P_1\,\mathrm{e}^{-\omega t}x + P_2\,\mathrm{e}^{\omega t}y + \frac{1}{4m\omega}({P_1}^2\,\mathrm{e}^{-2\omega t} - {P_2}^2\,\mathrm{e}^{2\omega t})\right]. \tag{19}$$

Each wavefunction (19) is a common eigenfunction of the conserved operators $\mathrm{e}^{\omega t}(p_x - m\omega x)$ and $\mathrm{e}^{-\omega t}(p_y + m\omega y)$ with eigenvalues P_1 and P_2, respectively.

4.2. The central potential $V = -k/r^2$

The function

$$f(z,t) = \sqrt{2mk}\,\ln z,$$

where k is a constant, is an analytic function of z (except at $z=0$) and leads to the central potential

$$V(x,y) = -\frac{k}{x^2+y^2}. \tag{20}$$

Thus,

$$S = \sqrt{\frac{mk}{2}}\,\ln(x^2+y^2)$$

is a solution of the HJ equation with the potential (20). By contrast with the previous examples, this solution does not contain arbitrary parameters and does not depend on the time.

5. Examples in higher dimensions

Apart from the potentials that can be defined directly with the aid of Eq. (6), making use of a solution of the Laplace equation, as in the foregoing sections, we can also combine the potentials already obtained in low dimensions to construct potentials in two or more dimensions. For instance, from Eqs. (10) and (18) we can form the function

$$S(x,y,z,t) = \frac{m\omega}{2}(x^2-y^2) + P_1\,\mathrm{e}^{-\omega t}x + P_2\,\mathrm{e}^{\omega t}y + \frac{1}{4m\omega}({P_1}^2\,\mathrm{e}^{-2\omega t} - {P_2}^2\,\mathrm{e}^{2\omega t}) + (Ft+P_3)\,z - \frac{(Ft+P_3)^3}{6mF}$$

which satisfies the Laplace equation in three dimensions and leads to the potential [*cf.* Eqs. (9) and (17)]

$$V(x,y,z) = -\frac{m\omega^2}{2}(x^2+y^2) - Fz$$

and, therefore, $\exp(\mathrm{i}S/\hbar)$ is a solution of the Schrödinger equation with this last potential, which contains three quantum numbers P_1, P_2, and P_3.

6. Concluding remarks

It may be remarked that, for a given potential, the existence of a solution of the HJ equation that also satisfies the Laplace equation, does not imply that all solutions of this HJ equation will satisfy the Laplace equation. In the examples of Section 3, the principal functions (10) and (13) (which contain an arbitrary parameter, P) are, up to an additive constant, the *only* common solutions of the corresponding HJ equation and the Laplace equation.

One open question is what are the physical or geometrical properties of a potential such that the corresponding HJ equation admits solutions that also satisfy the Laplace equation. Another question is the meaning of the quantum states obtained in this manner.

The eikonal equation of geometrical optics can be regarded as an approximation to the scalar Helmholtz equation, in a similar manner as the HJ equation is an approximation to the Schrödinger equation and, therefore, one can expect that, in some cases, a solution of the eikonal equation would produce an exact solution of the Helmholtz equation. This connection is currently under investigation [5].

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