Scattering of a ball by a bat in the limit of small deformation

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The problem of the mechanical evolution of a shock between a cylindrically symmetric object and a spherical one is solved in the strict rigid (small deformations) approximation for arbitrary values of the initial conditions. The friction during the impact is assumed to satisfy the standard rules. Firstly, when it is assumed that the only source of energy dissipation is friction, the problem is fully solved by determining the conditions at the separation point between the two bodies. A relation determining whether the contact points of the two bodies slides between them or become at rest (to be $pure\, rotation$ state) at the end of the impact, is determined for this case of the purely frictional energy dissipation. In second place, the solution is generalized to include losses in addition to the frictional ones. It follows that, whatever the mechanism of the additional form of dissipation is, assumed that it did not affects the usual forms of the laws of friction, the complementary losses only can change the ending value of the impulse I done by the normal force of the bat on the ball at separation. Then, the dynamical evolution of all the mechanical quantities with the value of I during the shock process remains invariable. Thus, under the adopter assumptions of strict rigidity and validity of the standard rules for friction, the solution of the problem is also exactly found, whenever the total amount of dissipated energy is considered as known (by example, measuring the ending mechanical energy of the system). The analysis allows to determine the values of the tangential and normal coefficients of restitution for the class of shocks examined. Finally, the results are applied to the description of experimental measures of the slow motion scattering of a baseball by a bat. The evaluations satisfactorily reproduce the measured curves for the output center of mass and angular velocities of the ball as functions of the scattering angle and the impact parameter, respectively.

Keywords: Mechanical impact; shock; scattering; physics of baseball.

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1. Introduction

Classical Mechanic is an ancient field of Physics [1,7]. An innumerable amount of problems had been already solved, which by now constitute a main database for technological applications. In particular, the scattering problem in the framework of Particle, Nuclear and Atomic Physics has been the subject of intense investigation along centuries of research [8-10]. On another hand, as stated in Ref. 11, at difference with the situation in microscopic Physics, the scattering between macroscopic bodies, had not been a similarly attended area of study. However, in relatively recent times, and as motivated by the relevance within the baseball of the shocking process between the bat and the ball in the baseball sport, a research activity on the theme had been stimulated [11-14]. An extended study of the physical process in impact mechanics can be found in Ref. 13. References 11 and 14 presented detailed studies of the shock problem of a bat and a ball directed to investigate the optimal batting configurations and the scattering results of the impact process at low velocities. The solution of the shock process given in those works were found under the restrictions of: 1) an assumed two dimensional character of the shock, and 2) the use of particular models for the impulse forces appearing during the impact. The work presented in Ref. 12 is devoted to investigate the influences of the lack of rigidity of the bat and the ball, and the dynamical evolution of the ball in the air on the bating process results. In these cited works references to a number of additional studies stimulated by the relevance of mechanical processes in sports can be found.

In the present study we investigate the problem of the shock between a rigid and spherical object (to be called as the "ball") and an also rigid body (to be named as the "bat") showing a cylindrical symmetry axis. The general motivation of the work is to clarify up to what extent the assumption of rigidity of the bodies, when taken in conjunction with the satisfaction of the usual laws for static and dynamic friction could allow for a full solution of the problem. When considered for possible practical applications, it can be noted that the strict rigidity approximation adopted determines that the results might be of use to study real shock events for which the ball and the bat are not appreciable deformed. This is the case of impacts occurring at small velocities in baseball, for slow direct normal relative velocities at the contact points between the bat and the ball, with possible higher values of the relative tangential velocity. Such a study is done in the last section of the work. In the case of billiard balls the rigidity approximation seems to be obeyed for most of the shocks occurring in normal playing.

The initial conditions for the shock in this work also generalize the ones employed in Ref. 11 and 14, by including arbitrary starting data for the center of mass and angular velocities. In addition the values of the normal and tangential coefficients of restitution are predicted. The results are also applied to describe the experimental measures of the slow motion scattering of a ball by a bat presented in Ref. 11. For bookkeeping purposes, the simpler explicit solution of the conservative and friction less impact is also presented in an appendix.

The discussion starts by considering the case in which sliding frictional forces in the contact points develops in the assumed strictly rigid bat and ball. As mentioned above, the situation generalizes the one studied in Refs. 11 and 14, by removing their two main assumptions: 1) The shock will considered as fully three dimensional with arbitrary initial conditions, and the bat form is only restricted to be a solid of rotation showing a cylindrical symmetry axis, 2) No model about the nature of the normal impact force will be adopted. Only the standard connection between the sliding friction and the normal force will be assumed. That is, the modulus of the sliding friction force will be equal to the friction coefficient μ times the magnitude of the instantaneous normal force at the contact point. As usual, the friction force over one of the bodies will be directed in the opposite sense to the tangent component for the relative velocity of the contact point on that object with respect to the contact point on the other body.

In addition, a criterium is found for deciding about whether the ending state of the shock corresponds to sliding tangent surfaces, or to a $pure\ rotation$ state. The $pure\ rotation$ state will be called that one in which the tangent surfaces end the shocking process by showing null tangential relative velocity. Analytic and integral expressions of the ending values of the center of mass and angular velocities are given, in terms of the solution of a simple differential equation for the two components of the relative velocities between the tangent points. It is an interesting outcome that the impulse done by the normal force of the bat on the ball can be employed in place of the time in describing the dynamical evolution during the shock. This property was noted in the work [14].

The solution for the ending velocities of the ball and the bat presents two kinds of behavior: In one case, the friction force is unable to reduce the tangential component of the relative velocity to zero during the small time interval of the shock. In this option, the two bodies end the impact with a remaining sliding between their contact surfaces.

In the alternative ending state, the frictional force becomes able in reducing to zero the relative velocity of the two contact points between the bodies at an intermediate instant of the shocking interval. In general, at this time instant t_{rp} in which the sliding between surfaces ends, the normal force is not yet vanishing. This means that a finite portion of the initial mechanical energy of the system is yet stored in the form of $elastic$ deformation energy. Therefore, after the attaining of the $pure\ rotation$ state, the system, which is yet within the shocking process, evolves in an alternative way than in the previous $sliding$ period. In this second interval, the evolution becomes conservative, and the equations are similar but not identical to the ones corresponding to the friction less conservative problem solved in the Appendix A. Their difference rests in that within this period, a static kind of frictional force can contribute to the conservative mutual impulses between the bodies.

As it was remarked before, in order to make the solution applicable to the realistic cases (in which there exist appreciable energy losses due heat, deformations, sound, etc., in addition to the frictional one), the solution of the problem allowing only frictional dissipation is here generalized to include alternative energy losses. The generalization was suggested and helped by two factors: a) The possibility of properly identifying the amount of stored elastic energy in the system at any moment when the energy losses are purely due to friction; b) The helpful technical fact that due to the assumed extreme rigidity approximation, the exact evolution of the system, no matter the nature of the energy losses, only depends on one single variable: the net impulse $I(t)$ transmitted up to a given instant t by the normal force exerted by the bat on the ball [14]. It should be underlined, that the found solution for this general case of dissipation, can be considered as an exact one, when one considers as a known quantity the dissipated mechanical energy at the end of the shock (by example by measuring the total energy at the end of the impacts in repeated experiments). Since all the components for the relative normal and tangential velocities between the contact points are predicted for this general solution, it follows that the values of important phenomenological quantities as the tangential and normal coefficients of restitution are also predicted, assumed that the total energy loss is known. In the case in that the friction is the only source of dissipation, the energy loss is also following and is not needed to be measured.

The generalized analysis is then applied to the description of the experiments on the scattering of a ball by a bat reported in Ref. 11. After phenomenologically fixing a single experimentally measured quantity: the value of the center of mass velocity of the ball at zero values of the impact parameter and its initial angular velocity, the calculated results furnish a satisfactory description of the reported measurements. In particular, the curves for the ending velocity of the ball as functions of the scattering angle coincided with the measured ones within the range of the dispersion of the data, for each of the three values of the initial angular velocity of the ball employed in the experiences.

The exposition of the work proceeds as follows. In Sec. 2, the starting equations for the shock problem are presented and the notation and basic definitions are given. Afterwards, the solution of the impact problem for the case in which the ending state is assumed to correspond to sliding contact surfaces is exposed. The Sec. 3 continues by presenting the solution for the situation in which the contact point of the ball and the bat finish the shock process in the $pure\ rotation$ state. Section 4 exposes how the solution for the case in which the losses are only due to friction can be readily generalized for the more general situation including additional types of energy dissipation. Finally, in Sec. 5 the results of previous sections are applied to solve the scattering problem of the ball by the bat in the particular configuration considered in the experiments reported in Ref. 11. The results for the description of the measurement reported in that reference are presented. Finally, in Sec. 5 a summary of the results is given.

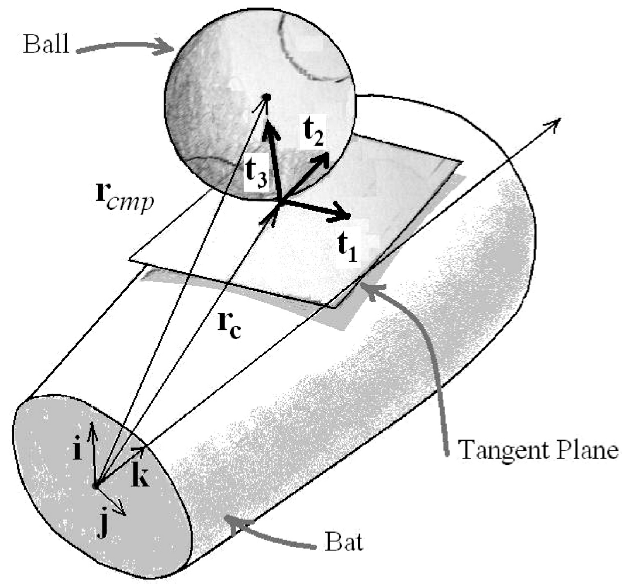

FIGURE 1. The figure illustrate the defined systems of reference to be systematically employed along the work. The ball and the bat are shown in the moment at which the impact starts.

2. Only frictional dissipative impact: the sliding final state

This section will expose some basic considerations and definitions which will be of use along the presentation. The Fig. 1, illustrates the shock process between the ball and the bat in the precise instant at which they become in contact. The adopted laboratory system of reference (to be named as the Lab system in what follows), will be situated on the center of mass of the bat and having its z (x_3) coordinate axis being collinear with the cylindrical symmetry axis of the bat. The three unitary vectors of the Lab reference frame axes will be $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. In what follows, bold letters will indicate vectors. The unit vector $\mathbf{k}$ along the z axis, will point in the direction of the barrel of the bat, and thus, the vectors $\mathbf{i}, \mathbf{j}$ will be contained in a transversal section of the bat as illustrated in Fig. 1, and they are chosen to form a direct triad with $\mathbf{k}$. The vector $\mathbf{r}_c$ depicted in Fig. 1, defines the position of the contact point of the two bodies in the above defined reference frame. Note that the adoption of the Lab frame does not restrict the generality of the discussion. The vector $\mathbf{r}_{\text{cmp}}$ gives the position of the center of mass of the ball in the Lab coordinate system. The set of three unit vectors $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\}$ are defined as follows: $\mathbf{t}_3$ is normal to the common tangent surface of both bodies at the contact point, and is directed as pointing outside the volume of the bat. Further, $\mathbf{t}_2$ can be defined as a tangential unit vector being contained in a common plane with the unit vector $\mathbf{k}$ and having a positive scalar product with it. Finally, $\mathbf{t}_1$ is defined as being orthogonal to $\mathbf{t}_2$ and $\mathbf{t}_3$ by also forming with them a direct triad.

Let us specify now few physical assumptions that will be adopted for the solution of the problem. Firstly, as it was already stated, it will be considered that the ball and the bat are ideally rigid bodies. That is, the elastic forces are supposed as being so strong, that the spacial forms of both objects remain almost the same during the whole shocking time interval δt_{ch}. Naturally, this lapse will be supposed as being extremely short. These assumptions will be reflected in the discussion to follow, in which the whole geometry of the arrangement will be supposed to be invariable during the time interval δt_{ch}. The only quantities that will allowed to change are the linear and angular velocities of the two bodies.

Let us write the general equations of motions for the evolution of the center of mass linear momenta $\mathbf{P}_p$, $\mathbf{P}_b$ and the angular momenta $\mathbf{L}_p$, $\mathbf{L}_b$ of the ball and the bat, respectively, during the shocking interval δt_{ch}. They can be written as

$$\frac{d}{dt}\mathbf{P}_p(t) = \mathbf{F}(t), \tag{1}$$

$$\frac{d}{dt}\mathbf{P}_b(t) = -\mathbf{F}(t), \tag{2}$$

$$\frac{d}{dt}\mathbf{L}_p(t) = (\mathbf{r}_c - \mathbf{r}_{\text{cmb}}) \times \mathbf{F}(t), \tag{3}$$

$$\frac{d}{dt}\mathbf{L}_b(t) = -\mathbf{r}_c \times \mathbf{F}(t), \tag{4}$$

in which $\mathbf{F}(t)$ is the very rapidly varying impulsive force which is exerted by the bat on the ball. The definitions of the momenta are

$$\begin{aligned}
\mathbf{P}_b(t) &= m_b \mathbf{v}_{\text{cmb}} = m_b \frac{d}{dt}\mathbf{r}_{\text{cmb}}(t), \\
\mathbf{P}_p(t) &= m_p \mathbf{v}_{\text{cmp}} = m_p \frac{d}{dt}\mathbf{r}_{\text{cmp}}(t), \\
\mathbf{L}_b(t) &= \widehat{\mathbf{I}}_b \cdot \mathbf{w}_b, \\
\mathbf{L}_p(t) &= \widehat{\mathbf{I}}_p \cdot \mathbf{w}_p(t) + \mathbf{r}_{\text{cmp}}(t) \times m_p\ \dot{\mathbf{v}}_{\text{cmp}}\ (t),
\end{aligned} \tag{5}$$

in which m_b and m_p are the masses of the bate and the ball, respectively. The angular momenta of the bat is defined with respect to the Lab reference frame, and for the simplicity of the further discussion the angular momenta of the ball was defined with respect to its center of mass. Note that the angular impulse in Eq. (3) is consistent with this definition. The inertia tensor of the ball $\widehat{\mathbf{I}}_p$ is the identity matrix and the one associated to the bat $\widehat{\mathbf{I}}_b$ has cylindrical symmetry in the Lab reference system. The explicit forms of the inertia tensor are

$$\widehat{\mathbf{I}}_p = I_p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\widehat{\mathbf{I}}_b = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = \begin{pmatrix} I_t & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \quad (6)$$

Note, that the vector $\mathbf{r}_{\text{cmb}}(t)$ is the position of the center of mass of the bat, which during the shocking interval δt_{ch} remains to be very close to the origin of the Lab reference frame depicted in Fig. 1, if the bodies are sufficiently rigid for the given initial relative velocities and angular momenta. That is, in the here adopted strict rigidity approximation, in which the two bodies are assumed be completely invariable in form and position during the impact, it will assumed that $\mathbf{r}_{\text{cmb}}(t) = 0$.

In a first instance, we will include energy dissipation only through the presence of sliding friction during the impact. The inclusion of additional sources of energy losses will be incorporated in last sections. As stated in the introduction the usual laws of friction will be assumed. That is, we will consider that the modulus of the sliding friction vector will be the sliding friction coefficient μ, times the modulus of the normal force between the bodies. Further, the direction of the friction force over one of the two bodies contact point, will be opposite to the relative velocity of this point with respect to the contact point of the other body.

Let us note that, because the normal force grows starting from zero at the beginning of the shocking period, in general the first stage of the shock process should correspond to a situation in which the contact points of the ball and the bat slide between them at the beginning of the process. Only in the particular situation in which the tangent velocity is already vanishing at the beginning of the impact, this period will not exist. In such a case the solution is directly given by the one presented in the next section.

Then, consider the equations of motion as written for this initial process. For an instant t being inside the very small time interval δt_{ch} during which the bodies are in contact, the equations can be written in the form

$$m_p\, d\mathbf{v}_{\text{cmp}}(t) = \left(-\mu \frac{\mathbf{v}_{pb}^{(t)}}{\mid \mathbf{v}_{pb}^{(t)} \mid} + \mathbf{t}_3\right) dI\ , \quad (7)$$

$$m_b\, d\mathbf{v}_{\text{cmb}}(t) = \left(\mu \frac{\mathbf{v}_{pb}^{(t)}}{\mid \mathbf{v}_{pb}^{(t)} \mid} - \mathbf{t}_3\right) dI\ , \quad (8)$$

$$I_p\ d\mathbf{w}_p(t) = -\mu dI (\mathbf{r}_c - \mathbf{r}_{\text{cmp}}) \times \frac{\mathbf{v}_{pb}^{(t)}}{\mid \mathbf{v}_{pb}^{(t)} \mid}, \quad (9)$$

$$d\,\mathbf{w}_b(t) = \mu dI \widehat{\mathbf{I}}_b^{-1} \cdot \mathbf{r}_c \times \frac{\mathbf{v}_{pb}^{(t)}}{\mid \mathbf{v}_{pb}^{(t)} \mid} - \frac{1}{I_t} dI \mathbf{r}_c \times \mathbf{t}_3, \quad (10)$$

where a new magnitude appearing is the differential impulse $dI = N_{b\to p} dt$ done by the normal force $N_{b\to p}$ exerted by the ball on the bat. It allows to define the net impulse done by this force up to a given time t by

$$I(t) = \int_0^t N_{b\to p} dt. \quad (11)$$

Another quantity appearing is the tangential component of the relative velocity $\mathbf{v}_{pb}^{(t)}$ between the contact point of the ball and the corresponding contact point of the bat (for which $\mid \mathbf{v}_{pb}^{(t)} \mid$ means its modulus). As defined above, μ is the sliding friction coefficient and the inverse of the inertia moment tensor of the bat $\widehat{\mathbf{I}}_p^{-1}$ can be explicitly written as follows

$$\widehat{\mathbf{I}}_b^{-1} = \begin{pmatrix} \frac{1}{I} & 0 & 0 \\ 0 & \frac{1}{I} & 0 \\ 0 & 0 & \frac{1}{I_3} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{I} & 0 & 0 \\ 0 & \frac{1}{I} & 0 \\ 0 & 0 & \frac{1}{I} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{I_3} - \frac{1}{I} \end{pmatrix}$$
$$= \frac{1}{I}\delta + \left(\frac{1}{I_3} - \frac{1}{I}\right) \mathbf{k}\,\mathbf{k},$$
$$\mathbf{i} = (1,0,0),\ \ \mathbf{j} = (0,1,0) \ \text{ and } \ \mathbf{k} = (0,0,1), \quad (12)$$

where $\mathbf{k}\,\mathbf{k}$ means the diadic tensor $\mathbf{k}\ \mathbf{k} \equiv k_i\, k_j$.

Let us define now a simplified notation for the tangential component $\mathbf{v}_{pb}^{(t)}(t)$ of the full relative velocity $\mathbf{v}_{pb}(t)$ between the contact points of the ball the bat as follows

$$\mathbf{v}_{pb}^{(t)}(t) \equiv \mathbf{v}(t) = v_1(t)\ \mathbf{t}_1 + v_2(t)\ \mathbf{t}_2,$$
$$\mid \mathbf{v} \mid \equiv v(t) = \sqrt{(v_1)^2 + (v_1)^2}. \quad (13)$$

Then, $\mathbf{v}_{pb}(t)$ can be written as follows

$$\mathbf{v}_{pb}(t) = \mathbf{v}_{pb}^{(t)} + (\mathbf{v}_{pb}(t) \cdot \mathbf{t}_3)\mathbf{t}_3$$
$$= v_1(t)\mathbf{t}_1 + v_2(t)\mathbf{t}_2 + (\mathbf{v}_{pb}) \cdot \mathbf{t}_3\mathbf{t}_3$$
$$= \mathbf{v}_{\text{cmp}}(t) - \mathbf{v}_{\text{cmb}}(t) + \mathbf{w}_p(t)$$
$$\times (\mathbf{r}_c - \mathbf{r}_{\text{cmp}}) - \mathbf{w}_b(t) \times \mathbf{r}_c, \quad (14)$$

which allows to write for the variation of $\mathbf{v}_{pb}(t)$ in a time interval dt in the considering sliding interval the expression

$$d\mathbf{v}_{pb}(t) = d\mathbf{v}_{\text{cmp}}(t) - d\mathbf{v}_{\text{cmb}}(t) + d(\mathbf{w}_p(t))$$
$$\times (\mathbf{r}_c - \mathbf{r}_{\text{cmp}}) - d\left(\mathbf{w}_b(t)\right) \times \mathbf{r}_c,$$

where, as described before, the geometry of the system has been assumed as invariant due to the perfect rigidity of the ball and the bat. This assumption can be satisfied, in particular, if all the velocities are chosen to be scaled to sufficiently small values producing small enough deformations. Employing the equations of motion allows to express the above variation in terms of the differential impulse of the normal forces in the following linear form

$$d\mathbf{v}_{pb}(t) = -\mu\left(\frac{1}{m_p}+\frac{1}{m_b}\right)\frac{\mathbf{v}(t)}{v(t)}dI + \left(\frac{1}{m_p}+\frac{1}{m_b}\right)\mathbf{t}_3 dI$$
$$-\frac{\mu}{I_p}dI(\mathbf{r}_c-\mathbf{r}_{\text{cmp}})\times\left((\mathbf{r}_c-\mathbf{r}_{\text{cmp}})\times\frac{\mathbf{v}(t)}{v(t)}\right)$$
$$+\mu dI\widehat{\mathbf{I}}_b^{-1}\cdot\mathbf{r}_c\times\frac{\mathbf{v}(t)}{v(t)}-\frac{1}{I_t}\,dI\;\mathbf{r}_c\times\mathbf{t}_3. \tag{15}$$

After projecting the above relation over $\mathbf{t}_1$ and $\mathbf{t}_2$, the following set of two differential equations for the variation of the relative tangential velocity components as functions of the impulse of the normal force of the bat on the ball can be obtained

$$dv_1(I) = -s_1\frac{v_1(I)}{\sqrt{(v_1(I))^2+(v_2(I))^2}}dI,$$
$$dv_2(I) = -s_2\frac{v_2(I)}{\sqrt{(v_1(I))^2+(v_2(I))^2}}dI + s_0\,dI\,, \tag{16}$$

in which the parameters appearing depend on the system's properties as follows

$$s_1 = \mu\Bigg(\frac{1}{m_p}+\frac{1}{m_b}+\frac{(\mathbf{r}_c-\mathbf{r}_{\text{cmp}})^2}{I_p}$$
$$+\frac{(\mathbf{r}_c)^2}{I_t}+\left(\frac{1}{I_3}-\frac{1}{I_t}\right)(\mathbf{t}_1\cdot(\mathbf{r}_c\times\mathbf{k}))^2\Bigg), \tag{17}$$

$$s_2 = \mu\Bigg(\frac{1}{m_p}+\frac{1}{m_b}+\frac{(\mathbf{r}_c-\mathbf{r}_{\text{cmp}})^2}{I_p}$$
$$+\frac{(\mathbf{r}_c)^2}{I_t}-\frac{(\mathbf{r}_c\cdot\mathbf{t}_2)^2}{I_t}\Bigg), \tag{18}$$

$$s_0 = (\mathbf{r}_c\cdot\mathbf{t}_3)(\mathbf{r}_c\cdot\mathbf{t}_2). \tag{19}$$

At this point it can be underlined that the time had disappeared form the equations of motion for the tangent velocities. That is, the time dependence is all embodied in the time dependence of net impulse of the normal force of the bat on the ball $I(t)$ defined by (11). This functional relation between the impulse I and the time t will be assumed in what follows for studying the evolution of the system during the shock interval in terms the impulse I in place of the time t. A very important consequence of the unique dependence of the evolution on the net impulse I, is the fact the presence of any form of dissipation in addition to friction will not affect the result of the evolution of all the properties up to a given state with definite value of I. Therefore, the only effect of the presence of such supplementary forms of dissipation will be to determine a different instants in which separation (end of the impact) will occur. This property will be employed in last sections to extend the solution to include non dissipation effects.

The solution of the above set of equations for the relative velocities, after determining their initial values allows to determine the evolution with the impulse I of all these velocities. As it will be verified in the particular examples solved in next sections, the general behavior of the solutions is such that both components evolve with I in a continuous way up to a critical value of the normal impulse, at which both components simultaneously tend to vanish. This point correspond to the attainment of the $pure\ rotation$ state. However, whether this critical situation would be approached or not depends on the dynamical equations of motion: they in fact should determine whether or not positive values of the normal force done by the bat on the ball can exist up to the arriving to the $pure\ rotation$ state. Verifying the above remarks, the set of equations of motion (7-10) during the impact process can be written in terms of the most appropriate evolution parameter I as follows

$$m_p\,\frac{d\mathbf{v}_{\text{cmp}}(I)}{dI} = \left(-\mu\frac{\mathbf{v}(I)}{v(I)}+\mathbf{t}_3\right), \tag{20}$$

$$m_b\,\frac{d\mathbf{v}_{\text{cmb}}(I)}{dI} = \left(\mu\frac{\mathbf{v}(I)}{v(I)}-\mathbf{t}_3\right), \tag{21}$$

$$I_p\,\frac{d\mathbf{w}_p(I)}{dI} = -\mu\,(\mathbf{r}_c-\mathbf{r}_{\text{cmp}})\times\frac{\mathbf{v}(I)}{v(I)}, \tag{22}$$

$$\frac{d\,\mathbf{w}_b(I)}{dI} = \mu\;\widehat{\mathbf{I}}_b^{-1}\cdot\mathbf{r}_c\times\frac{\mathbf{v}(I)}{v(I)}-\frac{1}{I_t}\,\mathbf{r}_c\times\mathbf{t}_3. \tag{23}$$

These equation clearly evidence that the evaluation of all the velocities is only determined by the value of the impulse of the normal force I being exerted by the bat on the ball. This impulse is defined by Eq. (11). Since the solution of the differential Eqs. (16) determines the tangential velocity as a function of I, the Eqs. (20)-(23) can be integrated to find out all the velocities in terms of I as follows

$$\mathbf{v}_{\text{cmp}}(I) = \mathbf{v}_{\text{cmp}}(0)+\frac{1}{m_p}(-\mathbf{I}_{fr}(I)+I\mathbf{t}_3), \tag{24}$$

$$\mathbf{v}_{\text{cmb}}(I) = \mathbf{v}_{\text{cmb}}(0)+\frac{1}{m_b}(\mathbf{I}_{fr}(I)-I\mathbf{t}_3), \tag{25}$$

$$\mathbf{w}_p(I) = \mathbf{w}_p(0)-\frac{1}{I_p}(\mathbf{r}_c-\mathbf{r}_{\text{cmp}})\times\mathbf{I}_{fr}(I), \tag{26}$$

$$\mathbf{w}_b(I) = \mathbf{w}_b(0)+\widehat{\mathbf{I}}_b^{-1}\cdot(\mathbf{r}_c\times\mathbf{I}_{fr}(I))-\frac{1}{I_t}I\mathbf{r}_c\times\mathbf{t}_3, \tag{27}$$

where the impulse done by the tangential frictional force is defined as a function of I by the following integral

$$\mathbf{I}_{fr}(I) = \mu\int_0^I dI'\;\frac{\mathbf{v}(I')}{v(I')}. \tag{28}$$

For a coming reference to them, let us define now the increments in the velocities with respect to their initial values $\mathbf{v}_{\text{cmp}}^{(\text{in})}, \mathbf{v}_{\text{cmb}}^{(\text{in})}, \mathbf{w}_{p}^{(\text{in})}$ and $\mathbf{w}_{b}^{(\text{in})}$ when the impulse rises to a definite value $I(t)$ by:

$$\begin{aligned}
\Delta\, \mathbf{v}_{\text{cmp}}(I) &= \mathbf{v}_{\text{cmp}}(I) - \mathbf{v}_{\text{cmp}}(0),\\
\Delta\, \mathbf{v}_{\text{cmb}}(I) &= \mathbf{v}_{\text{cmb}}(I) - \mathbf{v}_{\text{cmb}}(0),\\
\Delta \mathbf{w}_{p}(I) &= \mathbf{w}_{p}(I) - \mathbf{w}_{p}(0),\\
\Delta \mathbf{w}_{b}(I) &= \mathbf{w}_{b}(I) - \mathbf{w}_{b}(0), \qquad (29)
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_{\text{cmp}}(0) &= \mathbf{v}_{\text{cmp}}^{(\text{in})}, \mathbf{v}_{\text{cmb}}(0) = \mathbf{v}_{\text{cmb}}^{(\text{in})},\\
\mathbf{w}_{p}(0) &= \mathbf{w}_{p}^{(\text{in})},\ \mathbf{w}_{b}(0) = \mathbf{w}_{b}^{(\text{in})}. \qquad (30)
\end{aligned}$$

The above formulae indicate that the shock problem in this assumed initial sliding process, will become solved, after finding a condition determining the final value of the tangent velocity component at the moment of separation of the bodies. Let us consider this point in what follows.

In the assumed in this section case, in which the shock is finalized when the contact points are yet sliding between them, the posed equations remain being valid along all the shock interval. In this situation, the appropriate condition for fixing the ending value of the impulse I is that total dissipative work W done (due to friction or non elastic processes) up to the value of the impulse I, should be equal to the decrease of the total kinetic energy ΔE_{kin} along the evolution up to the same value of I. The physical reason for this condition is that when the normal force vanishes, which defines the separation of the bat and the ball, the conservation of energy implies that all the non already dissipated part of the mechanical energy should appear in the form of the translational and rotational kinetic energies of the bat and the ball.

Therefore, since as it has been concluded, the evolution as a function of I is completely independent of the nature of the dissipation, it follows that the solution of the problem in this period is only depending of the fraction of the initial kinetic energy of the two bodies which becomes dissipated in the shocking process, a quantity which only will determine the value of the impulse transmitted at separation I_{out}. This condition leads to the equation for I_{out}

$$W(I_{\text{out}}) = \Delta E_{\text{kin}}(I_{\text{out}}). \qquad (31)$$

The part of the total dissipative work W which is done by the friction up to the value of the time t, for which the impulse has the value $I(t)$ can be calculated to be

$$\begin{aligned}
W_{fr}(I(t)) &= -\int_0^t \mu N_{b\to p} \mid \mathbf{v} \mid dt\\
&= -\mu \int_0^{I(t)} \mid \mathbf{v}(I) \mid dI. \qquad (32)
\end{aligned}$$

In what follows, we will also consider also the existence of additional sources of dissipation, as the one associated to the non complete elastic behavior of the ball and the bat. Then, the total dissipative work $W(I)$ will be written as

$$W(I) = W_{fr}(I) + W_{\text{add}}(I), \qquad (33)$$

where $W_{\text{add}}(I)$ represents the amount of mechanical energy dissipated in the system up to the instant in which the normal impulse takes the value I due to mechanisms additional to the frictional one. It can be understood that the dependence on I of $W_{\text{add}}\ (I)$ will depend on the concrete forms of the dissipation process acting in the bat at the ball. However, it is a remarkable property that, assumed that if the system becomes able to arrive to the pure rotation state, the velocities at this point result to be (in the here considered perfect rigidity situation and only at this particular instant) completely independent of the existence of non frictional kinds of dissipation. The particular cases of the experimental results to be considered in next sections belong to the situation in which $pure$ $rotation$ is established.

The explicit form of the condition (31) determining the separation point under sliding regime (in case that it effectively occurs) is completed after defining the formula for the increase in the total kinetic energy as a function of I. It follows after expressing the values of the final lineal and angular velocities in terms of their initial values plus their increment. Making use of the Eqs. (29) this quantity has the expression

$$\begin{aligned}
\Delta E_{\text{kin}}(I) = &\frac{m_p}{2}\left(2\mathbf{v}_{\text{cmp}}^{(\text{in})}\cdot \Delta \mathbf{v}_{\text{cmp}}(I) + (\Delta \mathbf{v}_{\text{cmp}}(I))^2\right)\\
&+ \frac{m_b}{2}\left(2\mathbf{v}_{\text{cmb}}^{(\text{in})}\cdot \Delta \mathbf{v}_{\text{cmb}}(I) + (\Delta \mathbf{v}_{\text{cmb}}(I))^2\right)\\
&+ \frac{I_p}{2}\left(2\mathbf{w}_{p}^{(\text{in})}\cdot \Delta \mathbf{w}_{p}(I) + (\Delta \mathbf{w}_{p}(I))^2\right)\\
&+ \frac{1}{2}\Big(2\mathbf{w}_{b}^{(\text{in})}\cdot \mathbf{I}_b \cdot . \Delta \mathbf{w}_{b}(I)\\
&+ \Delta \mathbf{w}_{b}(I)\cdot \mathbf{I}_b \cdot \Delta \mathbf{w}_{b}(I)\Big), \qquad (34)
\end{aligned}$$

in which all the increments in the linear and angular velocities are given by the formulae furnished by the integrated equations of motion (24-27) and the solution of the differential equations for the tangent velocity (16).

Therefore, the full solution of the problem in the assumed situation in which along all the shock process the contact points of the ball and bat slide between them, can be obtained after finding the value of I_{out} making equal to zero the function

$$S(I_{\text{out}}) = W(I_{\text{out}}) - \Delta E_{\text{kin}}(I_{\text{out}}). \qquad (35)$$

It should be remarked that for the case in which all the dissipation is determined by friction, the problem is completely solved because the evaluation of the work of friction as a function of I_{out} is completely defined by (32) in terms of the found solutions of the tangential velocities as functions of the impulse I. The case of additional sources of losses, for to be analogously solved needs for a definition of the specific mechanism leading to dissipation. A model for taking in consideration such a mechanism will be constructed in the coming Sec. 4 in order to apply the results to the description of realistic experimental measures given in Ref. 11.

This completes the finding of the solution for the evolution of the impact process in this firstly assumed situation. It can be helpful to recall that all the shocking events in which the contact points slide at the beginning of the impact, start evolving as guided by this just considered *sliding* case. In the case that no solution exists for the above equation, the situation should correspond to a problem in which the work which is done by the friction along the whole initial sliding period is not able to become equal to the decrease in the kinetic energy of the bat and the ball. Let us consider the finding of the solution in this second regime.

3. Only frictional dissipative impact: the "pure rotation" final state

As followed from the previous section, we will now consider the situation in which the maximal work that can be done by the friction up to the point in which the sliding between the contact points vanish, is not able to dissipate the mechanical energy down to the value needed to coincide with the total kinetic energy at this same sliding state. In other words, at the instant t_{rp} (or the corresponding impulse of the bat on the ball I_{rp}) at which sliding stops, and the *pure rotation* state is attained, a portion of the total initial energy should be yet stored in the form of deformation energy. In addition, the presence of deformation energies forces is represented in the considered problem by a non vanishing normal force.

Thus, the shock process is divided in two parts, each one being governed by different dynamical equations. The first one was discussed in the past section, in which sliding occurs and stops at the impulse value I_{rp} at which the slice between the contact points of the ball and bat ends. At this value of the impulse the increments in the velocities are given by the formulae (24-27), in which the tangential sliding velocity vanishes. These linear and angular velocity increments have the explicit expressions

$$\mathbf{v}_{\text{cmp}}(I_{rp}) = \mathbf{v}_{\text{cmp}}(0) + \frac{1}{m_p}(-\mathbf{I}_{fr}(I_{rp}) + I_{rp}\mathbf{t}_3), \quad (36)$$

$$\mathbf{v}_{\text{cmb}}(I_{rp}) = \mathbf{v}_{\text{cmb}}(0) + \frac{1}{m_b}(\mathbf{I}_{fr}(I_{rp}) - I_{rp}\mathbf{t}_3), \quad (37)$$

$$\mathbf{w}_p(I_{rp}) = \mathbf{w}_p(0) - \frac{1}{I_p}\,(\mathbf{r}_c - \mathbf{r}_{\text{cmp}}) \times \mathbf{I}_{fr}(I_{rp}), \quad (38)$$

$$\mathbf{w}_b(I_{rp}) = \mathbf{w}_b(0) \; + \; \widehat{\mathbf{I}}_b^{-1} \cdot (\mathbf{r}_c \times \mathbf{I}_{fr}(I_{rp})) - \frac{1}{I_t}\, I_{rp}\, \mathbf{r}_c \times \mathbf{t}_3, \quad (39)$$

These quantities fully determine the linear and angular velocities of the ball and the bat in the next intermediate *pure rotation* state in which a portion of the total mechanical energy is yet stored in the form of elastic deformation energy. The value of the total kinetic energy at this point $E^{(rp)}$ can be calculated through the formula

$$E^{(rp)} = \frac{m_p}{2}(\mathbf{v}_{\text{cmp}}(I_{rp}))^2 + \frac{m_b}{2}(\mathbf{v}_{\text{cmb}}(I_{rp}))^2 + \frac{I_p}{2}(\mathbf{w}_p(I_{rp}))^2 + \frac{1}{2}\mathbf{w}_b(I_{rp}) \cdot \mathbf{I}_b \cdot \mathbf{w}_b(I_{rp}). \quad (40)$$

As remarked before, the existence of deformation energy at the transmitted impulse I_{rp} is represented by the existence of a yet not vanishing normal force, which should tend to zero in the new stage of evolution. In this second period of the impact process, the stored elastic energy transforms in a contribution to the kinetic energy of the ball and the bat at the real ending state of the shock, in which the normal force tends to vanish. Deformation losses can occurs also in this process.

Therefore, after the instant at which the impulse is I_{rp}, the system will be governed by a similar, but not identical, set of equations valid for the conservative shock studied in appendix A. The difference is related with the fact that during this last interval, a *static* frictional force can be dynamically required to remain acting. This possibility was excluded in the case of the absence of friction of the conservative case, but here it can occurs due to the possible existence of a static friction. Then, the increments of the center of mass velocities and angular velocities up to the value of the impulse I_f at any moment t_f within this final pure rotation period can be written in the form

$$m_p \Delta\mathbf{v}_{\text{cmp}}^{(rp)}(I_f) = I_f\;\mathbf{t}_3 + \mathbf{I}^{fr}, \quad (41)$$

$$m_b \Delta\mathbf{v}_{\text{cmp}}^{(rp)}(I_f) = -I_f\;\mathbf{t}_3 - \mathbf{I}^{fr}, \quad (42)$$

$$I_p \Delta\mathbf{w}_p^{(rp)} = (\mathbf{r}_c - \mathbf{r}_p) \times \mathbf{I}^{fr}, \quad (43)$$

$$\widehat{\mathbf{I}}_b \cdot \Delta\mathbf{w}_b^{(rp)}(I_f) = -\mathbf{r}_c \times \mathbf{t}_3 I_f - \mathbf{r}_c \times \mathbf{I}^{fr}, \quad (44)$$

$$\mathbf{I}_{\text{imp}}^{fr} = I_1^{fr}\mathbf{t}_1 + I_2^{fr}\mathbf{t}_2. \quad (45)$$

The parameters I_f, I_1^{fr} and I_2^{fr} are the impulses of the normal and frictional forces produced by the bat on the ball starting from the time t_{rp} up to the instant t_f, and are defined by the integrals

$$I_f(t_f) = \int_{t_{rp}}^{t_f} N_{b\to p}(t)\; dt, \quad (46)$$

$$I_1^{fr}(t_f) = \int_{t_{rp}}^{t_f} fr_{b\to p}(t)\; dt, \quad (47)$$

$$I_2^{fr}(t_f) = \int_{t_{rp}}^{t_f} fr_{b\to p}(t)\; dt. \quad (48)$$

A complete set equations for determining the evolution of all the quantities can now be defined, after assuming that we are effectively in the $pure\ rotation$ regime occurring before the ending of the shock. The additional conditions for completely fixing the solution of the equations of motion are basically the vanishing of the two tangential components of the relative velocities of the contact points at any moment during this ending process. This condition for the relative velocity at the tangential point can be written as follows

$$0=\Delta\mathbf{v}_{\text{cmp}}^{(rp)}(I_f)\cdot\mathbf{t}_i-\Delta\mathbf{v}_{\text{cmb}}^{(rp)}(I_f)\cdot\mathbf{t}_i+\Delta\mathbf{w}_p^{(rp)}(I_f)\times(\mathbf{r}_c-\mathbf{r}_{\text{cmp}})\cdot\mathbf{t}_i-\Delta\mathbf{w}_b^{(rp)}(I_f)\times\mathbf{r_c}\cdot\mathbf{t}_i,\qquad i=1,2. \tag{49}$$

In these two equations, all the increments in the velocities can be substituted in terms of the just defined three values of the impulses. They in turns can be solved for the two values of the frictional impulses in terms of the unique values of the normal force impulse as follows

$$I_i^{fr}(t_f)=I_f\sum_{j=1,2}S_{ij}v_j, \tag{50}$$

$$S_{ij}=\frac{\mathbf{t}_i\cdot\mathbf{t}_1\mathbf{t}_1\cdot\mathbf{t}_j}{D_1}+\frac{\mathbf{t}_i\cdot\mathbf{t}_2\mathbf{t}_2\cdot\mathbf{t}_j}{D_2}, \tag{51}$$

$$D_1=\frac{1}{m_b}+\frac{1}{m_p}+\frac{(\mathbf{r}_c-\mathbf{r}_p)^2}{I_p}+\frac{(\mathbf{r}_c)^2}{I_t}+\left(\frac{1}{I_3}-\frac{1}{I_t}\right)(\mathbf{k}\times\mathbf{r}_c)^2, \tag{52}$$

$$D_2=\frac{1}{m_b}+\frac{1}{m_p}+\frac{(\mathbf{r}_c-\mathbf{r}_p)^2}{I_p}+\frac{(\mathbf{r}_c)^2}{I_t}-\frac{1}{I_t}(\mathbf{r}_c\cdot\mathbf{t}_2)^2. \tag{53}$$

Therefore, all the velocity increments in (36-39) can be expressed as linear functions of the impulse done by the normal force of the bat on the ball I_f. Note that this impulse is defined as the one transmitted after the instant in which the system arrives to the pure rotation state up to any moment t_f. In explicit form the values of the velocities as functions of I_f are defined by Eqs. (41-45) in the form

$$\mathbf{v}_{\text{cmp}}^{(rp)}(I_f)=\mathbf{v}_{\text{cmp}}^{(rp)}(0)+\frac{I_f}{m_p}\left(\mathbf{t}_3+\sum_{i=1,2}\sum_{j=1,2}S_{ij}v_j\mathbf{t}_i\right), \tag{54}$$

$$\mathbf{v}_{\text{cmp}}^{(rp)}(I_f)=\mathbf{v}_{\text{cmp}}^{(rp)}(0)-\frac{I_f}{m_b}\left(\mathbf{t}_3+\sum_{i=1,2}\sum_{j=1,2}S_{ij}v_j\mathbf{t}_i\right), \tag{55}$$

$$\mathbf{w}_p^{(rp)}(I_f)=\mathbf{w}_p^{(rp)}(0)+\frac{I_f}{I_p}(\mathbf{r}_c-\mathbf{r}_p)\times\left(\sum_{i=1,2}\sum_{j=1,2}S_{ij}v_j\mathbf{t}_i\right), \tag{56}$$

$$\mathbf{w}_b^{(rp)}(I_f)=\mathbf{w}_b^{(rp)}(0)-I_f\widehat{\mathbf{I}}_b^{-1}\cdot\left(\mathbf{r}_c\times\mathbf{t}_3+\mathbf{r}_c\times\sum_{i=1,2}\sum_{j=1,2}S_{ij}v_j\mathbf{t}_i\right). \tag{57}$$

In terms of these velocities, the total amount of kinetic energy $E_f(I_f)$ at a fixed value of the impulse I_f is defined by the formula

$$E_f(I_f)=E^{(rp)}+\frac{m_p}{2}\left(2\mathbf{v}_{\text{cmp}}(I_{rp})\cdot\Delta\mathbf{v}_{\text{cmp}}^{(rp)}(I_f)+(\Delta\mathbf{v}_{\text{cmp}}^{(rp)}(I_f))^2\right)+\frac{m_b}{2}\left(2\mathbf{v}_{\text{cmb}}(I_{rp})\cdot\Delta\mathbf{v}_{\text{cmb}}^{(rp)}(I_f)+(\Delta\mathbf{v}_{\text{cmb}}^{(rp)}(I_f))^2\right)+\frac{I_p}{2}\left(2\mathbf{w}_p(I_{rp})\cdot\Delta\mathbf{w}_p^{(rp)}(I_f)+(\Delta\mathbf{w}_p^{(rp)}(I_f))^2\right)+\frac{1}{2}\left(2\left(\mathbf{w}_b(I_{rp})\cdot\widehat{\mathbf{I}}_b\cdot\Delta\mathbf{w}_b^{(rp)}(I_f)+\Delta\mathbf{w}_b^{(rp)}(I_f)\cdot\widehat{\mathbf{I}}_b\cdot\Delta\mathbf{w}_b^{(rp)}(I_f)\right)\right). \tag{58}$$

This completes the solution for the evolution equations for the second process in which pure rotation occurs. In order to fully define the solution it only rests to determine the separation point. The condition for determining it is presented in the next subsection for the case in which dissipation is only produced by friction.

However, it needs for a clear definition about the concrete mechanisms of energy dissipation in the system. However, as it was already remarked, one helpful outcome is that, under the assumed rigidity assumptions, no matter the form of the existing dissipation mechanisms, their differences only can alter a single parameter of the problem: the particular value of the total impulse done by the normal force of the bat on the ball at the just end of the whole impact. This strong property will be used in next section to define a general solution of the problem in which additional sources of dissipation exist.

3.1. A criterium for determining the shock case from the initial conditions when dissipation is only frictional

Let us consider in this subsection the condition allowing to determine in advance which kind of state will show the ball and the bat at the end of the shock event, by only knowing the initial conditions. This completes the determination of the solution for the case in which dissipation is only implemented by the friction and the bodies attain the pure rotation state. It can be noticed that under the assumption of existence of additional non frictional sources of dissipation, such a criterium is expected to depend on the specific mechanism to be considered. However, assuming that we know the total amount of energy dissipated in the shock, the solution of the mechanical problem will be also found in the next section.

In this case of the sole presence of frictional losses, the criterium is directly given by the sign of the quantity

$$C\left(\mathbf{v}_{\text{cmp}}^{(\text{in})},\mathbf{v}_{\text{cmb}}^{(\text{in})},\mathbf{w}_{p}^{(\text{in})},\mathbf{w}_{b}^{(\text{in})}\right)$$
$$=E^{(\text{in})}\left(\mathbf{v}_{\text{cmp}}^{(\text{in})},\mathbf{v}_{\text{cmb}}^{(\text{in})},\mathbf{w}_{p}^{(\text{in})},\mathbf{w}_{b}^{(\text{in})}\right)+W_{fr}(I_{\text{rp}})-E^{(\text{rp})}, \quad (59)$$

where $W(0)$ and $E^{(rp)}$ are implicit functions of the initial velocities $\mathbf{v}_{\text{cmp}}^{(\text{in})},\mathbf{v}_{\text{cmb}}^{(\text{in})},\mathbf{w}_{p}^{(\text{in})},\mathbf{w}_{b}^{(\text{in})}$, which were obtained in the course of the previous discussion. If the sign of C is positive, then the total energy after the bat and the ball arrives to the $pure\ rotation$ state

$$E^{(\text{in})}\left(\mathbf{v}_{\text{cmp}}^{(\text{in})},\mathbf{v}_{\text{cmb}}^{(\text{in})},\mathbf{w}_{p}^{(\text{in})},\mathbf{w}_{b}^{(\text{in})}\right)+W_{fr}(I_{rp})$$

is larger than the kinetic energy of the two bodies in this same state $E^{(rp)}$. Therefore, at this moment the system has energy stored in the form elastic deformations, and a non vanishing normal force should remain existing. That is, the shock is not yet ended and the second kind of the solution should be considered.

In another hand, if C is negative, it indicates that the total energy of the system after attaining $pure\ rotation$ results to be smaller than the kinetic energy in the same $pure\ rotation$ state. This means either, that energy is not being conserved in the process, or that the system could not in fact attains the $pure\ rotation$ state, and the shock ended with sliding contact points. In this case, the kind of solution to employ should be the one discussed in Sec. 2. The case $C=0$ indicates that the shock ends precisely at the moment in which the system arrives to the $pure\ rotation$ state.

4. General dissipative impact "solution"

In this short section we describe how the solution found in previous sections lend the basis for a more general solution in which losses in addition to the frictional ones are present in the process. Let us only assume that the new kind of losses do not affects the rules governing the relations between the normal impact force and the tangential frictional one. Then, it can be observed that along the rapid mechanical evolution during the impact process, all the equations of motions for the two bodies are exactly the same at any instant at which the bodies are not yet separated. That is, the mechanical state of the ball and the bat at any moment within the shock time interval, is fully determined by the total amount of linear momentum which had been transmitted up to this moment by the normal force of the ball on the bat. Therefore, under the adopter assumptions of strict rigidity and validity of the standard rules for friction, all the mechanical quantities of the problem coincide with the ones in the case of the sole presence of frictional dissipation, at any given instant before the separation between the bodies. Then, let us consider the exact separation instant. It is clear that the condition of separation will be that the at this precise moment, the (already known) frictional dissipation energy, plus the additional energy lost (by heat, sound, plastic deformations, ...) should be equal to the difference between the initial value of the total mechanical energy (at the beginning of the shock) and the final value of this same quantity at the separation instant. Therefore, if we consider the amount of mechanical energy dissipated during the shock (certainly a measurable quantity) as a known value, then the full solution of the impact problem is obtained. That is, all the mechanical properties are determined at all times. Since all the tangential and normal to the contact plane velocities can be calculated, it also follows that results predicts the tangential and normal coefficients of restitution for the class of shocks studied. This analysis will be employed in the next section to describe experimental data of the slow motion impacts of a ball and a bat. Before, in the next subsection, let us present the explicit separation conditions.

4.1. The condition for determining the separation point

Let us discuss now the explicit condition to be imposed for determining the moment in which the ball and the bat start to separate. It can be constructed in terms of the energy balance in the system. The separation point can be defined as that one at which the initial total mechanical energy minus all the mechanical energy lost up to this point, just becomes equal to the total kinetic energy of the ball and the bat. In explicit terms, it can be written in the form

$$E^{(\text{in})}+W_{fr}(I_{rp})+W_{\text{add}}(I_{\text{rp}}+I_{\text{out}})=E_f(I_{\text{out}}) \quad (60)$$

where the total dissipative work of the friction W_{fr} is given by the general expression (32) after substituting $I=I_{\text{rp}}$

$$W_{fr}(I_{rp})=-\mu\int_0^{I_{rp}}|\,\mathbf{v}\,|\,dI, \quad (61)$$

and the losses due to other sources of dissipation in addition to friction up the total value of the impulse done by the normal force at the separation point $I_{\text{rp}}+I_{\text{out}}$, is the term $W_{\text{add}}(I_{\text{rp}}+I_{\text{out}})$. The right hand side of the equation is the total kinetic energy of the ball and the bat at the separation instant. That is, the total kinetic energy at the point in which pure rotation is established plus the increment due to the changes in all the velocities during the pure rotation evolution up to the separation point. This kinetic energy is defined by Eq. (58) as evaluated at the impulse done by the normal forces at the separation point.

As it was remarked at the start of this section, without specifying the nature of the additional sources of dissipation in addition to the frictional one, is not possible to define the separation point assumed that we only know the initial data. However, if we consider as a known quantity the total mechanical energy after the shock (or alternatively, the amount of energy lost in the impact, the problem can be considered as solved in this section.

Let us now consider again the simpler case in the only energy dissipating source is the friction. Then the additional losses term $W_{\text{add}}(I_{\text{rp}}+I_f)$ vanishes. In this case relation (60) becomes a simple quadratic equation for I_{out} after substituting the expressions (54-57) for the velocity increments, which all are homogeneous linear functions of I_{out}. In Eq. (60) all the quantities $E^{(\text{in})}, W_{fr}(0)$, $E^{(\text{rp})}, \mathbf{v}_{\text{cmp}}(I_{\text{rp}})$, $\mathbf{v}_{\text{cmb}}(t_{\text{rp}}), \mathbf{w}_p(I_{\text{rp}})$ and $\mathbf{w}_b(I_{\text{rp}})$ are already known from the solutions just obtained in the previous stage of the shock. Henceforth, the complete analytic solution of the shock problem follows in the considered case in which $pure\ rotation$ is attained and friction is the only source of energy losses. The application of the analysis to describe experiments in which additional sources of losses add to friction will be discussed in next section.

5. Description of experimental measures

In this section we will consider the application of the results presented before to the description of the measures relative to the scattering of a ball by a bat in Ref. 11. The experiment consisted in dropping a free falling ball at certain height which determines a vertical velocity of 4.0 m/sec at the instant in which it shocks with an horizontally oriented static bat. Then, a high video recording of the free fall of the ball allowed the authors to determine all the kinematic parameters before and after the impact. Experiments were done, in which the ball was pitched a number of times with fixed values of the angular velocities along the symmetry axis of the bat. Experiences for three values of the angular velocities $w_o=79$, 0, -72 rad/sec were done. The bat has a barrel diameter of 6.67 cm, a length of 84 cm and a mass of 0.989 kg. The center of mass of the bat is situated at 26.5 cm of its barrel end. The ball for which measures were done landed on the bat at distances along the bat axis ranging between 14-16 cm from the barrel end. Then, we will describe the shocks by assuming that the ball impacted the bat at an axial distance of 15 cm from the barrel end.

The parameters of the bat and the ball considered in that work determine the following values for the magnitudes defined in the previous sections

$$m_p = 0.145\ \text{kg},$$
$$m_b = 0.989\ \text{kg},$$
$$r_p = \sqrt{(\mathbf{r_c}-\mathbf{r_{cmp}})^2} = 0.036\ \text{m}$$
$$r_b = 0.03335\ \text{m}$$
$$I_p = (2/5)(0.036)^2\, m_p\ \text{kg m}^2,$$
$$I_t = 0.0460\ \text{kg m}^2,$$
$$I_3 = 4.39\times10^{-4}\ \text{kg m}^2,$$
$$\mu = 0.5 \tag{62}$$

The set of unit vectors sitting at the tangential point of the bodies were chosen in the following way

$$\mathbf{t}_1 = \cos(\theta)\,\mathbf{j} - \sin(\theta)\,\mathbf{i},$$
$$\mathbf{t}_2 = \mathbf{k},$$
$$\mathbf{t}_3 = \sin(\theta)\,\mathbf{j} + \cos(\theta)\,\mathbf{i},$$
$$\mathbf{r}_c = 0.115\,\mathbf{k} + 0.03335\,(\sin(\theta)\,\mathbf{j}+\cos(\theta)\,\mathbf{i}), \tag{63}$$

which reflect the fact that the barrel of the bat has been fixed as a cylinder of radius 0.03335 cm and that the center of mass of the bat has a minimum distance of 0.115 m from the plane being transversal to the bat axis and contains the contact point. The angle θ is the one formed between a radius traced from the axis of the bat to the contact point. For the experimental arrangement, the initial velocities of the ball and the bat just an instant before the impact become

$$\mathbf{v}_{\text{cmp}}^{(\text{in})} = (-4,0,0),$$
$$\mathbf{v}_{\text{cmb}}^{(\text{in})} = (0,0,0),$$
$$\mathbf{w}_p^{(\text{in})} = (0,0,w_o),$$
$$\mathbf{w}_b^{(\text{in})} = (0,0,0). \tag{64}$$

5.1. The solution of the shock problem for nearly vanishing impact parameter

In starting, let us exemplify the solution of the impact problem for the situation in which the vertically falling ball makes contacts with the horizontally oriented bat at a very small value of the impact parameter. That is, when the vertical line of falling passes very close to the bat symmetry axis. For concreteness let us suppose that the impact occurs at the small value of the angle $\theta=\pi/400$. This configuration will serve two purposes of the presentation. In first place it will illustrate the application of the formal solutions found in previous sections to a concrete shock process. In second hand this particular solution for scattering at zero impact parameter case will serve for phenomenologically constructing a description of the experimental data presented in Ref. 11. Firstly, consider the solution of the Eqs. (16) for the evolution for the tangential velocities in the firstly occurring sliding period. The evaluation of the parameters s_1, s_2 and s_o defined in relations (17-19) leads to the explicit form of the equations

$$\frac{dv_1(I)}{dI} = -13.98\frac{v_1(I)}{\sqrt{v_1^2(I)+v_2^2(I)}}, \tag{65}$$

$$\frac{dv_2(I)}{dI} = -12.58\frac{v_2(I)}{\sqrt{v_1^2(I)+v_2^2(I)}} - 0.0833, \tag{66}$$

$$v_1(0) = 0.0314 \ \frac{\text{meter}}{\text{sec}}, \tag{67}$$

$$v_2(0) = 0. \tag{68}$$

The employed initial values of the components of the tangent velocities $v_1(0)$ and $v_2(0)$ (along the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively) were determined by projecting the relative velocity at the just beginning of the impact, which is defined by Eq. (14), on each of these vectors. Note that the starting value of the tangent velocity is small due to the assumptions of vanishing angular velocity of the ball in common with the very small selection of the impact parameter.

The solutions of these equations for $v_1(I)$, $v_2(I)$ and modulus of the tangent velocity $v(I) = \sqrt{v_1^2(I) + v_2^2(I)}$ are depicted in the Figs. 2-4. Note that the v_2 component is very small, although non vanishing, which is consistent with the fact that the shock is not strictly two dimensional, because the center of mass of the bat is out of the plane which is orthogonal to the symmetry axis of the bat and passes through the center of mass of the ball. The v_2 component, although initially vanishing, develops values which grow up to a maximal one for tending to zero again. On another hand the v_1 component of velocity start decreasing from the start to vanish exactly at the same value of the impulse I, for which the v_2 component also becomes equal to zero. Therefore, the system of equations predict that both components simultaneously tend to approach a vanishing value. This property is exhibited by all the solutions of the scattering problem found in this work to describe the experimental results in Ref. (11). The Fig. 4 clearly illustrates the vanishing of the modulus of the tangent velocity. From the Figs. 2-4 it can be seen that the value of the impulse I_{rp} of the normal force on the ball for which the system arrives to pure rotation for this special scattering configuration is

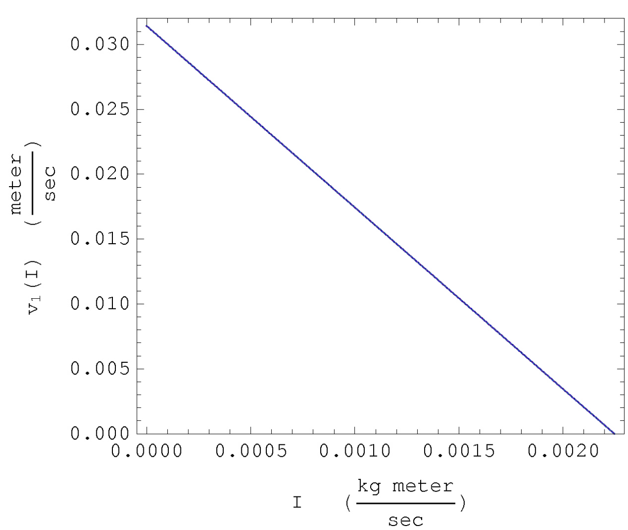

FIGURE 2. The figure shows the evolution with the impulse I of the component $v_1(I)$ of the tangent relative velocity between the contact points on the bat and the ball. It was evaluated by solving the corresponding differential equations.

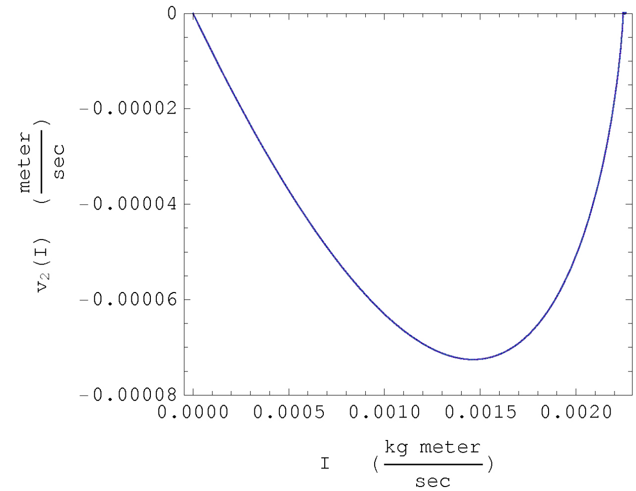

FIGURE 3. The figure shows the evolution with the impulse I of the component $v_2(I)$ of the relative velocity between the contact points on the bat and the ball. It was evaluated by solving the corresponding differential equations.

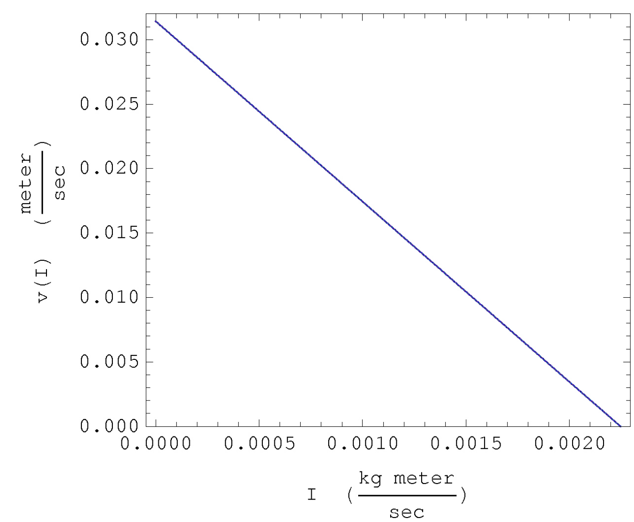

FIGURE 4. The figure shows the evolution with the impulse I of the modulus $v(I)$ of the tangent relative velocity between the contact points in the bat and the ball. It was also evaluated by solving the corresponding differential equations.

$$I_{rp=}0.00224 \ \frac{\text{kg meter}}{\text{sec}}. \tag{69}$$

Having found the evolution of the tangent velocities with the variation of the impulse of the normal forces I, we become able to check whether the shock process will end in pure rotation state or in a sliding condition between the contact points of the bat and the ball. For this purpose let us evaluate the relation (35) by substituting the above defined data for the velocities valid for the experiment and evaluating $W_{\text{fr}}(I_{\text{rp}})$ and $E^{(\text{rp})}$ through their respective formulae (61) and (40). The evolution of the four velocities of the ball and the bat from the starting of the shock up to the moment in which the impulse at which *pure rotation* could be attained, was evaluated from the formulae (24-27) after determining the impulse of the friction from its definition (28).

The evaluation results in a positive value for the C function

$$C\left(\mathbf{v}_{\text{cmp}}^{(\text{in})},\mathbf{v}_{\text{cmb}}^{(\text{in})},\mathbf{w}_{p}^{(\text{in})},\mathbf{w}_{b}^{(\text{in})}\right) = E^{(\text{in})}\left(\mathbf{v}_{\text{cmp}}^{(\text{in})},\mathbf{v}_{\text{cmb}}^{(\text{in})},\mathbf{w}_{p}^{(\text{in})},\mathbf{w}_{b}^{(\text{in})}\right) + W_{\text{fr}}(I_{\text{rp}}) - E^{(\text{rp})} = 0.00896 \quad \text{Joules}, \tag{70}$$

$$E^{(\text{in})}\left(\mathbf{v}_{\text{cmp}}^{(\text{in})},\mathbf{v}_{\text{cmb}}^{(\text{in})},\mathbf{w}_{p}^{(\text{in})},\mathbf{w}_{b}^{(\text{in})}\right) = 1.16, \tag{71}$$

$$E^{(\text{rp})} = 1.1510 \quad \text{Joules}, \tag{72}$$

$$W_{\text{fr}}(I_{\text{rp}}) = -0.0000176 \quad \text{Joules} . \tag{73}$$

Therefore, assumed that the only source of energy losses is the friction dissipation, since C results to be positive at the instant in which the transmitted impulse is I_{rp} for which $pure\ rotation$ is attained, the total mechanical energy is larger that the kinetic energy $E^{(\text{rp})}$ of the two bodies. In addition the values of the dissipated energy by the friction $W_{fr}(I_{\text{rp}})$ results to be very small when $pure\ rotation$ is established. Since the kinetic energy $E^{(\text{rp})}$ at pure rotation has a value close to the total initial mechanical energy, it is clear that the pure rotation state is attained just at the very beginning of the impact. This is compatible with the very small impulse I_{rp} (see 69) transmitted by the normal force in arriving to vanishing sliding of the contact surfaces.

Having defined that the scattering situation corresponds to a pure rotation final state, let us consider now the evolution of the system after the impulse done by the normal force continues to be growing under the pure rotation state. The change of all the velocities of the bat and the ball in this new process are determined as simple linear functions of the impulse I_f by equations (54-57).

Let us consider first the case in which only friction is able to dissipate energy. Then the condition for the separation of the ball and the bat (60) gives a simple quadratic equation for the determination of the value of I_f at which the two bodies separate. The condition for separation and its explicit form are

$$E^{(\text{in})} + W_{fr}(I_{\text{rp}}) = E_f(I_f) \tag{74}$$

$$1.159 = 1.151 + I_f\,(-3.981 + 4.097\,I_f), \tag{75}$$

which give for the value of the impulse of the normal force during the pure rotation interval up to the separation point, assumed that the only existing energy losses are associated to friction, the result

$$I_f^{(\text{out},1)} = 0.9739 \quad \frac{\text{kg meter}}{\text{sec}}. \tag{76}$$

This value compared with $I_{\text{rp}} = 0.00224$ evidences that the pure rotation state was directly established at the beginning of the impact.

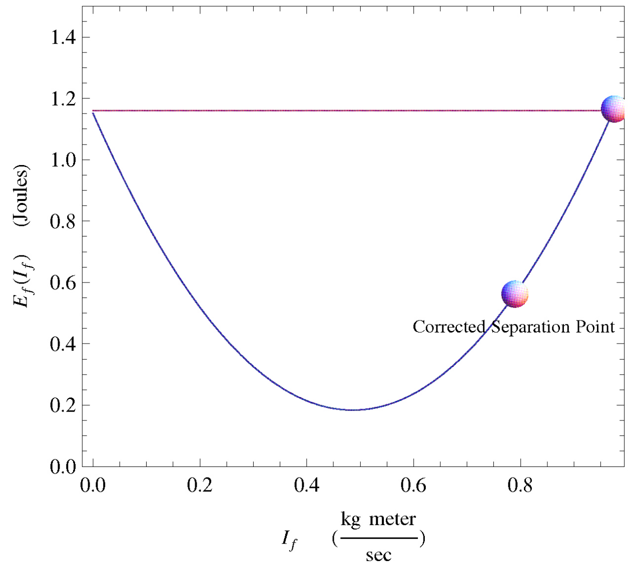

FIGURE 5. The plot of the total kinetic energy of the bat and the ball E_f as a function of the impulse of the normal force of the bat on the ball I in the ending pure rotation process. The horizontal line defines the conserved value of the total mechanical energy after the pure rotation state is established. The difference between the horizontal line and $E_f(i)$ gives the amount of energy stored in the form of elastic deformation at any value of the impulse I. The ball depicted on the horizontal indicates the separation point between the ball and the bat when the only source of dissipation is the friction. The similar ball laying on the curve of E_f indicates the separation point to be defined by the model here constructed for evaluating the non frictional energy losses.

The determination of I_f finishes the solution of the shock problem in this case, since substituting this value in the expressions (54-57) determines all the center of mass and angular velocities of the bat and the ball at the separation instant.

5.1.1. *Consideration of the losses due to inelastic processes*

Let us consider now the situation in which there exist energy dissipation sources in addition to the friction. For this purposes consider Fig. 5 which illustrates the solution the just discussed solution. The parabolic curve shows the dependence of the kinetic energy E_f as a function of I_f. The horizontal line indicates the value of the conserved mechanical energy $E^{(\text{in})}+W_{\text{fr}}(I_{\text{rp}})$ in the considered pure rotation period. The figure shows how, as the system evolves from the instant in which $pure\ rotation$ was established, it accumulates energy in elastic form as signaled by the difference between the kinetic energy E_f and the conserved mechanical energy $E^{(\text{in})} + W_{\text{fr}}(I_{\text{rp}})$, up to a maximum value, which afterwards starts to decrease. This behavior will be taken into account in what follows to construct a model for the non elastic dissipation processes different from frictional one. The basic purpose will be to apply the analysis to the description of the measures of scattering of a ball by a bat given in Ref. 11.

Assume that we are already in the pure rotation state, as the former evaluations in this section had stated. Then, let us consider the more general condition (60) for determining the separation point between the ball and bat, in which the additional losses term $W_{\text{add}}(I_{\text{rp}}+I_f)$ was introduced. It is clear that the function $W_{\text{add}}(I_{\text{rp}}+I_f)$ depends on the types of materials constituting the bat and the ball, in particular on their properties under the large local deformations occurring near the impact point. Therefore, we have not at hand well defined information about how the non elastic dissipation is occurring as the impulse of the normal force is growing when the shock develops. Therefore, we will employ a global condition for the determination of the amount of dissipation in addition to the frictional ones. As remarked before, this condition was suggested by the data depicted in Fig. 5.

The condition adopted an intuitively motivated notion: that the amount of non elastic losses in any type of shock will be given by a fixed fraction e^2 of the maximal amount of elastic energy which is accumulated along the evolution of the system, when dissipation is only given by friction. In explicit terms

$$W_{\text{add}}\left(I_{\text{rp}}+\ I_f^{(\text{out},2)}\right)=-e^2\Big(E^{(\text{in})}+W_{\text{fr}}(I_{\text{rp}})-E_f(I_f^{\max})\Big), \qquad (77)$$

That is, the additional energy losses at the value of the impulse at which the bodies separate $I_f^{(\text{out},2)}$, will be chosen to be a fraction e^2 of the difference between the total mechanical energy after pure rotation is attained $E^{(\text{in})}+W_{\text{fr}}(I_{\text{rp}})$ (a quantity which is conserved in the assumed case in the above definition of pure frictional losses) and the total kinetic energy $E_f(I_f^{\max})$ at the value of the impulse $I_f^{\max}$. This value $I_f^{\max}$ correspond to the impulse at which the stored elastic energy $(E^{(\text{in})}+W_{\text{fr}}(I_{\text{rp}})-E_f(I_f))$ is maximal as a function of I_f when pure frictional dissipation is assumed. Then the condition for separation (60) gets the general form

$$E_f\left(I_f^{(\text{out},2)}\right)=E^{(\text{in})}+W_{\text{fr}}(I_{\text{rp}})-e^2\left(E^{(\text{in})}+W_{\text{fr}}(I_{\text{rp}})-E_f(I_f^{\max})\right), \qquad (78)$$

from which the value of $I_f^{(\text{out},2)}$ can be directly obtained because $E_f(I_f^{(\text{out},2)})$ is a quadratic function of $I_f^{(\text{out},2)}$ defined by (75).

Once the value of $I_f^{(\text{out},2)}$ is at hand, its substitution in (54) and (56) allows to evaluate for the absolute values of the ball center of mass and angular velocities which are basic quantities measured in Ref. 11, the expressions

$$|\mathbf{v}_p^{f\text{in}}|=\sqrt{\mathbf{v}_{\text{cmp}}^{(\text{rp})}(I_f^{(\text{out},2)})\cdot\mathbf{v}_{\text{cmp}}^{(\text{rp})}(I_f^{(\text{out},2)})}, \qquad (79)$$

$$\mathbf{v}_{\text{cmp}}^{(\text{rp})}(I_f^{(\text{out},2)})=\mathbf{v}_{\text{cmp}}^{(\text{rp})}(0)+\frac{I_f^{(\text{out},2)}}{m_p}\left(\mathbf{t}_3+\sum_{i=1,2}\sum_{j=1,2}S_{\text{ij}}v_j\mathbf{t}_i\right), \qquad (80)$$

$$|\mathbf{w}_p^{f\text{in}}|=\sqrt{\mathbf{w}_p^{(\text{rp})}\left(I_f^{(\text{out},2)}\right)\cdot\mathbf{w}_p^{(\text{rp})}\left(I_f^{(\text{out},2)}\right)}, \qquad (81)$$

$$\mathbf{w}_p^{(\text{rp})}\left(I_f^{(\text{out},2)}\right)=\mathbf{w}_p^{(\text{rp})}(0)+\frac{I_f^{(\text{out},2)}}{I_p}(\mathbf{r}_c-\mathbf{r}_p)\times\left(\sum_{i=1,2}\sum_{j=1,2}S_{ij}v_j\mathbf{t}_i\right). \qquad (82)$$

These quantities were calculated for a set of values of the impact parameter defined as the minimal distance between vertical line along which the center of the ball was falling and the initially horizontally oriented symmetry axis of the bat.

The value of the constant e^2 was determined by fixing the measured value of the output center of mass velocity in Ref. 11, at the particular condition of scattering considered at the beginning of this section. That is when the angular velocity of the falling ball is zero and the center of mass velocity is 4.0 meter/sec at a nearly vanishing value of the impact parameter. The condition (78) for determining the separation in this case gets the form

$$r_1+r_2\,e^2=r_3+I_f^{(\text{out},2)}(r_4+r_5I_f^{(\text{out},2)}) \qquad (83)$$

$$r_1=1.159, \qquad (84)$$

$$r_2=0.9761, \qquad (85)$$

$$r_3=1.151 \qquad (86)$$

$$r_4=-3.981 \qquad (87)$$

$$r_5=4.097 \qquad (88)$$

where the values of the impulse at the minimum value of the kinetic energy E_f for the assumed set of initial data is $I_f^{\max}=0.4858$ and the corresponding value of the kinetic energy at this point is $E_f(I_f^{\max})=0.1838$.

The fixation of e^2 proceeded by assuming some trial values of this quantity and solving the equation for $I_f^{(\text{out},2)}$ for each one of them, by further evaluating the absolute value of the final ball be velocity by using (79). The trials were repeated after to arrive to a final output velocity of the ball being around a value of $|\mathbf{v}_p^{f\text{in}}|=1.446$ meter/sec, which is close to the one measured in Ref. 11 for the assumed scattering conditions. The resulting value of e^2 was 0.618.

Once this parameter was determined, we performed evaluations of the output angular and center of mass velocities of the ball for various values of the angle θ. Different sets of evaluations were done for these quantities, one for each of three values of the initial angular velocity of the ball, for which measures were done in the experiments: $w_o=+79,0,-72$ rad/sec. The results for the absolute values of the center mass velocities of the ball $|\mathbf{v}_p^{f\text{in}}|$ were plotted as functions of the scattering angle α (expressed in degrees)

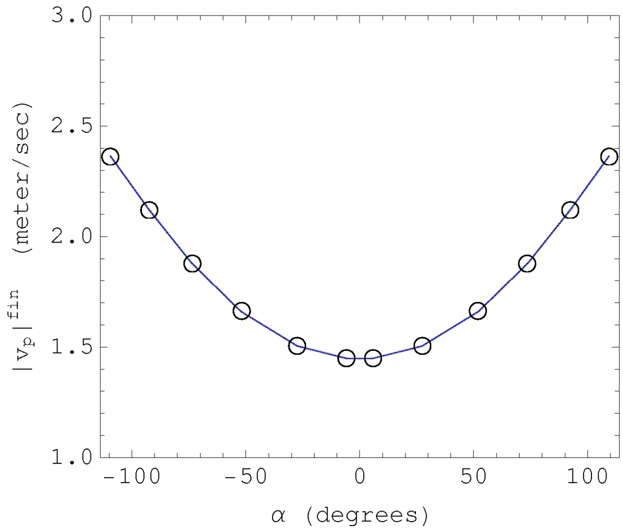

FIGURE 6. The figure show the calculated absolute value of the center of mass velocity of the ball after the impact as a function of the scattering angle α. The value at $\alpha = 0$ was fixed to be close to the measured one in Ref. 11, which determined the value of the constant e^2 defining the non-frictional energy losses.

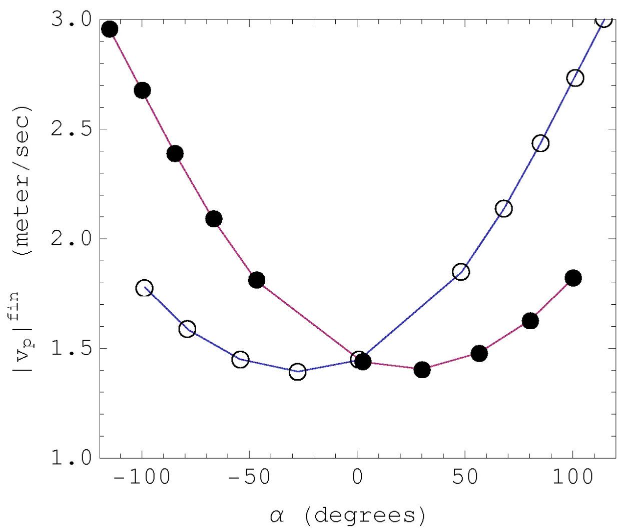

FIGURE 7. The figure shows two sets of calculated absolute values of the center of mass velocity of the ball after the impact as functions of the scattering angle α. The open circles depict the velocity values when the initial angular velocity of the ball is +79 rad/sec. The filled ones indicate the velocities for an initial angular velocity of -72 rad/sec

formed by the output ball velocity and its corresponding input value. This angle is defined by

$$\alpha = \frac{180}{\pi} \text{ArcSin}\left(-\frac{\mathbf{k} \cdot \mathbf{v}_p^{fin} \times \mathbf{i}}{|\mathbf{k} \cdot \mathbf{v}_p^{fin} \times \mathbf{i}|} \right). \tag{89}$$

We also evaluated the ending angular velocity of the ball as functions in this case of the impact parameter E (expressed in inches) defined by

$$E = (r_p + r_b)\cos(\theta). \tag{90}$$

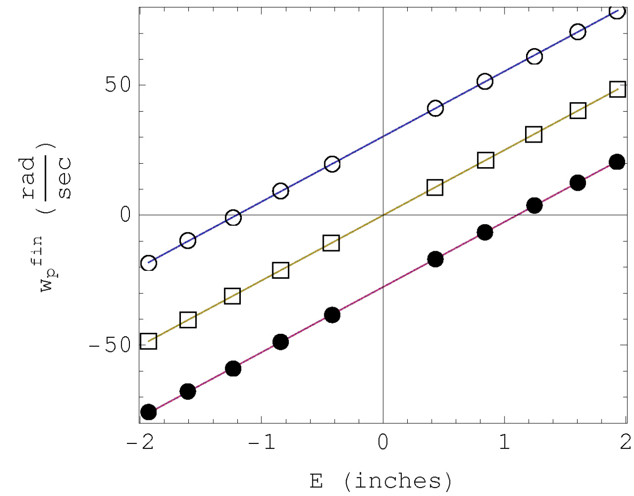

FIGURE 8. The figure illustrate the variation with the impact parameter of three sets of values of the calculated angular velocity of the ball after the shock. The open circles correspond to an initial angular velocity equals to w_o=+79 rad/sec . The squares show the evaluated angular velocities for $w_o = 0$. The filled circles indicate the calculated angular velocities for w_o =-72 rad/sec.

The results for the final absolute value of the ball velocity $|\mathbf{v}_p^{fin}|$ as a function of α when its initial angular velocity is taken as vanishing are depicted in Fig. 6. As described before the value of the final center of mass velocity of the ball was phenomenologically fixed to approximately reproduce the measured value near 1.5 meter/sec at $\alpha = 0$ for the zero initial angular momentum of the ball experiment. No other parameter fixation was additionally done. Therefore all the shown data for the values of the ball velocities in dependence the scattering angle α represent predictions of the analysis done here. The comparison of the results with the ones plotted in the corresponding Fig. 2 (top) of Ref. 11 permits to conclude that the model solution found here satisfactorily reproduces the measured data.

The predicted values for the final center of mass velocities of the ball for the cases in which it shocks with a 4.0 meter/sec center of mass velocity with the static and horizontal bat, and having angular velocities of values $+79$ and -72 rad/sec, are presented in Fig. 7. These results again satisfactorily match the corresponding measurements shown in Fig. 3 (top) in Ref. 11. It can be noted that the same, natural to be expected, asymmetry of the velocities with respect to the change of the sign of the scattering angle α is exhibited and the quantitative values also approach the measured ones within the experimental errors. Finally, the ending angular velocities of the ball for each of the three values of the initial angular velocities are plotted in Fig. 8. In this case the nearly linear dependence for the three experiments measured in Ref. 11 and shown in Fig. 5 of that work, are satisfactorily reproduced in slope and values within the precision allowed by the degree of dispersion of the measured values.

6. Summary

The work firstly presented a solution of the general problem of the scattering between a spherical object and a cylindrically symmetric one, when both of them are assumed as perfectly rigid systems and the friction is defined by the standard rules and is the only source of energy dissipation. It is perhaps useful to underline that the adopted rigidity assumption corresponds to assume that the bodies are not deformed during the short time interval of the impact. This is clearly occurring in situations in which the relative velocities are sufficiently small. The yet wide character of this class of problems confers interest to the study. A simple criterium is determined allowing to decide from the beginning whether the final states of the bodies will correspond or not to sliding contact surfaces or to the contact points being at rest at the end of the shocking process.

Further, the exact solution for the evolution of all the physical quantities during the shock is also found, when other types of energy dissipation in addition to the frictional one are present. In this case, it follows that the dynamical evolution of all the mechanical quantities along all the time in which the two bodies remain in contact, fully coincide with the one associated to the pure frictional case. This follows under the unique assumption about that the additional dissipation mechanism do not alter the laws relating the frictional force with normal one at the contact point. Then, if we only consider the standard initial data as known, the only lacking information in the solution is the concrete value of the net impulse done by the ball on the bat at the precise separation point. The determination of this point needs of detailed information on the additional sources of dissipation if only the initial data are assumed as known.

However, in an alternative way, we can consider that a full solution of the general problem is found, after assuming that the total final energy of the system is known. This not an impractical supposition, because the total mechanical energy at the end is certainly a measurable property in experiments that can be repeated. Thus, we interpret that a full solution of the mechanical problem is determined in the general problem including other kind of losses by assuming the final energy as a known quantity.

Afterwards, the mentioned general method of solution is applied to the description of the experimental measurements of the slow motion scattering of a ball by a bat presented in Ref. 11. These experiences show dissipation mechanisms additional to the frictional one, and basically consisted in measuring a vertically falling ball which impacts at 4.0 meter/sec an horizontally laying and non rotating static bat. The solution of the problem satisfactorily reproduced the measured dependence of the final velocity of the ball as a function of the scattering angle. This happens for each of the three values of the initial angular velocity of the ball employed in the experiments. The behavior of the final angular velocity of the ball on the impact parameter for each of the cited values of the initial angular velocity of the ball are also appropriately described.

7. The rigid and frictionless shock solution

In this appendix, we will consider the solution of the shock problem for the case in which the interaction force between the bodies $\mathbf{F}(t)$ during the impact is conservative and normal. Due to its impact nature, let us consider the force as given by a Dirac delta distribution

$$\mathbf{F}(t) = \mathbf{I}_{\text{imp}}\delta(t-t_0),$$

in which $\mathbf{I}_{\text{imp}}$ is the total impulse vector transmitted by the force. Then, the Newton equations for the problem can be written as follows

$$\begin{aligned} m_p\frac{d}{dt}\mathbf{v}_{\text{cmp}}(t) &= \mathbf{I}_{\text{imp}}\delta(t-t_0),\\ m_b\frac{d}{dt}\mathbf{v}_{\text{cmb}}(t) &= -\mathbf{I}_{\text{imp}}\delta(t-t_0),\\ \widehat{\mathbf{I}}_p\cdot\frac{d}{dt}\mathbf{w}_p(t) &= (\mathbf{r}_c-\mathbf{r}_{\text{cmp}})\times\mathbf{I}_{\text{imp}}\delta(t-t_0),\\ \widehat{\mathbf{I}}_b\cdot\frac{d}{dt}\mathbf{w}_b(t) &= -\mathbf{r}_c\times\mathbf{I}_{\text{imp}}\delta(t-t_0). \end{aligned} \tag{91}$$

Note that the third equation was expressed in terms of the angular impulse respect to the center of mass of the ball. This was done by using the definition (5) of the angular momentum of the ball respect to the reference frame sitting at the center of mass of the bat, and the first of the equations in (91). Integrating the above equations over time, it follows

$$\begin{aligned} m_p\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right) &= \mathbf{I}_{\text{imp}},\\ m_b\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}-\mathbf{v}_{\text{cmb}}^{(\text{in})}\right) &= -\mathbf{I}_{\text{imp}},\\ \widehat{\mathbf{I}}_p\cdot\left(\mathbf{w}_p^{(\text{out})}-\mathbf{w}_p^{(\text{in})}\right) &= (\mathbf{r}_c-\mathbf{r}_{\text{cmp}})\times\mathbf{I}_{\text{imp}},\\ \widehat{\mathbf{I}}_b\cdot\left(\mathbf{w}_b^{(\text{out})}-\mathbf{w}_b^{(\text{in})}\right) &= -\mathbf{r}_c\times\mathbf{I}_{\text{imp}}, \end{aligned} \tag{92}$$

where the superindices (in) and (out), indicate the values of the magnitudes at an instant before and after the start of the shock, respectively.

Let us consider now the condition satisfied by the impulse of the interaction force in order to implement our two suppositions: conservation of energy and the absence of friction between the contact surfaces. Its is clear that if there is no friction between the contact planes there will be no projection of the forces in the tangent planes and therefore:

$$\begin{aligned} \mathbf{t}_i\cdot\mathbf{I}_{\text{imp}} = 0 &= m_p\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{t}_i, \quad i=1,2,\\ \mathbf{t}_i\cdot\mathbf{I}_{\text{imp}} = 0 &= m_b\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}-\mathbf{v}_{\text{cmb}}^{(\text{in})}\right)\cdot\mathbf{t}_i, \quad i=1,2. \end{aligned} \tag{93}$$

Thus, the tangent components of the center of mass velocities after the shock are exactly the same as themselves before the impact. Therefore, these two quantities are already determined. For the normal to the tangent plane of the center of mass components it follows,

$$m_p\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{t}_3=-m_b\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}-\mathbf{v}_{\text{cmb}}^{(\text{in})}\right)\cdot\mathbf{t}_3, \quad (94)$$

which coincides in form with the usual result for the simple collinear and conservative shock between two bodies.

For the ball, the simplification is stronger, because the impact force, as having no tangent component, has a vanishing angular impulse

$$I_p\left(\mathbf{w}_p^{(\text{out})}-\mathbf{w}_p^{(\text{out})}\right)=-(\mathbf{r}_c-\mathbf{r}_{\text{cmp}})\times\mathbf{I}_{\text{imp}},=0, \quad (95)$$

which directly implies that the angular velocity vector of the ball is conserved during the shock:

$$\mathbf{w}_p^{(\text{out})}=\mathbf{w}_p^{(\text{in})}, \quad (96)$$

furnishing the solution for these variables after the shock is finished.

Further, the normal direction of the conservative impulsive force implies that its angular impulse on the bat is directed in the $\mathbf{t}_1$ direction. This property, then implies the conservation of the components of the initial angular velocity along the $\mathbf{t}_2$ and $\mathbf{t}_3$ spacial directions:

$$\left(\mathbf{w}_b^{(\text{out})}-\mathbf{w}_b^{(\text{in})}\right)\cdot\mathbf{t}_2=0, \quad (97)$$

$$\left(\mathbf{w}_b^{(\text{out})}-\mathbf{w}_b^{(\text{in})}\right)\cdot\mathbf{t}_3=0. \quad (98)$$

The remaining two integrated Newton equations, constitute a set of two equations for the yet undetermined variables $\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}-\mathbf{v}_{\text{cmb}}^{(\text{in})}\right)\cdot\mathbf{t}_3$, $\left(\mathbf{w}_b^{(\text{out})}-\mathbf{w}_b^{(\text{in})}\right)\cdot\mathbf{t}_1$ and $\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{t}_3$, that can be written in the forms

$$-m_b\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}-\mathbf{v}_{\text{cmb}}^{(\text{in})}\right)\cdot\mathbf{t}_3=m_p\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{t}_3,$$
$$\left(\mathbf{w}_b^{(\text{out})}-\mathbf{w}_b^{(\text{in})}\right)\cdot\mathbf{t}_1=-m_p(\mathbf{t}_1\times\mathbf{t}_2\cdot\mathbf{t}_3)$$
$$\times(\mathbf{r}_{\text{cmp}}\cdot\mathbf{t}_2)\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{t}_3.$$

These equations state that the discontinuities in the normal and angular velocities of the bat, are both expressed in terms of the discontinuity of the normal velocity of the ball. Thus, after finding another equation being able in determining this unique ball velocity discontinuity, the problem will become solved.

This additional condition, should correspond to impose the conservation of the energy after the end of the shock. Its expression is

$$\frac{m_p}{2}\sum_{i=1}^{3}\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}\cdot\mathbf{t}_i\right)^2+\frac{I_p}{2}\sum_{i=1}^{3}\left(\mathbf{w}_p^{(\text{out})}\cdot\mathbf{t}_i\right)^2$$
$$+\frac{m_b}{2}\sum_{i=1}^{3}\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}\cdot\mathbf{t}_i\right)^2+\frac{1}{2}\sum_{i=1}^{3}I_i\left(\mathbf{w}_b^{(\text{out})}\cdot\mathbf{t}_i\right)^2$$
$$=\frac{m_p}{2}\sum_{i=1}^{3}\left(\mathbf{v}_{\text{cmp}}^{(\text{in})}\cdot\mathbf{t}_i\right)^2+\frac{I_p}{2}\sum_{i=1}^{3}\left(\mathbf{w}_p^{(\text{in})}\cdot\mathbf{t}_i\right)^2$$
$$+\frac{m_b}{2}\sum_{i=1}^{3}\left(\mathbf{v}_{\text{cmb}}^{(\text{in})}\cdot\mathbf{t}_i\right)^2+\frac{1}{2}\sum_{i=1}^{3}I_i\left(\mathbf{w}_b^{(\text{in})}\cdot\mathbf{t}_i\right)^2. \quad (99)$$

After using the known information about the variables which have been already determined, all the quantities entering these relation can be expressed as functions of the only three remaining unknown quantities in the following way:

$$\mathbf{v}_{\text{cmb}}^{(\text{out})}=\sum_{i=1}^{3}\left(\mathbf{v}_{\text{cmb}}^{(\text{in})}\cdot\mathbf{t}_i\right)\ \mathbf{t}_i+\left(\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}-\mathbf{v}_{cmb}^{(\text{in})}\right)\cdot\mathbf{t}_3\right)\ \mathbf{t}_3$$
$$=\mathbf{v}_{\text{cmb}}^{(\text{in})}+\left(\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}-\mathbf{v}_{\text{cmb}}^{(\text{in})}\right)\cdot\mathbf{t}_3\right)\ \mathbf{t}_3,$$
$$\mathbf{v}_{\text{cmp}}^{(\text{out})}=\sum_{i=1}^{3}\left(\mathbf{v}_{\text{cmp}}^{(\text{in})}\cdot\mathbf{t}_i\right)\ \mathbf{t}_i+\left(\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{t}_3\right)\ \mathbf{t}_3$$
$$=\mathbf{v}_{\text{cmp}}^{(\text{in})}+\left(\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{t}_3\right)\ \mathbf{t}_3,$$
$$\mathbf{w}_p^{(\text{out})}=\mathbf{w}_p^{(\text{in})},$$
$$\mathbf{w}_b^{(\text{out})}=\sum_{i=1}^{3}\left(\mathbf{w}_b^{(\text{in})}\cdot\mathbf{t}_i\right)\ \mathbf{t}_i+\left(\left(\mathbf{w}_b^{(\text{out})}-\mathbf{w}_b^{(\text{in})}\right)\cdot\mathbf{t}_1\right)\ \mathbf{t}_1$$
$$=\mathbf{w}_b^{(\text{in})}+\left(\left(\mathbf{w}_b^{(\text{out})}-\mathbf{w}_b^{(\text{in})}\right)\cdot\mathbf{t}_1\right)\ \mathbf{t}_1. \quad (100)$$

Henceforth, the equations for the three remaining variables to be determined, take the forms

$$m_p\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{n}_c=-m_b\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}-\mathbf{v}_{\text{cmb}}^{(\text{in})}\right)\cdot\mathbf{n}_c,$$
$$I\left(\mathbf{w}_b^{(\text{out})}-\mathbf{w}_b^{(\text{in})}\right)\cdot\mathbf{t}_1=-m_p(\mathbf{r}_{\text{cmp}}\cdot\mathbf{t}_2)\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{t}_3,$$
$$0=\frac{m_p}{2}\left(\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}\cdot\mathbf{t}_3\right)^2-\left(\mathbf{v}_{\text{cmp}}^{(\text{in})}\cdot\mathbf{t}_3\right)^2\right)$$
$$+\frac{m_b}{2}\left(\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}\cdot\mathbf{t}_3\right)^2-\left(\mathbf{v}_{\text{cmb}}^{(\text{in})}\cdot\mathbf{t}_3\right)^2\right)$$
$$+\frac{1}{2}I\left(\left(\mathbf{w}_b^{(\text{out})}\cdot\mathbf{t}_1\right)^2-\left(\mathbf{w}_b^{(\text{out})}\cdot\mathbf{t}_1\right)^2\right). \quad (101)$$

After defining the three quantities

$$x=\left(\mathbf{v}_{\text{cmp}}^{(\text{out})}-\mathbf{v}_{\text{cmp}}^{(\text{in})}\right)\cdot\mathbf{t}_3,$$
$$y=\left(\mathbf{v}_{\text{cmb}}^{(\text{out})}-\mathbf{v}_{\text{cmb}}^{(\text{in})}\right)\cdot\mathbf{t}_3,$$
$$z=\left(\mathbf{w}_b^{(\text{out})}-\mathbf{w}_b^{(\text{in})}\right)\cdot\mathbf{t}_1, \quad (102)$$

the equations get the simple forms

$$x = -\frac{m_b}{m_p} y,$$

$$z = -\frac{m_p}{I}(\mathbf{r}_{\text{cmp}} \cdot \mathbf{t}_2) x,$$

$$0 = \frac{m_p}{2}\left(2\left(\mathbf{v}_{\text{cmp}}^{(\text{in})} \cdot \mathbf{t}_3\right) x + x^2\right) + \frac{m_b}{2}\left(2\left(\mathbf{v}_{\text{cmb}}^{(\text{in})} \cdot \mathbf{t}_3\right) y + y^2\right) + \frac{1}{2} I \left(2\left(\mathbf{w}_b^{(\text{in})} \cdot \mathbf{t}_1\right) z + z^2\right), \tag{103}$$

which after eliminating y and z give the following quadratic equation for x

$$x \left\{ \left(\frac{m_p}{2} + \frac{m_p^3}{2m_b^2} + \frac{I}{2}\left(\frac{m_p}{I}\right)^2 (\mathbf{r}_{\text{cmp}} \cdot \mathbf{t}_2)^2 \right) x + m_p \mathbf{v}_{\text{cmp}}^{(\text{in})} \cdot \mathbf{t}_3 - \frac{m_p^2}{m_b} \mathbf{v}_{\text{cmb}}^{(\text{in})} \cdot \mathbf{t}_i - m_p \left(\mathbf{w}_b^{(\text{in})} \cdot \mathbf{t}_1\right)(\mathbf{r}_{\text{cmp}} \cdot \mathbf{t}_2) \right\} = 0. \tag{104}$$

After solving the equation for x, the solutions for the three remaining quantities can be explicitly obtained in the forms

$$x = -\frac{2\left(\mathbf{v}_{\text{cmp}}^{(\text{in})} \cdot \mathbf{t}_3 - \frac{m_p}{m_b}\mathbf{v}_{\text{cmb}}^{(\text{in})} \cdot \mathbf{t}_i - (\mathbf{w}_b^{(\text{in})} \cdot \mathbf{t}_1)(\mathbf{r}_{\text{cmp}} \cdot \mathbf{t}_2)\right)}{\left(\frac{m_p}{2} + \frac{m_p^3}{2m_b^2} + \frac{I}{2}(\frac{m_p}{I})^2 (\mathbf{r}_{\text{cmp}} \cdot \mathbf{t}_2)^2\right)}, \tag{105}$$

$$y = \frac{m_p}{m_b} \frac{2\left(\mathbf{v}_{\text{cmp}}^{(\text{in})} \cdot \mathbf{t}_3 - \frac{m_p}{m_b}\mathbf{v}_{\text{cmb}}^{(\text{in})} \cdot \mathbf{t}_i - (\mathbf{w}_b^{(\text{in})} \cdot \mathbf{t}_1)(\mathbf{r}_{\text{cmp}} \cdot \mathbf{t}_2)\right)}{\left(\frac{m_p}{2} + \frac{m_p^3}{2m_b^2} + \frac{I}{2}(\frac{m_p}{I})^2 (\mathbf{r}_{cmp} \cdot \mathbf{t}_2)^2\right)}, \tag{106}$$

$$z = \frac{m_p}{I}(\mathbf{r}_{\text{cmp}} \cdot \mathbf{t}_2) \frac{2\left(\mathbf{v}_{\text{cmp}}^{(\text{in})} \cdot \mathbf{t}_3 - \frac{m_p}{m_b}\mathbf{v}_{\text{cmb}}^{(\text{in})} \cdot \mathbf{t}_i - (\mathbf{w}_b^{(\text{in})} \cdot \mathbf{t}_1)(\mathbf{r}_{\text{cmp}} \cdot \mathbf{t}_2)\right)}{\left(\frac{m_p}{2} + \frac{m_p^3}{2m_b^2} + \frac{I}{2}(\frac{m_p}{I})^2 (\mathbf{r}_{\text{cmp}} \cdot \mathbf{t}_2)^2\right)}. \tag{107}$$

Finally, the searched final state quantities become expressed in terms of the initial ones by the formulae

$$\mathbf{v}_{\text{cmb}}^{(\text{out})} = \mathbf{v}_{\text{cmb}}^{(\text{in})} + x\, \mathbf{t}_3, \qquad \mathbf{v}_{\text{cmp}}^{(\text{out})} = \mathbf{v}_{\text{cmp}}^{(\text{in})} + y\, \mathbf{t}_3,$$

$$\mathbf{w}_p^{(\text{out})} = \mathbf{w}_p^{(\text{in})}, \qquad \mathbf{w}_b^{(\text{out})} = \mathbf{w}_b^{(\text{in})} + z\, \mathbf{t}_1, \tag{108}$$

which define the solution of the conservative shock problem. It seems helpful to underline that for finding the given solution, we have assumed energy conservation. However, even by considering the frictionless case, can instead assume that a fraction of the total mechanical energy could have been dissipated in other forms of energy (vibrations, deformations, heat, etc.) during the impact. For this purpose it only needed to added a dissipation term to the energy conservation Eq. (99).

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