Fractional mechanical oscillators

J.F. Gómez-Aguilar[a,*], J.J. Rosales-García[b], J.J. Bernal-Alvarado[a],
T. Córdova-Fraga[a], and R. Guzmán-Cabrera[b]
*[a]Departamento de Física, División de Ciencias e Ingenierías Campus León, Universidad de Guanajuato,
Lomas del Bosque s/n, Lomas del Campestre, León Guanajuato, México,
e-mail: (jfga, bernal, teo)@fisica.ugto.mx*
*[b]Departamento de Ingeniería Eléctrica, División de Ingenierías Campus Irapuato-Salamanca, Universidad de Guanajuato,
Carretera Salamanca-Valle de Santiago, km. 3.5 + 1.8 km, Comunidad de Palo Blanco, Salamanca Guanajuato México,
e-mail: (rosales, guzmanc)@ugto.mx
Tel: +52 (477) 788-5100 ext. 8449.

In this contribution we propose a new fractional differential equation to describe the mechanical oscillations of a simple system. In particular, we analyze the systems mass-spring and spring-damper. The order of the derivatives is $0 < \gamma \le 1$. In order to be consistent with the physical equation a new parameter σ is introduced. This parameter characterizes the existence of fractional structures in the system. A relation between the fractional order time derivative γ and the new parameter σ is found. Due to this relation the solutions of the corresponding fractional differential equations are given in terms of the Mittag-Leffler function depending only on the parameter γ. The classical cases are recovered by taking the limit when $\gamma = 1$.

Keywords: Fractional calculus; mechanical oscillators; caputo derivative; fractional structures.

En esta contribución se propone una nueva ecuación diferencial fraccionaria que describe las oscilaciones mecánicas de un sistema simple. En particular, se analizan los sistemas masa-resorte y resorte-amortiguador. El orden de las derivadas es $0 < \gamma \le 1$. Para mantener la consistencia con la ecuación física se introduce un nuevo parámetro σ. Este parámetro caracteriza la existencia de estructuras fraccionarias en el sistema. Se muestra que existe una relación entre el orden de la derivada fraccionaria γ y el nuevo parámetro σ. Debido a esta relación las soluciones de las correspondientes ecuaciones diferenciales fraccionarias estan dadas en terminos de la función de Mittag-Leffler, cuyas soluciones dependen solo del orden fraccionario γ. Los casos clásicos son recuperados en el límite cuando $\gamma = 1$.

Descriptores: Calculo fraccionario; oscilaciones mecanicas; derivada de caputo; estructuras fraccionarias.

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1. Introduction

Although the application of Fractional Calculus (FC) has attracted interest of researches in recent decades, it has a long history when the derivative of order 0.5 has been described by Leibniz in a letter to L'Hospital in 1695. A reviewing paper on applications and the formalism can be found in [1]. FC, involving derivatives and integrals of non-integer order, is historically the first generalization of the classical calculus [2-5]. Many physical phenomena have "intrinsic" fractional order description, hence, FC is necessary in order to explain them. In many applications FC provides a more accurate model of physical systems than ordinary calculus do. Since its success in the description of anomalous diffusion [6], non-integer order calculus, both in one dimension and in multidimensional space, has become an important tool in many areas of physics, mechanics, chemistry, engineering, finances and bioengineering [7-10]. Fundamental physical considerations in favor of the use of models based on derivatives of non-integer order are given in [11-13]. Another large field which requires the use of FC is the theory of fractals [14]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [15]. This is the main advantage of FC

in comparison with the classical integer-order models, in which such effects are in fact neglected.

In a paper of Ryabov it is discussed the fractional oscillator equation involving fractional time derivatives of the Riemann-Liouville type [16]. Naber in [17], studied the linearly damped oscillator equation, written as a fractional derivative in the Caputo representation. The solution is found analytically and a comparison with the ordinary linearly damped oscillator is made. In [18] was considered the fractional oscillator, being a generalization of the conventional linear oscillator, in the framework of fractional calculus. It is interpreted as an ensemble of ordinary harmonic oscillators governed by a stochastic time arrow. Despite introducing the fractional time derivatives the cases mentioned above seem to be justified, there is no clear understanding of the basic reason for fractional derivation in physics. Therefore, it is interesting to analyze a simple physical system and try to understand their fully behavior given by a fractional differential equation.

The aim of this work is to give a simple alternative to construct fractional differential equations for physical systems. In particular, we analyze the systems mass-spring and

spring-damper in terms of the fractional derivative of the Caputo type. The analytical solutions are given in terms of the Mittag-Leffler function depending on the parameter γ.

2. Fractional oscillator system

We propose a simple alternative procedure for constructing the fractional differential equation for the fractional oscillator system. To do that, we replace the ordinary time derivative operator by the fractional one in the following way:

$$\frac{d}{dt} \to \frac{d^\gamma}{dt^\gamma}, \qquad 0 < \gamma \leq 1 \qquad (1)$$

It can be seen that (1) is not quite right, from a physical point of view, because the time derivative operator d/dt has dimension of inverse seconds s^{-1}, while the fractional time derivative operator d^γ/dt^γ has, $\mathrm{s}^{-\gamma}$. In order to be consistent with the time dimensionality we introduce the new parameter σ in the following way

$$\left[\frac{1}{\sigma^{1-\gamma}}\frac{d^\gamma}{dt^\gamma}\right] = \frac{1}{\mathrm{s}}. \qquad 0 < \gamma \leq 1 \qquad (2)$$

where γ is an arbitrary parameter which represents the order of the derivative. In the case $\gamma = 1$ the expression (2) becomes an ordinary derivative operator d/dt. In this way (2) is dimensionally consistent if and only if the new parameter σ, has dimension of time $[\sigma] = \mathrm{s}$. Then, we have a simple procedure to construct fractional differential equations. It consists in the following; in an ordinary differential equation replace the ordinary derivative by the following fractional derivative operator

$$\frac{d}{dt} \to \frac{1}{\sigma^{1-\gamma}}\frac{d^\gamma}{dt^\gamma}, \qquad 0 < \gamma \leq 1. \qquad (3)$$

The expression (3) is a time derivative in the usual sense, because its dimension is s^{-1}. The parameter σ (auxiliary parameter) represents the *fractional time components* in the system. This non-local time is called *the cosmic time* [19]. Another physical and geometrical interpretation of the fractional operators is given in [20].

To analyze the dynamical behavior of a fractional system it is necessary to use an appropriate definition of fractional derivative. In fact, the definition of the fractional order derivative is not unique and there exist several definitions, including: Grünwald-Letnikov, Riemann-Liouville, Weyl, Riesz and the Caputo representation. In the Caputo case, the derivative of a constant is zero and we can define, properly, the initial conditions for the fractional differential equations which can be handled by using an analogy with the classical case (ordinary derivative). Caputo derivative implies a memory effect by means of a convolution between the integer order derivative and a power of time. For this reason, in this paper we prefer to use the Caputo fractional derivative.

The Caputo fractional derivative for a function of time, $f(t)$, is defined as follows [5]

$${}_{0}^{C}D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\gamma-n+1}}d\eta, \qquad (4)$$

where $n = 1, 2, \ldots \in N$ and $n-1 < \gamma \leq n$. We consider the case $n = 1$, *i.e.*, in the integrand there is only a first derivative. In this case, $0 < \gamma \leq 1$, is the order of the fractional derivative.

The Caputo derivative operator satisfies the following relations

$${}_{0}^{C}D_t^\gamma[f(t) + g(t)] = {}_{0}^{C}D_t^\gamma f(t) + {}_{0}^{C}D_t^\gamma g(t),$$

$${}_{0}^{C}D_t^\gamma c = 0, \text{ where } c \text{ is constant.} \qquad (5)$$

For example, in the case $f(t) = t^k$, where k is arbitrary number and $0 < \gamma \leq 1$ we have the following expression for the fractional derivative operation,

$${}_{0}^{C}D_t^\gamma t^k = \frac{k\Gamma(k)}{\Gamma(k+1-\gamma)}t^{k-\gamma}, \qquad (0 < \gamma \leq 1) \qquad (6)$$

where $\Gamma(k)$ and $\Gamma(k+1-\gamma)$ are the Gamma functions. If $\gamma = 1$ the expression (6) yields the ordinary derivative

$${}_{0}^{C}D_t^1 t^k = \frac{dt^k}{dt} = kt^{k-1}. \qquad (7)$$

During the recent years the Mittag-Leffler function has caused extensive interest among physicist due to its role played in describing realistic physical systems with memory and delay. The Mittag-Leffler function is defined by the series expansion as

$$E_a(t) = \sum_{m=0}^\infty \frac{t^m}{\Gamma(am+1)}, \qquad (a > 0), \qquad (8)$$

where $\Gamma(\cdot)$ is the Gamma function. When $a = 1$, from (8) we have

$$E_1(t) = \sum_{m=0}^\infty \frac{t^m}{\Gamma(m+1)} = \sum_{m=0}^\infty \frac{t^m}{m!} = e^t. \qquad (9)$$

Therefore, the Mittag-Leffler function is a generalization of the exponential function.

Now, we can write a fractional differential equation corresponding to the mechanical system, Fig. 1, in the following way

$$\frac{m}{\sigma^{2(1-\gamma)}}\frac{d^{2\gamma}x(t)}{dt^{2\gamma}} + \frac{\beta}{\sigma^{1-\gamma}}\frac{d^\gamma x(t)}{dt^\gamma} + kx(t) = 0,$$

$$0 < \gamma \leq 1 \qquad (10)$$

where m is the mass, measured in Kg, β is the damped coefficient, measured in $\mathrm{N} \cdot \mathrm{s/m}$ and k is the spring constant, measured in $\mathrm{N/m}$ [5].
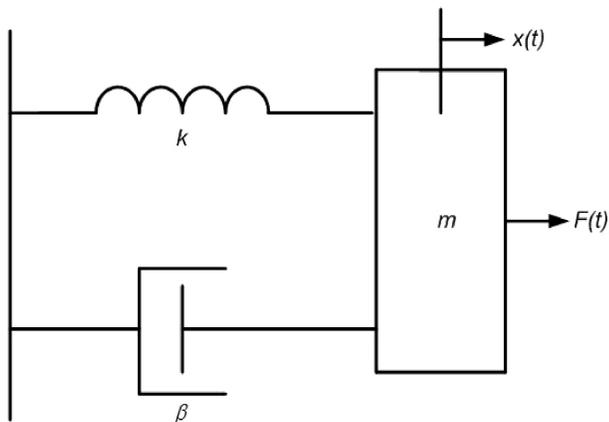
FIGURE 1. Damped oscillator.

From Eq. (10) we obtain the particular cases: when

1. $\beta = 0$

$$\frac{m}{\sigma^{2(1-\gamma)}}\frac{d^{2\gamma}x(t)}{dt^{2\gamma}} + kx(t) = 0, \qquad 0 < \gamma \le 1 \quad (11)$$

and

2. $m = 0$

$$\frac{\beta}{\sigma^{1-\gamma}}\frac{d^{\gamma}x(t)}{dt^{\gamma}} + kx(t) = 0, \qquad 0 < \gamma \le 1, \quad (12)$$

Equation (11) may be written as follows

$$\frac{d^{2\gamma}x}{dt^{2\gamma}} + \omega^2 x(t) = 0, \quad (13)$$

where

$$\omega^2 = \frac{k\sigma^{2(1-\gamma)}}{m} = \omega_0^2\sigma^{2(1-\gamma)}, \quad (14)$$

is the angular frequency for different values of γ, and $\omega_0^2 = k/m$ is the fundamental frequency of the system (*i.e*, when $\gamma = 1$). The solution for the Eq. (13) with $x(0) = x_0$ and $\dot{x}(0) = 0$ as the initial conditions, is given by

$$x(t) = x_0 E_{2\gamma}\left\{-\omega^2 t^{2\gamma}\right\}, \quad (15)$$

where

$$E_{2\gamma}\left\{-\omega^2 t^{2\gamma}\right\} = \sum_{n=0}^{\infty}\frac{\left(-\omega^2 t^{2\gamma}\right)^n}{\Gamma(2\gamma n + 1)}, \quad (16)$$

is the Mittag-Leffler function.

In the case $\gamma = 1$ from (14) we have $\omega^2 = \omega_0^2 = k/m$ and (15) becomes hyperbolic cosine

$$E_2\left\{-\frac{k}{m}t^2\right\} = \text{ch}\left(\sqrt{-\frac{k}{m}t^2}\right) = \text{ch}\left(i\sqrt{\frac{k}{m}}t\right)$$

$$= \text{ch}(i\omega_0 t) = \cos\omega_0 t. \quad (17)$$

Then in the case $\gamma = 1$ the solution of the Eq. (13) is a periodic function given by

$$x(t) = x_0\cos\omega_0 t. \quad (18)$$

Expression (18) is the well known solution for the case of integer differential Eq. (11) with $\gamma = 1$.

Note that the parameter γ, which characterizes the fractional order time derivative can be related to the σ parameter, which characterizes the existence, in the system, of fractional structures (components that show an intermediate behavior between a system conservative (spring) and dissipative (damper)). For example, for the system described by the fractional equation (11), we can write the relation

$$\gamma = \frac{\sigma}{\sqrt{\frac{m}{k}}} = \sigma\omega_0, \qquad 0 < \sigma \le \sqrt{\frac{m}{k}}. \quad (19)$$
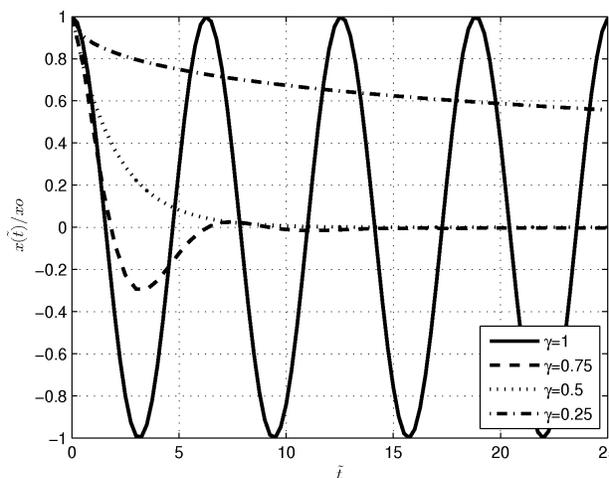

FIGURE 2. Mass-Spring system, $\gamma = 1$, $\gamma = 0.75$, $\gamma = 0.5$ and $\gamma = 0.25$.
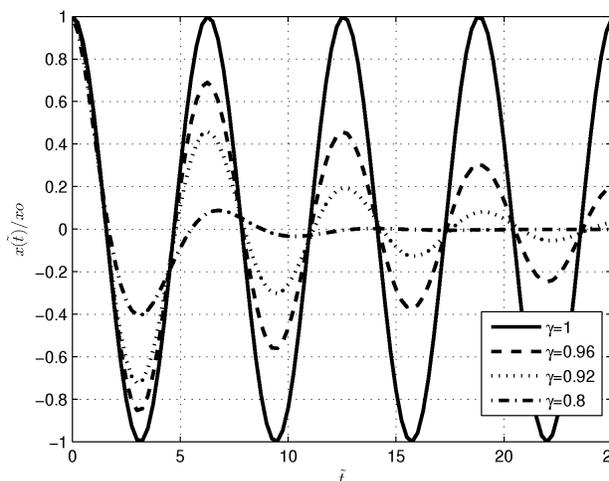

FIGURE 3. Mass-Spring system, $\gamma = 1$, $\gamma = 0.96$, $\gamma = 0.92$ and $\gamma = 0.8$.
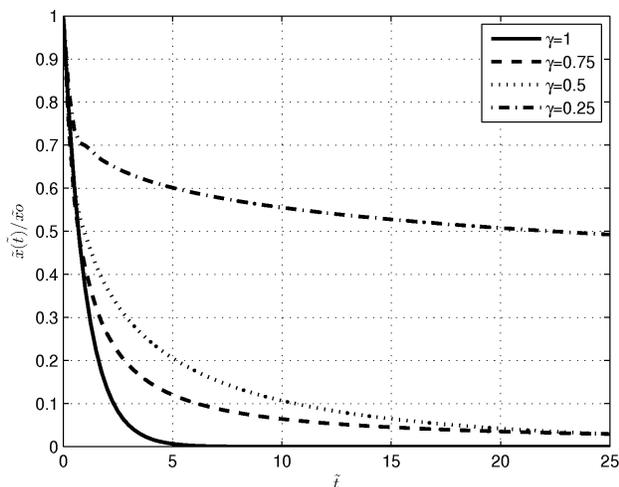
FIGURE 4. Damper-Spring system, $\gamma = 1$, $\gamma = 0.75$, $\gamma = 0.5$ and $\gamma = 0.25$.
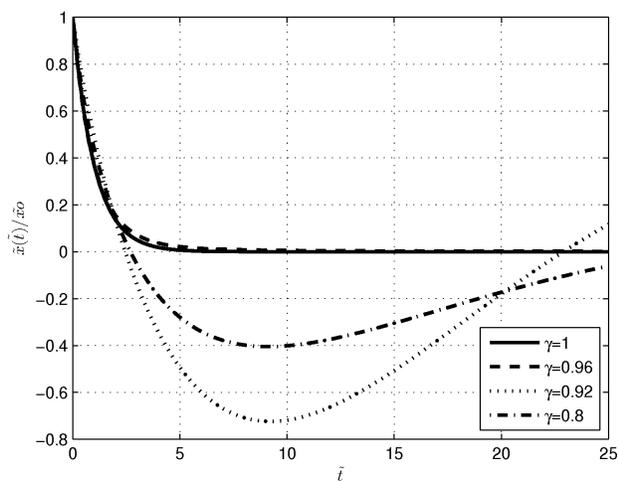

FIGURE 5. Damper-Spring system, $\gamma = 1$, $\gamma = 0.96$, $\gamma = 0.92$ and $\gamma = 0.8$.

Then, the magnitude $\delta = 1 - \gamma$ characterizes the existence of fractional structures in the system. It is easy to see that, when $\gamma = 1$, equivalently $\sigma = 1/\omega_0 = \sqrt{m/k}$, the value of δ is zero, which means that in the system there are no fractional structures. However, in the interval $0 < \gamma < 1$, δ grows and tends to unity, fractional structures appear in the mechanical system.

Taking into account the expression (19), the solution (15) of the Eq. (11) can be rewritten through γ by

$$x(\tilde{t}) = x_0 E_{2\gamma}\Big\{ -\gamma^{2(1-\gamma)}\tilde{t}^{2\gamma} \Big\}, \qquad (20)$$

where $\tilde{t} = t\omega_0$ is a dimensionless parameter. Plots for different values of γ are shown in the Fig. 2 and 3.

As we can see from (20), the displacement of the fractional oscillator is essentially described by the Mittag-Leffler function

$$E_{2\gamma}\Big\{ -\gamma^{2(1-\gamma)}\tilde{t}^{2\gamma} \Big\}.$$

Also it is proved that, if γ is less than 1 the displacement shows the behavior of a damped harmonic oscillator. The

damping of fractional oscillator is intrinsic to the equation of motion and not by introducing an additional force as in the case of an ordinary damping harmonic oscillator. The fractional oscillator should be considered as an ensemble average of harmonic oscillators.

On the other hand, solution of the Eq. (12) is given by

$$\tilde{x}(t) = \tilde{x}_0 E_{\gamma}\Big\{ -\frac{k\sigma^{1-\gamma}}{\beta} t^{\gamma} \Big\}, \qquad (21)$$

where $E_{\gamma}\{\}$ is the Mittag-Leffler function defined above.

For the case $\gamma = 1$, the expression (21) becomes

$$\tilde{x}(t) = \tilde{x}_0 e^{-\frac{k}{\beta}t}, \qquad (22)$$

which is the well-known solution for the integer differential Eq. (12). In this case the relation between γ and σ is given by

$$\gamma = \frac{k}{\beta}\sigma, \qquad 0 < \sigma \le \frac{k}{\beta}. \qquad (23)$$

The solution (21) of the fractional Eq. (12), taking into account the relation (23), may be written as follows

$$\tilde{x}(\tilde{t}) = \tilde{x}_0 E_{\gamma}\Big\{ -\gamma^{(1-\gamma)}\tilde{t}^{\gamma} \Big\}, \qquad (24)$$

where $\tilde{t} = \frac{k}{\beta}t$ is a dimensionless parameter. Figures 4 and 5, show the solution of (24) for different values of γ.

3. Conclusion

In this work we have proposed a new fractional differential equation of order $0 < \gamma \le 1$ to describe the mechanical oscillations of a simple system. In particular, we analyze the systems mass-spring and spring-damper. In order to be consistent with the physical equation the new parameter σ is introduced. The proposed equation gives a new universal behavior for the oscillating systems, Eqs. (20) and (24), for equal value of the magnitude, $\delta = 1 - \gamma$ characterizing the existence of the fractional structures on the system. We also found that there is a relation between γ and σ depending on the system studied, see the Eqs. (19) and (23). The analytical solutions are given in terms of the Mittag-Leffler function. They depend on the parameter γ and preserve physical units in the system parameters. The classical cases are recovered by taking the limit when $\gamma = 1$.

The general case of the Eq. (10) with respect to the parameter γ and the classification of fractional systems depending on the magnitude δ will be made in a future paper.

We hope that this way of dealing with fractional differential equations can help us to understand better the behavior of the fractional order systems.

Acknowledgments

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