

Higher dimensional gravity and Farkas property in oriented matroid theory

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We assume gravity in a d -dimensional manifold M and consider a splitting of the form $M = M_p \times M_q$, with $d = p + q$. The most general two-block metric associated with M_p and M_q is used to derive the corresponding Einstein-Hilbert action \mathcal{S} . We focus on the special case of two distinct conformal factors ψ and φ (ψ for the metric in M_p and φ for the metric in M_q), and we write the action \mathcal{S} in the form $\mathcal{S} = \mathcal{S}_p + \mathcal{S}_q$, where \mathcal{S}_p and \mathcal{S}_q are actions associated with M_p and M_q , respectively. We show that a simplified action is obtained precisely when $\psi = \varphi^{-1}$. In this case, we find that under the duality transformation $\varphi \leftrightarrow \varphi^{-1}$, the action \mathcal{S}_p for the M_p -space or the action \mathcal{S}_q for the M_q -space remain invariant, but not both. This result establishes an analogy between Farkas property in oriented matroid theory and duality in general relativity. Furthermore, we argue that our approach can be used in several physical scenarios such as 2t physics and cosmology.

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Duality has been an important concept in several physical scenarios, including string theory [1], M -theory [2], Matroid theory [3], MacDowell-Mansouri gravity (see Refs. 4 to 5 and references therein) and cosmological models [6,7]. Here, we explore the possible relevance of duality in higher dimensional gravity. We first consider general relativity in a d -dimensional manifold M with arbitrary space-time signature. We then proceed to split such manifold in the form $M = M_p \times M_q$, with $p + q = d$, assuming a two-block diagonal metric. Considering such splitting, we obtain the Einstein-Hilbert action \mathcal{S} . For the special case in which the two metrics associated with M_p and M_q are expressed in terms of the conformal factors ψ and φ , respectively, we show that the action \mathcal{S} can be written as $\mathcal{S} = \mathcal{S}_p + \mathcal{S}_q$, where \mathcal{S}_p and \mathcal{S}_q are actions associated with M_p and M_q , respectively. Taking $\psi = \varphi^{-1}$, the reduced action is analysed. The key idea is to consider the invariance of such reduced action under the duality transformation $\varphi \leftrightarrow \varphi^{-1}$. In this context, we find that the invariance of \mathcal{S} does not follow, but rather we discover that, under this duality transformation, the action \mathcal{S}_p is invariant, provided $p \rightleftharpoons q + 2$, while \mathcal{S}_q is not. And conversely, the action \mathcal{S}_q is invariant, provided $p \rightleftharpoons q - 2$, while \mathcal{S}_p is not. We argue that this result establishes an analogy between Farkas property in oriented matroid theory [8] and duality in higher dimensional gravity. It is worth mentioning that the Farkas property has been used as an alternative to define the concept of oriented matroid [9]. In turn, oriented matroid theory has been proposed [10-11] as the appropriate mathematical framework for considering duality in several physical scenarios, including p-branes and M-theory. Therefore, our work may be useful in the analysis of some aspects of duality in p-branes and M-theory. Furthermore, we argue that our approach can also be relevant in the context of 2t physics and cosmology [12-18].

We start the analysis with a metric γ_{AB} in a d -dimensional manifold M , which will be splitted into a

two-blocks metric corresponding to $M = M_p \times M_q$. The first block metric, of dimension p , will be denoted by $g_{\mu\nu}$, and the second of dimension $q = d - p$, will be denoted by g_{ij} , where Greek indices (α, β, \dots) run from 1 to p , lower-case Latin indices (i, j, \dots) from $p + 1$ to d , and capital Latin indices (A, B, \dots) from 1 to d . With this prescription, we have [12-14]:

$$\gamma_{AB} = \begin{pmatrix} g_{\mu\nu}(x, y) & 0 \\ 0 & g_{ij}(x, y) \end{pmatrix}, \quad (1)$$

where for consistency, the upper zero 0 corresponds to a $p \times q$ -matrix and the lower zero 0 corresponds to a $q \times p$ -matrix. Here, x refers to coordinates in M_p , while y refers to coordinates in M_q .

As usual, the Riemann tensor is defined in terms of the Christoffel symbols as

$$\mathcal{R}^A_{BCD} = \partial_C \Gamma^A_{DB} - \partial_D \Gamma^A_{CB} + \Gamma^A_{CF} \Gamma^F_{DB} - \Gamma^A_{DF} \Gamma^F_{CB}, \quad (2)$$

where in turn, the Christoffel symbols are

$$\Gamma^A_{BC} = \frac{1}{2} g^{AF} (g_{BF,C} + g_{CF,B} - g_{BC,F}). \quad (3)$$

From (3), using the distinction of the indices $\mu, \nu, \dots etc.$ and $i, j, \dots etc.$, we find that the Christoffel symbols can be splitted in six classes:

$$\begin{aligned} \Gamma^{\mu}_{\alpha\beta} &= \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}, & \Gamma^{\mu}_{\alpha i} &= \frac{1}{2} g^{\mu\lambda} g_{\alpha\lambda, i}, \\ \Gamma^{\mu}_{ij} &= -\frac{1}{2} g^{\mu\lambda} g_{ij, \lambda}, & \Gamma^i_{jk} &= \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \\ \Gamma^i_{j\alpha} &= \frac{1}{2} g^{i\lambda} g_{j\lambda, \alpha}, & \Gamma^i_{\alpha\beta} &= -\frac{1}{2} g^{i\lambda} g_{\alpha\beta, \lambda}. \end{aligned} \quad (4)$$

Here, $\left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\}$ and $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ refer to Christoffel symbols in terms of the metric $g_{\alpha\beta}$ and g_{ij} , respectively.

Now, the components of the Riemann tensor can also be splitted in similar classes. For example:

$$\mathcal{R}^\mu_{\nu\alpha\beta} = R^\mu_{\nu\alpha\beta} + \Gamma^\mu_{\alpha k} \Gamma^k_{\beta\nu} - \Gamma^\mu_{\beta k} \Gamma^k_{\alpha\nu}, \quad (5)$$

where

$$R^\mu_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\beta\nu} - \partial_\beta \Gamma^\mu_{\alpha\nu} + \Gamma^\mu_{\alpha\lambda} \Gamma^\lambda_{\beta\nu} - \Gamma^\mu_{\beta\lambda} \Gamma^\lambda_{\alpha\nu}. \quad (6)$$

Using (4), the relation (5) becomes

$$\begin{aligned} \mathcal{R}^\mu_{\nu\alpha\beta} &= R^\mu_{\nu\alpha\beta} \\ &+ \frac{1}{4}(g^{\mu\lambda} g^{kl} g_{\beta\lambda,k} g_{\alpha\nu,l} - g^{\mu\lambda} g^{kl} g_{\alpha\lambda,k} g_{\beta\nu,l}). \end{aligned} \quad (7)$$

Similarly, we get the component of the Riemann tensor with Latin indices:

$$\begin{aligned} \mathcal{R}^i_{jkl} &= R^i_{jkl} \\ &+ \frac{1}{4}(g^{\lambda\tau} g^{im} g_{lm,\lambda} g_{kj,\tau} - g^{\lambda\tau} g^{im} g_{km,\lambda} g_{lj,\tau}). \end{aligned} \quad (8)$$

We also have

$$\begin{aligned} \mathcal{R}^\mu_{\nu j} &= \partial_\nu \Gamma^\mu_{ji} - \partial_j \Gamma^\mu_{\nu i} \\ &+ \Gamma^\mu_{\nu\lambda} \Gamma^\lambda_{ji} - \Gamma^\mu_{j\lambda} \Gamma^\lambda_{\nu i} + \Gamma^\mu_{\nu k} \Gamma^k_{ji} - \Gamma^\mu_{jk} \Gamma^k_{\nu i}. \end{aligned} \quad (9)$$

Defining the covariant derivatives $\mathcal{D}_\nu \Gamma^\mu_{ij} = \partial_\nu \Gamma^\mu_{ij} + \Gamma^\mu_{\nu\lambda} \Gamma^\lambda_{ij}$ and $\mathcal{D}_j \Gamma^\mu_{\nu i} = \partial_j \Gamma^\mu_{\nu i} - \Gamma^\mu_{\nu k} \Gamma^k_{ji}$, the expression (9) is reduced to

$$\mathcal{R}^\mu_{\nu j} = \mathcal{D}_\nu \Gamma^\mu_{ji} - \mathcal{D}_j \Gamma^\mu_{\nu i} - \Gamma^\mu_{j\lambda} \Gamma^\lambda_{\nu i} - \Gamma^\mu_{jk} \Gamma^k_{\nu i}. \quad (10)$$

Thus, by using (4), we find

$$\begin{aligned} \mathcal{R}^\mu_{\nu j} &= -\frac{1}{2} \mathcal{D}_\nu g_{ij}{}^{,\mu} - \frac{1}{2} \mathcal{D}_j (g^{\mu\lambda} g_{\lambda\nu,i}) \\ &- \frac{1}{4} g^{\mu\lambda} g^{\alpha\sigma} g_{\lambda\alpha,j} g_{\sigma\nu,i} + \frac{1}{4} g^{kl} g_{kj}{}^{,\mu} g_{li,\nu}. \end{aligned} \quad (11)$$

With a similar calculation, we obtain

$$\begin{aligned} \mathcal{R}^i_{\mu j\nu} &= -\frac{1}{2} \mathcal{D}_j g_{\mu\nu}{}^{,i} - \frac{1}{2} \mathcal{D}_\nu (g^{il} g_{lj,\mu}) \\ &- \frac{1}{4} g^{il} g^{km} g_{lk,\nu} g_{mj,\mu} + \frac{1}{4} g^{\alpha\lambda} g_{\alpha\nu}{}^{,i} g_{\lambda\mu,j}. \end{aligned} \quad (12)$$

From the Ricci tensor

$$\mathcal{R}_{AB} = \mathcal{R}^K_{AKB}, \quad (13)$$

we can construct the curvature scalar \mathcal{R} :

$$\mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu} + g^{ij} \mathcal{R}_{ij}. \quad (14)$$

We can explicitly rewrite (14) as follows:

$$\mathcal{R} = g^{\mu\nu} (\mathcal{R}^\alpha_{\mu\alpha\nu} + \mathcal{R}^k_{\mu k\nu}) + g^{ij} (\mathcal{R}^\alpha_{i\alpha j} + \mathcal{R}^k_{i k j}). \quad (15)$$

Using the symmetric property $g^{ij} \mathcal{R}^\alpha_{i\alpha j} = g^{\mu\nu} \mathcal{R}^k_{\mu k\nu}$, (15) can be simplified to

$$\mathcal{R} = g^{\mu\nu} \mathcal{R}^\alpha_{\mu\alpha\nu} + g^{ij} \mathcal{R}^k_{i k j} + 2g^{\mu\nu} g^{ij} \mathcal{R}_{\mu\nu ij}. \quad (16)$$

Considering (7), (8), (11) and (12), and writing as ${}^1R = g^{\mu\nu} R^\alpha_{\mu\alpha\nu}$ and ${}^2R = g^{ij} R^k_{i k j}$ the Ricci scalar in p and q dimensions, respectively, (16) leads to

$$\begin{aligned} \mathcal{R} &= {}^1R + {}^2R \\ &- \frac{1}{4} g^{\mu\nu} g^{\alpha\beta} g^{ij} g_{\mu\nu,i} g_{\alpha\beta,j} + \frac{1}{4} g^{\mu\nu} g^{\alpha\beta} g^{ij} g_{\mu\alpha,i} g_{\nu\beta,j} \\ &- \frac{1}{4} g^{\mu\nu} g^{ij} g^{kl} g_{ij,\mu} g_{kl,\nu} + \frac{1}{4} g^{\mu\nu} g^{ij} g^{kl} g_{ik,\mu} g_{jl,\nu} \\ &- \frac{1}{2} g^{ij} \mathcal{D}_\mu g_{ij}{}^{,\mu} - \frac{1}{2} \mathcal{D}_i (g^{\mu\nu} g_{\mu\nu}{}^{,i}) \\ &- \frac{1}{4} g^{\mu\nu} g^{\alpha\beta} g^{ij} g_{\mu\alpha,i} g_{\nu\beta,j} + \frac{1}{4} g^{\mu\nu} g^{ij} g^{kl} g_{ik,\mu} g_{jl,\nu} \\ &- \frac{1}{2} g^{\mu\nu} \mathcal{D}_i g_{\mu\nu}{}^{,i} - \frac{1}{2} \mathcal{D}_\mu (g^{ij} g_{ij}{}^{,\mu}) \\ &- \frac{1}{4} g^{\mu\nu} g^{ij} g^{kl} g_{ik,\mu} g_{jl,\nu} + \frac{1}{4} g^{\mu\nu} g^{\alpha\beta} g^{ij} g_{\mu\alpha,i} g_{\nu\beta,j}. \end{aligned} \quad (17)$$

This expression can be simplified to

$$\begin{aligned} \mathcal{R} &= {}^1R + {}^2R - \mathcal{D}_\mu (g^{ij} g_{ij}{}^{,\mu}) - \mathcal{D}_i (g^{\mu\nu} g_{\mu\nu}{}^{,i}) \\ &- \frac{1}{4} g^{\mu\nu} g^{\alpha\beta} g^{ij} g_{\mu\nu,i} g_{\alpha\beta,j} - \frac{1}{4} g^{\mu\nu} g^{\alpha\beta} g^{ij} g_{\mu\alpha,i} g_{\nu\beta,j} \\ &- \frac{1}{4} g^{\mu\nu} g^{ij} g^{kl} g_{ij,\mu} g_{kl,\nu} - \frac{1}{4} g^{\mu\nu} g^{ij} g^{kl} g_{ik,\mu} g_{jl,\nu}. \end{aligned} \quad (18)$$

Therefore, we obtain the action

$$\begin{aligned} \mathcal{S} &= \int_M \sqrt{g} \sqrt{2g} \mathcal{R} = \int_M [\sqrt{g} \sqrt{2g} {}^1R + \sqrt{g} \sqrt{2g} {}^2R \\ &- \sqrt{g} \mathcal{D}_\mu (\sqrt{2g} g^{ij} g_{ij}{}^{,\mu}) - \sqrt{2g} \mathcal{D}_i (\sqrt{g} g^{\mu\nu} g_{\mu\nu}{}^{,i}) \\ &+ \frac{1}{4} \sqrt{g} \sqrt{2g} (g^{\mu\nu} g^{\alpha\beta} g^{ij} g_{\mu\nu,i} g_{\alpha\beta,j} \\ &- g^{\mu\nu} g^{\alpha\beta} g^{ij} g_{\mu\alpha,i} g_{\nu\beta,j} \\ &+ g^{\mu\nu} g^{ij} g^{kl} g_{ij,\mu} g_{kl,\nu} - g^{\mu\nu} g^{ij} g^{kl} g_{ik,\mu} g_{jl,\nu})]. \end{aligned} \quad (19)$$

The third and fourth terms into the integral are regarded as boundary terms that won't take a role in our analysis, *i.e.* we assume that

$$\int_M \sqrt{g} \mathcal{D}_\mu (\sqrt{2g} g^{ij} g_{ij}{}^{,\mu}) = 0$$

and

$$\int_M \sqrt{2g} \mathcal{D}_i (\sqrt{g} g^{\mu\nu} g_{\mu\nu}{}^{,i}) = 0.$$

In fact, the vanishing of these total divergences can be justified by choosing proper boundary conditions. However, in general these total divergences terms could be considered.

This is, of course, still more important at the quantum level, where the boundary terms of an action may have nonvanishing contribution. Therefore (19) can be simplified as follows see Ref. 17 and references therein

$$\begin{aligned} \mathcal{S} &= \int_M \sqrt{^1g} \sqrt{^2g} \mathcal{R} = \int_M \sqrt{^1g} \sqrt{^2g} [^1R + ^2R \\ &+ \frac{1}{4} g^{\mu\nu} g^{\alpha\beta} g^{ij} (g_{\mu\nu,i} g_{\alpha\beta,j} - g_{\mu\alpha,i} g_{\nu\beta,j}) \\ &+ \frac{1}{4} g^{\mu\nu} g^{ij} g^{kl} (g_{ij,\mu} g_{kl,\nu} - g_{ik,\mu} g_{jl,\nu})]. \end{aligned} \quad (20)$$

Thus, we have achieved our first goal of splitting the action in terms of the M^p and M^q spaces. However, observe that the last two terms in (20) are interacting terms between the two metric fields g_{ij} and $g_{\alpha\beta}$.

As an application of (20), we shall now discuss several examples. First, one may assume that $g_{\mu\nu} = g_{\mu\nu}(x)$ and $g_{ij} = g_{ij}(y)$. In this case \mathcal{S} is reduced to

$$\mathcal{S} = \int_M \sqrt{^1g} \sqrt{^2g} (^1R + ^2R), \quad (21)$$

which is a well known result.

More interesting cases may arise if we assume that $g_{\mu\nu} = \psi^2(x, y) \tilde{g}_{\mu\nu}(x)$ and $g_{ij} = \varphi^2(x, y) \tilde{g}_{ij}(y)$. Let us first substitute this assumption in the last two terms of (20):

$$\begin{aligned} \mathcal{S} &= \int_M \sqrt{^1g} \sqrt{^2g} [^1R + ^2R \\ &+ p(p-1)\psi^{-2}\varphi^{-2}\tilde{g}^{ij}\psi_{,i}\psi_{,j} \\ &+ q(q-1)\psi^{-2}\varphi^{-2}\tilde{g}^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}]. \end{aligned} \quad (22)$$

Since the Ricci scalars 1R and 2R become

$$\begin{aligned} ^1R &= \psi^{-2} [^1\tilde{R} - (p-1)(p-4)\psi^{-2}\psi_{,\lambda}\psi^{,\lambda} \\ &- 2(p-1)\psi^{-1}\nabla_{\lambda}\psi^{,\lambda}] \end{aligned} \quad (23)$$

and

$$\begin{aligned} ^2R &= \varphi^{-2} [^2\tilde{R} - (q-1)(q-4)\varphi^{-2}\varphi_{,i}\varphi^{,i} \\ &- 2(q-1)\varphi^{-1}\nabla_i\varphi^{,i}] \end{aligned} \quad (24)$$

respectively, after some rearrangements, we obtain

$$\begin{aligned} \mathcal{S} &= \int_M \sqrt{^1\tilde{g}} \sqrt{^2\tilde{g}} [\psi^{p-2}\varphi^q ^1\tilde{R} + \psi^p\varphi^{q-2} ^2\tilde{R} \\ &- (p-1)(p-4)\psi^{p-4}\varphi^q\psi_{,\lambda}\psi^{,\lambda} \\ &- 2(p-1)\psi^{p-3}\varphi^q\nabla_{\lambda}\psi^{,\lambda} + p(p-1)\psi^{p-2}\varphi^{q-2}\psi_{,i}\psi^{,i} \\ &- (q-1)(q-4)\psi^p\varphi^{q-4}\varphi_{,i}\varphi^{,i} - 2(q-1)\psi^p\varphi^{q-3}\nabla_i\varphi^{,i} \\ &+ q(q-1)\psi^{p-2}\varphi^{q-2}\varphi_{,\lambda}\varphi^{,\lambda}]. \end{aligned} \quad (25)$$

Integrating by parts, (25) yields

$$\begin{aligned} \mathcal{S} &= \int_M \sqrt{^1\tilde{g}} \sqrt{^2\tilde{g}} [(\psi^{p-2}\varphi^q) ^1\tilde{R} + (\psi^p\varphi^{q-2}) ^2\tilde{R} \\ &+ (p-1)(p-2)\psi^{p-4}\varphi^q\psi_{,\lambda}\psi^{,\lambda} \\ &+ 2q(p-1)\psi^{p-3}\varphi^{q-1}\varphi_{,\lambda}\psi^{,\lambda} \\ &+ p(p-1)\psi^{p-2}\varphi^{q-2}\psi_{,i}\psi^{,i} \\ &+ (q-1)(q-2)\psi^p\varphi^{q-4}\varphi_{,i}\varphi^{,i} \\ &+ 2p(q-1)\psi^{p-1}\varphi^{q-3}\varphi_{,i}\psi^{,i} \\ &+ q(q-1)\psi^{p-2}\varphi^{q-2}\varphi_{,\lambda}\varphi^{,\lambda}]. \end{aligned} \quad (26)$$

Here, we are interested in exploring a possible duality symmetry in (26). For this purpose, let us consider the special case $\psi = \varphi^{-1}$. We have

$$\begin{aligned} \mathcal{S} &= \int_M \sqrt{^1\tilde{g}} \sqrt{^2\tilde{g}} [(\varphi^{q-p+2}) ^1\tilde{R} + (\varphi^{q-p-2}) ^2\tilde{R} \\ &+ (p-1)(p-2)\varphi^{q-p}\varphi_{,\lambda}\varphi^{,\lambda} + 2q(p-1)\varphi^{q-p}\varphi_{,\lambda}\varphi^{,\lambda} \\ &+ p(p-1)\varphi^{q-p-4}\varphi_{,i}\varphi^{,i} + (q-1)(q-2)\varphi^{q-p-4}\varphi_{,i}\varphi^{,i} \\ &+ 2p(q-1)\varphi^{q-p-4}\varphi_{,i}\varphi^{,i} + q(q-1)\varphi^{q-p}\varphi_{,\lambda}\varphi^{,\lambda}], \end{aligned} \quad (27)$$

which can also be rewritten as

$$\begin{aligned} \mathcal{S} &= \int_M \sqrt{^1\tilde{g}} \sqrt{^2\tilde{g}} [(\varphi^{q-p+2}) ^1\tilde{R} + (\varphi^{q-p-2}) ^2\tilde{R} \\ &+ [(p-1)(p-2) + 2q(p-1) \\ &+ q(q-1)]\varphi^{q-p}\varphi_{,\lambda}\varphi^{,\lambda} + [(q-1)(q-2) \\ &+ 2p(q-1) + p(p-1)]\varphi^{q-p-4}\varphi_{,i}\varphi^{,i}]. \end{aligned} \quad (28)$$

Since

$$\begin{aligned} (p-1)(p-2) + 2q(p-1) + q(q-1) \\ = (p+q-1)(p+q-2) \end{aligned} \quad (29)$$

and

$$\begin{aligned} (q-1)(q-2) + 2p(q-1) + p(p-1) \\ = (p+q-1)(p+q-2), \end{aligned} \quad (30)$$

we can further simplify (28) to

$$\mathcal{S} = \mathcal{S}_p + \mathcal{S}_q, \quad (31)$$

where

$$\begin{aligned} \mathcal{S}_p &= \int_M \sqrt{^1\tilde{g}} \sqrt{^2\tilde{g}} [(\varphi^{q-p+2}) ^1\tilde{R} \\ &+ (p+q-1)(p+q-2)\varphi^{q-p}\varphi_{,\lambda}\varphi^{,\lambda}] \end{aligned} \quad (32)$$

and

$$S_q = \int_M \sqrt{1\tilde{g}}\sqrt{2\tilde{g}}[(\varphi^{q-p-2})^2 \tilde{R} + (p+q-1)(p+q-2)\varphi^{q-p-4}\varphi_{,i}\varphi^{,i}]. \quad (33)$$

We are interested in the possible invariance of the action (31) under the duality transformation $\varphi \rightarrow \varphi^{-1}$. Applying this transformation to (31), we obtain

$$S_p \rightarrow \int_M \sqrt{1\tilde{g}}\sqrt{2\tilde{g}}[\varphi^{p-q-2} \tilde{R} + (p+q-1)(p+q-2)\varphi^{p-q-4}\varphi_{,\lambda}\varphi^{,\lambda}], \quad (34)$$

while

$$S_q \rightarrow \int_M \sqrt{1\tilde{g}}\sqrt{2\tilde{g}}[\varphi^{p-q+2} \tilde{R} + (p+q-1)(p+q-2)\varphi^{p-q}\varphi_{,i}\varphi^{,i}]. \quad (35)$$

Therefore we observe that if $p \rightarrow q+2$ and $q \rightarrow p-2$, then S_p remains invariant, but

$$S_q = \int_M \sqrt{1\tilde{g}}\sqrt{2\tilde{g}}[\varphi^{p-q+6} \tilde{R} + (p+q-1)(p+q-2)\varphi^{p-q+4}\varphi_{,i}\varphi^{,i}]; \quad (36)$$

that is, S_q is not invariant. Conversely, if $p \rightarrow q-2$ and $q \rightarrow p+2$, then S_q is invariant, while S_p becomes

$$S_p = \int_M \sqrt{1\tilde{g}}\sqrt{2\tilde{g}}[\varphi^{p-q-6} \tilde{R} + (p+q-1)(p+q-2)\varphi^{p-q-4}\varphi_{,\lambda}\varphi^{,\lambda}], \quad (37)$$

which means that S_p is not invariant. Therefore, we have the peculiar situation that under the duality transformation $\varphi \rightarrow \varphi^{-1}$, S_p or S_q remains invariant, but not both, depending on having $p \rightarrow q+2$ and $q \rightarrow p-2$, (that is $p \rightleftharpoons q+2$), or $p \rightarrow q-2$ and $q \rightarrow p+2$ ($p \rightleftharpoons q-2$), respectively. This in turn implies that, in order to preserve the symmetry, the total dimension d should be even. It is important to emphasize that one should start fixing $d = p+q$, and then to explore for which values of p and q the actions S_p or S_q are invariants. For instance assume $d = 6$ then the only possibility for S_p to be invariant is with $p = 4$ and $q = 2$. But for this values S_q is not invariant. Now assume $d = 10$ in this case one can make S_q invariant by taking $p = 4$ and $q = 6$, but in these values S_p is not invariant.

This duality property for the action (31) resembles the Farkas property in oriented matroid theory [8]. In order to fully appreciate this comment, let us explain briefly what the Farkas property means. First, let us consider a total tangent bundle $T = (H, V)$ associated with M , where H and V

denote the horizontal and vertical parts of T . Assume that $L \subseteq M$ corresponds to H and that the orthogonal complement L_\perp corresponds to V . Now, just as (H, V) determine the structure of T , the dual pair (L, L_\perp) determines the structure of the total space M . It turns out that one can introduce the concept of an oriented matroid in terms of the structure (L, L_\perp) , rather than only in terms of the subspace L . (For details in oriented matroids, see Ref. 8 and 9.) One can prove that a transition of the form

$$(L, L_\perp) \rightarrow (H, V), \quad (38)$$

ensures a duality symmetry, which is one of the main subjects of oriented matroid theory. In fact, there exist a formal definition of an oriented matroid in terms of the analogue of (L, L_\perp) . Such a definition uses the concept of Farkas property, which we shall now proceed to discuss briefly (see Ref. 5 for details).

Let us first describe the sign vector concept. Let $E \neq \emptyset$ be a finite set. An element $X \in \{+, -, 0\}^E$ is called a sign vector. The positive, negative and zero parts of X are denoted by X^+ , X^- and X^0 respectively. Further, we define $suppX \equiv X^+ \cup X^-$. Consider two sets S and S' of signed vectors. The pair (S, S') is said to have the Farkas property, if $\forall e \in E$ either

$$(Fa) \exists X \in S, e \in suppX \text{ and } X \geq 0$$

or

$$(Fb) \exists Y \in S', e \in suppY \text{ and } Y \geq 0,$$

but not both. Here, $X \geq 0$ means that X has a positive (+) or a zero (0) entry in every coordinate. Observe that (S, S') has the Farkas property if and only if (S', S) has it. Let S be a set of signed vectors, and let I and J denote disjoint subsets of E . Then

$$S \setminus I / J = \{\tilde{X} | \exists X \in S, X_I = 0, X_J = *, X = \tilde{X} \text{ on } E \setminus (I \cup J)\}, \quad (39)$$

is called a minor of S (obtained by deleting I and contracting J). Here, the symbol "*" denotes an arbitrary value. If S and S' are sets of sign vectors on E , then $(S \setminus I / J, S' \setminus J / I)$ is called minor of (S, S') . Similarly,

$${}_I S = \{\tilde{X} | \exists X \in S, X_I = -\tilde{X}_I, X_{E \setminus I} = \tilde{X}_{E \setminus I}\} \quad (40)$$

is called the reorientation of S on I . Further, $({}_I S, {}_I S')$ is the reorientation of (S, S') on I . Moreover, S is symmetric if $S = -S$, where $-S$ is the set of signed vectors which are opposite to the signed vectors of S .

We can now give a definition of oriented matroids in terms of the Farkas property. Let $E \neq \emptyset$ be a finite set and let S and S' two sets of sign vectors. The pair (S, S') is called an oriented matroid on E , if

(O1) S and S' are symmetric, and

(O2) every reorientation of every minor of (S, S') has the Farkas property.

From this definition it follows that (S', S) is also an oriented matroid as it is every reorientation and every minor of (S, S') .

Two sign vectors X and Y are orthogonal if

$$(X^+ \cap Y^+) \cup (X^- \cap Y^-) \neq \emptyset \Leftrightarrow (X^+ \cap Y^-) \cup (X^- \cap Y^+) \neq \emptyset. \quad (41)$$

Accordingly, we denote the orthogonal complement of S by S_\perp , and it is defined by

$$S_\perp = \{Y | Y \perp X \text{ for all } X \in S\}. \quad (42)$$

If $S' \subseteq S_\perp$, then S and S' can be considered as orthogonal.

Coming back to our problem at hand we have that, due to the form of the metric (1), S_p and S_q can be associated with the two orthogonal subspaces M_p and M_q of M , respectively. This suggests to introduce the analogue of the Farkas property for the action (31): For every transformation $\varphi \rightarrow \varphi^{-1}$,

(Ha) $\exists p$ for M_p , such that S_p is invariant

or

(Hb) $\exists q$ for M_q , such that S_q is invariant,

but not both.

It is interesting to write (31) in the alternative form

$$S = \int_M \sqrt{1\tilde{g}} \sqrt{2\tilde{g}} [\varphi^{q-p} (\gamma^{\mu\nu} \tilde{R}_{\mu\nu} + \gamma^{ij} \tilde{R}_{ij}) + (d-1)(d-2)\varphi^{q-p-2} (\gamma^{\lambda\sigma} \varphi_{,\lambda} \varphi_{,\sigma} + \gamma^{ij} \varphi_{,i} \varphi_{,j})]. \quad (43)$$

Here, we considered the fact that ${}^1\tilde{R} = \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu}$, ${}^2\tilde{R} = \tilde{g}^{ij} \tilde{R}_{ij}$, $\gamma^{\mu\nu} = \varphi^2(x, y) \tilde{g}^{\mu\nu}(x)$ and $\gamma^{ij} = \varphi^{-2}(x, y) \tilde{g}^{ij}(x)$. This expression can be written in the more compact form

$$S = \int_M \sqrt{1\tilde{g}} \sqrt{2\tilde{g}} \varphi^{q-p} [\gamma^{AB} \tilde{R}_{AB} + (d-1)(d-2)\varphi^{-2} \gamma^{AB} \varphi_{,A} \varphi_{,B}]. \quad (44)$$

Let us make some final comments. Usually, one is interested in the invariance of an action S under certain infinitesimal transformations. Here we have shown that, if an action can be divided in two complementary ‘‘orthogonal’’ actions S_p and S_q , then the invariance of the total action $S = S_p + S_q$ under duality transformations is not what really matters, but rather whether S_p or S_q are invariant, but not both, as the analogue of the Farkas property should require. Since the results obtained above are valid for various higher dimensions (with even d) -albeit the conditions imposed for the metric-, they can be of interest for cosmology with extra dimensions, in particular when those extra dimensions are non-compact [15-17]. Also, since *a priori* it has been not chosen any signature for the metric γ_{AB} , our analysis may give some insight in 2t physics [18].

It may be interesting if we also comment about a possible generalization of the action (31) in terms of its associated effective action. In the case of QED and QCD the effective action arises when one considers radiative non linear corrections to the original action [19-21]. Thinking from this perspective it is expected that in the case of the action (31) one may obtain a non linear interacting potential associated to both S_p and S_q . In this case one may expect that the action (31) no longer remains invariant under the interchange $\varphi \leftrightarrow \varphi^{-1}$. This resembles the gravitational anomalies that arises in several gravitational theories. In turn it is known that gravitational anomalies are related to several topological invariants such the Euler characteristic or Pontrjagin topological invariant (see for instance [22] and references therein). In this way, our approach may lead to interesting topological invariants when taking into account quantum corrections, but this goes beyond the aims of the present work.

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