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Bound state solutions of schrodinger equation with modified hylleraas plus exponential rosen morse potential

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We present solutions of the Schrödinger equation with modified Hylleraas plus exponential Rosen Morse potential within the framework of the elegant approximation to deal with the centrifugal term for arbitrary orbital angular quantum number. We obtain the energy spectrum and the corresponding wave function using the parametric form of the Nikiforov-Uvarov method. In many cases of interest the energy eigenvalues and eigen functions have been discussed for different potentials.

Keywords: Nikiforov-Uvarov method; Schrodinger equation; centrifugal term.

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1. Introduction

The bound state solutions of Schrodinger equation in quantum mechanics is very difficult to solve with some physical central potentials [1-2]. The analytical solution of Schrodinger equation plays a vital role in quantum mechanics and solving the Schrödinger equation is still an interesting work in the existing literature [3-6]. Recently, the study of exponential-type potential has attracted a lot of interest by many authors [7-8]. However, the bound state solutions of the Schrödinger equation of some of these potentials are possible for few cases such as harmonic oscillator [9], Coulomb potential [10], Woods-Saxon [11], Hulthen [12], Manning-Rosen [13] and others [14].Moreover, when the arbitrary angular momentum quantum number, one can only solve the Schrödinger equation approximately using a suitable approximation scheme [15]. One of such approximation includes conventional approximation scheme proposed by Greene and Aldrich [16], improved approximation scheme by Jia *et al* [17], elegant approximation scheme [18] and a new approximation scheme by Dong et al [19]. These approximations are used to deal with the centrifugal term and many authors have investigated approximately the bound state solutions of the Schrödinger equation with exponential-like potentials. For further details readers can refer to most recent works [20-22].

In this work, we used elegant approximation scheme proposed in Ref. 18 to deal with the centrifugal term and solve Schrödinger equation with modified Hylleraas plus exponential Rosen-Morse potential defined as [23-24].

$$V(r) = \frac{-V_0}{b}\left[\frac{a+e^{-2\alpha(r-r_c)}}{1+e^{-2\alpha(r-r_c)}}\right] - \frac{4V_1 e^{-2\alpha(r-r_c)}}{(1+e^{-2\alpha(r-r_c)})^2} + V_2\left(\frac{1-e^{-2\alpha(r-r_c)}}{1+e^{-2\alpha(r-r_c)}}\right), \quad (1)$$

where V_0, V_1, V_2 are the potential depths and a, b are the Hylleraas parameters, α is the adjustable parameter and r_c is the distance from the equilibrium position.

2. Review of nu method and its parametric form

The NU method is based on the solution of a generalized second order linear differential equation with special orthogonal function [25]. The Schrödinger equation

$$\psi''(x) + (E - V(x))\psi(x) = 0 \quad (2)$$

can be solved by this method. This can be done by transforming this equation into equation of hypergeometric type with appropriate transformation, $s = s(x)$

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0. \quad (3)$$

In order to find the exact solution to Eq. (3), we set the wave function as

$$\psi(s) = \phi(s)\chi(s) \quad (4)$$

and on substituting Eq. (4) into Eq. (3), then Eq. (3) reduces to hypergeometric type,

$$\sigma(s)\chi''(s) + \tau(s)\chi'(s) + \lambda\chi(s) = 0 \quad (5)$$

where the wave function $\phi(s)$ is defined as the logarithmic derivative [25],

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \quad (6)$$

where $\pi(s)$ is at most first order polynomials.

Likewise, the hypergeometric type function $\phi(s)$ in Eq. (5) for a fixed n is given by the Rodriques relation as

$$\chi_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}[\sigma^n(s)\rho(s)] \quad (7)$$

where B_n is the normalization constant and the weight function $\rho(s)$ must satisfy the condition

$$\frac{d}{ds}(\sigma^n(s)\rho(s)) = \tau(s)\rho(s) \tag{8}$$

with

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s). \tag{9}$$

In order to accomplish the condition imposed on the weight function $\rho(s)$, it is necessary that the classical or polynomials $\tau(s)$ be equal to zero to some point of an interval (a, b) and its derivative at this interval at $\sigma(s) > 0$ will be negative, that is

$$\frac{d\tau(s)}{ds} < 0. \tag{10}$$

Therefore, the function $\pi(s)$ and the parameters λ required for the NU method are defined as follows:

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma} \tag{11}$$

$$\lambda = k + \pi'(s). \tag{12}$$

The s-values in Eq. (11) are possible to evaluate if the expression under the square root be square of polynomials. This is possible, if and only if its discrimnant is zero. With this, the new eigenvalues equation becomes

$$\lambda = \lambda_n = -\frac{nd\tau}{ds} - \frac{n(n-1)}{2}\frac{d^2\sigma}{ds^2}, \quad n = 0, 1, 2, \ldots \tag{13}$$

On comparing Eq. (12) and Eq. (13), we obtain the energy eigenvalues

The parametric generalization of the NU method is given by the generalized hypergeometric-type equation as [26]

$$\psi''(s) + \frac{(c_1 - c_2 s)}{s(1 - c_3 s)}\psi'(s) + \frac{1}{s^2(1 - c_3 s)^2} \times [-\xi_1 s^2 + \xi_2 s - \xi_3]\psi(s) = 0. \tag{14}$$

Equation (14) is solved by comparing it with Eq. (3) and the following polynomials are obtained:

$$\tilde{\tau}(s) = (c_1 - c_2 s), \quad \sigma(s) = s(1 - c_3 s), \quad \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3. \tag{15}$$

Now substituting Eq. (15) into Eq. (11), we find

$$\tilde{\sigma}(s) = c_4 + c_5 s \pm [(c_6 - c_3 k_\pm)s^2 + (c_7 + k_\pm)s + c_8]^{\frac{1}{2}} \tag{16}$$

where

$$c_4 = \frac{1}{2}(1 - c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3), \quad c_6 = c_5^2 + \xi_1 \quad c_7 = 2c_4 c_5 - \xi_2, \quad c_8 = c_4^2 + \xi_3. \tag{17}$$

The resulting value of k in Eq. (16) is obtained from the condition that the function under the square root be square of a polynomials and it yields,

$$k_\pm = -(c_7 + 2c_3 c_8) \pm 2\sqrt{c_8 c_9} \tag{18}$$

where

$$c_9 = c_3 c_7 + c_2^2 c_8 + c_6. \tag{19}$$

The new $\pi(s)$ for k_{-}becomes

$$\pi(s) = c_4 + c_5 s - [(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}] \tag{20}$$

for the k_{-} value,

$$k_{-} = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9}. \tag{21}$$

Using Eq. (9), we obtain

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}]. \tag{22}$$

The physical condition for the bound state solution is $\tau' < 0$ and thus

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0. \tag{23}$$

With the aid of Eqs. (12) and (13), we obtain the energy equation as

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1) \times (\sqrt{c_9} + c_3\sqrt{c_8}) + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0. \tag{24}$$

The weight function $\rho(s)$ is obtained from Eq. (8) as

$$\rho(s) = s^{c_{10}-1}(1 - c_3 s)^{\frac{c_{11}}{c_3} - c_{10} - 1} \tag{25}$$

and together with Eq. (7), we have

$$\chi_n(s) = P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s) \tag{26}$$

where

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} \tag{27}$$

$$c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}) \tag{28}$$

and $P_n^{(\alpha,\beta)}(s)$ are the Jacobi Polynomials. The second part of the wave function is obtained from Eq. (6) as

$$\phi(s) = s^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} \tag{29}$$

where

$$c_{12} = c_4 + \sqrt{c_8}, \quad c_1 3 = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}). \tag{30}$$

Thus, the total wave function becomes

$$\psi(s) = N_n s^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} \times P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s), \tag{31}$$

where N_n is the normalization constant.

3. Factorization method

In spherical coordinate, the Schrödinger equation with the potential $V(r)$ is given as [27]

$$\frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\times\psi(r,\theta,\varphi)+V(r)\psi(r,\theta,\varphi)=E\psi(r,\theta,\varphi). \quad (32)$$

Using the common ansatz for the wave function as

$$\psi(r,\theta,\varphi)=\frac{R(r)}{r}Y_{lm}(\theta,\varphi) \quad (33)$$

and substituting Eq. (33) into Eq. (32), we obtain the following sets of equations:

$$\frac{d^2R_{nl}}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{\lambda\hbar^2}{2\mu r^2}\right]R_{nl}=0 \quad (34)$$

$$\frac{d^2\Theta(\theta)}{d\theta^2}+\cot\theta\frac{d\Theta(\theta)}{d\theta}\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\Theta(\theta)=0 \quad (35)$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m_l^2\Phi(\varphi)=0 \quad (36)$$

where $\lambda=l(l+1)$ and m_l^2 are the separation constants.

$Y_{ml}(\theta,\varphi) = \Theta(\theta)\Phi(\varphi)$ is the solution of Eqs. (35) and (36). $Y_{ml}(\theta,\varphi)$ are the spherical harmonic and their solutions are well known [28]. Equation (34) is the radial part of the Schrödinger equation which we are subject of discussion in the next session.

4. Solution of the radial part of Schrödinger equation

Substituting potential of Eq. (1) into radial Schrödinger equation of Eq. (34), we obtain

$$\frac{d^2R_{nl}}{dr^2}+\frac{2\mu}{\hbar^2}\left[E_{nl}+\frac{V_0}{b}\left(\frac{a+e^{-2\alpha(r-r_c)}}{1+e^{-2\alpha(r-r_c)}}\right)+\frac{4V_1e^{-2\alpha(r-r_c)}}{(1+e^{-2\alpha(r-r_c)})}^{2}-V_2\left(\frac{1-e^{-2\alpha(r-r_c)}}{1+e^{-2\alpha(r-r_c)}}\right)-\frac{l(l+1)\hbar^2}{2\mu r^2}\right]R_{nl}=0. \quad (37)$$

It is well known that the Schrödinger equation of Eq. (37) cannot be solved exactly for $l\neq 0$ by any known method. The way out is to use approximation for the centrifugal term. On this note, we invoke the elegant approximation [18] for the $1/r^2$ as

$$\frac{1}{r^2}=a_1+\frac{a_2}{(1+e^{-2\alpha(r-r_c)})}+\frac{a_3}{(1+e^{-2\alpha(r-r_c)})^2} \quad (38)$$

where,

$$a_1=\frac{1}{r_c^2}\left[1+\frac{3}{2\alpha r_c}+\frac{3}{4\alpha^2r_c^2}\right] \quad (39)$$

$$a_2=-\frac{1}{r_c^2}\left[\frac{2}{\alpha r_c}+\frac{3}{2\alpha^2r_c^2}\right] \quad (40)$$

$$a_3=\frac{1}{r_c^2}\left[\frac{1}{2\alpha r_c}+\frac{3}{4\alpha^2r_c^2}\right] \quad (41)$$

We compare the approximation of Eq. (38) with $1/r^2$ in Fig. 1 and this shows a good agreement for the centrifugal term for a short-range potential.

Substituting Eq. (38) into (37) and using the transformation $s=-e^{-2\alpha(r-r_c)}$, we have

$$\frac{d^2R(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR(s)}{ds}+\frac{1}{s^2(1-s)^2}[-(\varepsilon^2-A)s^2+(2\varepsilon^2+B)s-(\varepsilon^2-C)]R(s)=0, \quad (42)$$

where

$$-\varepsilon^2=\frac{\mu E}{2\alpha^2\hbar^2}, \quad (43)$$

$$A=\frac{\mu}{2\alpha^2\hbar^2b}(V_0+V_2b)-\frac{l(l+1)a_1}{4\alpha^2}, \quad (44)$$

$$B=-\frac{\mu V_0}{2\alpha^2\hbar^2b}(a+1)-\frac{2\mu V_1}{\alpha^2\hbar^2}+\frac{l(l+1)}{2\alpha^2}\left(a_1+\frac{a_2}{2}\right) \quad (45)$$

$$C=\frac{\mu}{2\alpha^2\hbar^2b}(V_0a-V_2b)-\frac{l(l+1)}{4\alpha^2}(a_1+a_2+a_3). \quad (46)$$

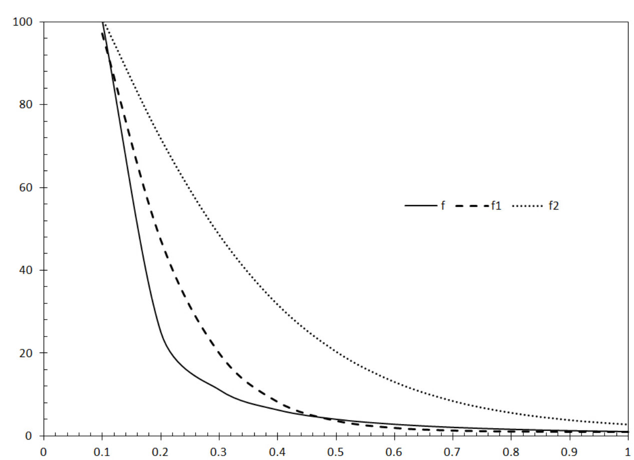

FIGURE 1. Comparison of the centrifugal term $f=1/r^2$ with the elegant approximation f_1 for $\alpha=0.1$ with $a_1=1$, $a_2=150$, $a_3=85$, $rc=0.1$ and f_2 for $\alpha=0.2$ with $a_1=1$, $a_2=150$, $a_3=100$ and $rc=0.1$.

Comparing Eq. (42) with Eq. (14), we obtain the following parameters:

$$\begin{aligned}
&\xi_1 = \varepsilon^2 - A; \quad \xi_2 = 2\varepsilon^2 + B, \quad \xi_3 = \varepsilon^2 - C\\
&c_1 = c_2 = c_3 = 1, \quad c_4 = 0, \quad c_5 = -1/2\\
&c_6 = \frac{1}{4} + \varepsilon^2 - A, \quad c_7 = -2\varepsilon^2 - B\\
&c_8 = \varepsilon^2 - C, \quad c_9 = \frac{1}{4} - (A + B + C)\\
&c_{10} = 1 + 2\sqrt{\varepsilon^2 - C},\\
&c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} - A - B - C} + \sqrt{\varepsilon^2 - C}\right)\\
&c_{12} = \sqrt{\varepsilon^2 - C},\\
&c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} - A - B - C} + \sqrt{\varepsilon^2 - C}\right)
\end{aligned} \tag{47}$$

Using Eq. (16), we calculate the $\pi(s)$ function as

$$\pi(s) = \frac{-s}{2} \pm \left[\left(\frac{1}{4} + \varepsilon^2 - A - k_\pm\right)s^2 + (2\varepsilon^2 - B + k_\pm)s + \varepsilon^2 - C\right]^{\frac{1}{2}}. \tag{48}$$

From Eq. (18), $k_\pm$ is determined to be

$$k_\pm = B + 2C \pm 2\sqrt{(\varepsilon^2 - C)\left(\frac{1}{4} - A - B - C\right)}. \tag{49}$$

Hence, we choose the proper value of $\pi(s)$ so that

$$\tau(s) = 1 - 2s - 2\left[\left(\sqrt{\left(\frac{1}{4} - A - B - C\right)} + \sqrt{(\varepsilon^2 - C)}\right)s - \sqrt{(\varepsilon^2 - C)}\right], \tag{50}$$

whose negative derivative is

$$\tau'(s) = -\left[2 + 2\left(\sqrt{\frac{1}{4} - A - B - C} + \sqrt{(\varepsilon^2 - C)}\right)\right] < 0. \tag{51}$$

The new $\pi(s)$ function for NU is chosen as

$$\pi(s) = \frac{-s}{2} \pm \begin{cases} \left(\sqrt{\frac{1}{4} - A - B - C} - \sqrt{(\varepsilon^2 - C)}\right)s + \sqrt{(\varepsilon^2 - C)} \\ \text{for} \quad k_+ = B + 2C + 2\sqrt{(\varepsilon^2 - C)\left(\frac{1}{4} - A - B - C\right)} \\ \left(\sqrt{\frac{1}{4} - A - B - C} - \sqrt{(\varepsilon^2 - C)}\right)s - \sqrt{(\varepsilon^2 - C)} \\ \text{for} \quad k_- = B + 2C - 2\sqrt{(\varepsilon^2 - C)\left(\frac{1}{4} - A - B - C\right)} \end{cases} \tag{52}$$

We find the physical solution from Eq. (52) as

$$\pi(s) = \frac{-s}{2} - \left[\left(\sqrt{\frac{1}{4} - A - B - C} + \sqrt{(\varepsilon^2 - C)}\right)s - \sqrt{(\varepsilon^2 - C)}\right] \tag{53}$$

for

$$k_- = B + 2C - 2\sqrt{(\varepsilon^2 - C)\left(\frac{1}{4} - A - B - C\right)}. \tag{54}$$

Now using Eq. (24), (43), (44), (45), (46) and (47), we obtain the energy spectrum for the modified Hylleraas plus exponential Rosen Morse potential as

$$E_{nl} = -\frac{V_0 a}{b} + V_2 + \frac{l(l+1)\hbar^2}{2\mu}(a_1 + a_2 + a_3) - \frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{\mu V_0(1-a)}{2\alpha^2\hbar^2 b} + \frac{\mu V_2}{\alpha^2\hbar^2} + \frac{l(l+1)}{4\alpha^2}(a_2 + a_3) + \left(n + \frac{1}{2} + \sqrt{\frac{2\mu V_1}{\alpha^2\hbar^2} + \frac{l(l+1)a_3}{4\alpha^2} + \frac{1}{4}}\right)^2}{n + \frac{1}{2} + \sqrt{\frac{2\mu V_1}{\alpha^2\hbar^2} + \frac{l(l+1)a_3}{4\alpha^2} + \frac{1}{4}}}\right]^2. \tag{55}$$

The weight function $\rho(s)$ can be obtained by using Eqs. (25) and parameters in Eq. (47) as

$$\rho(s) = s^{\epsilon}(1-s)^{\vartheta}. \tag{56}$$

Where $\epsilon = 2\sqrt{\varepsilon^2 - C}$ and $\vartheta = 2\sqrt{(1/4) - (A+B+C)}$ and using Eq. (26), we get the wave function $\chi_n(s)$ as

$$\chi_n(s) = P_n^{(\epsilon,\vartheta)}(1-2s) \tag{57}$$

where $P_n^{(\epsilon,\vartheta)}$ is the Jacobi polynomial. The other wave function $\phi(s)$ is obtained from Eq. (29) as

$$\phi(s) = s^{\epsilon/2}(1-s)^{\frac{1+\vartheta}{2}}. \tag{58}$$

The radial wave function is obtained from Eq. (31) as

$$R_{nl}(s) = N_{nl} s^{\frac{\epsilon}{2}}(1-s)^{\frac{1+\vartheta}{2}} P_n^{(\epsilon,\vartheta)}(1-2s), \tag{59}$$

where N_{nl} is the normalization constant. The normalization constant N_{nl} can be calculated in a compact form. In order to do this,we start by using the normalization condition,

$$\int_0^\infty |N_{nl}|^2 dr = 1$$

and under the change of coordinate $x = 1-2e^{-2\alpha(r-r_c)}$. We can rewrite Eq. (59) as,

$$R_{nl}(s) = N_{nl}\frac{n!\Gamma(\epsilon+1)}{\Gamma(n+\epsilon+1)}\left(e^{-\alpha(r-r_c)}\right)^{\frac{\epsilon}{2}} \times\left(1-e^{-\alpha(r-r_c)}\right)^{\frac{1+\vartheta}{2}} P_n^{(\epsilon,\vartheta)}\left(1-2e^{-\alpha(r-r_c)}\right). \tag{60}$$

Furthermore, the relation between the hypergeometric function and the Jacobi polynomials are [33-34],

$$F_1\left(-n, n+\nu+\mu+1, \nu+1; \frac{1-x}{2}\right) = \frac{n!\Gamma(\nu+1)}{\Gamma(n+\nu+1)} P_n^{(\nu,\mu)}(x) \tag{61}$$

$$\int_{-1}^{1}(1-x)^{\nu-1}(1+x)^{\mu}\left[P_n^{(\nu,\mu)}(x)\right]^2 dx = \frac{2^{\nu+\mu}\Gamma(n+\nu+1)\Gamma(n+\mu+1)}{n!\nu\Gamma(n+\nu+\mu+1)}. \tag{62}$$

By using

$$\frac{1+x}{2} = 1 - \frac{1-x}{2}$$

and the Eqs. (61-62), we obtain the normalization constant N_{nl} as,

$$N_{nl}=\frac{1}{\Gamma(\epsilon+1)}\left[\frac{\epsilon\Gamma(n+\epsilon+1)\Gamma(n+\epsilon+\vartheta+1)}{n!\Gamma(n+\vartheta+1)}\right]^{\frac{1}{2}}. \tag{63}$$

Finally, the total wave function $\psi(r,\theta,\varphi)$ of the modified Hylleraas plus exponential Rosen Morse potential is obtained using Eq. (33) as

$$\psi(r,\theta,\varphi) = \frac{1}{\Gamma(\epsilon+1)}\left[\frac{\epsilon\Gamma(n+\epsilon+1)\Gamma(n+\epsilon+\vartheta+1)}{n!\Gamma(n+\vartheta+1)}\right]^{\frac{1}{2}} \times\frac{1}{r}\left(e^{-2\alpha(r-r_c)}\right)^{\frac{\epsilon}{2}}\left(1+e^{-2\alpha(r-r_c)}\right)^{\frac{1+\vartheta}{2}} \times P_n^{(\epsilon,\vartheta)}\left(1+2e^{-2\alpha(r-r_c)}\right)Y_{lm}(\theta,\varphi). \tag{64}$$

5. Discussion

By choosing appropriate values in the modified Hylleraas plus Rosen Morse model, we obtain four types of potential models and then evaluate their energy eigenvalues equations and their corresponding eigen functions.

5.1. Woods-Saxon Potential

If we choose $V_1 = V_2 = a = 0$ and $b = 1$ in Eq. (1), we obtain Woods-Saxon potential [29]

$$V(r) = \frac{-V_0 e^{-2\alpha(r-r_c)}}{1+e^{-2\alpha(r-r_c)}}. \tag{65}$$

Substituting these parameters into Eqs. (55) and (60), we obtain the approximate energy eigenvalues and the corresponding wave function as

$$E_{nl} = \frac{l(l+1)\hbar^2}{2\mu}(a_1+a_2+a_3) - \frac{\alpha^2\hbar^2}{2\mu}, \tag{66}$$

and

$$\psi(r,\theta,\varphi) = N_{nl}\frac{1}{r}\left(e^{-2\alpha(r-r_c)}\right)^{\frac{\mu'}{2}}\left(1+e^{-2\alpha(r-r_c)}\right)^{\frac{1+\vartheta'}{2}} \times P_n^{(\mu',\vartheta')}\left(1+2e^{-2\alpha(r-r_c)}\right)Y_{lm}(\theta,\varphi), \tag{67a}$$

respectively where

$$\mu' = 2\sqrt{\varepsilon^2-\Lambda}, \quad \vartheta' = 2\sqrt{\frac{1}{4}-\delta-\gamma-\Lambda}$$
$$\Lambda = \frac{l(l+1)}{4\alpha^2}(a_2+a_3)$$
$$\delta = \frac{\mu V_0}{2\alpha^2\hbar^2} - \frac{l(l+1)a_1}{4\alpha^2}$$
$$\gamma = \frac{\mu V_0}{2\alpha^2\hbar^2} - \frac{l(l+1)}{2\alpha^2}\left(a_1+\frac{a_2}{2}\right). \tag{67b}$$

5.2. Rosen Morse Potential

If we replace $b = a = 1$ and $V_0 = 0$ in Eq. (1), we obtain the Rosen Morse potential [29]

$$V(r)=\frac{-4V_1 e^{-2\alpha(r-r_c)}}{(1+e^{-2\alpha(r-r_c)})^2}+V_2\left(\frac{1-e^{-2\alpha(r-r_c)}}{1+e^{-2\alpha(r-r_c)}}\right). \tag{68}$$

Now substituting these parameters into Eqs. (55) and (60), we get the energy spectrum and the corresponding wave function for the Rosen Morse potential as

$$E_{nl}=V_2+\frac{l(l+1)\hbar^2}{2\mu}(a_1+a_2+a_3)-\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{\mu V_2}{2\alpha^2\hbar^2}+\frac{l(l+1)}{4\alpha^2}(a_2+a_3)+\left(n+\frac{1}{2}+\sqrt{\frac{\mu V_1}{2\alpha^2\hbar^2}+\frac{l(l+1)a_3}{4\alpha^2}+\frac{1}{4}}\right)^2}{n+\frac{1}{2}+\sqrt{\frac{\mu V_1}{2\alpha^2\hbar^2}+\frac{l(l+1)a_3}{4\alpha^2}+\frac{1}{4}}}\right]^2 \tag{69}$$

and

$$\psi(r,\theta,\varphi)=N_{nl}\frac{1}{r}\left(e^{-2\alpha(r-r_c)}\right)^{\frac{\tilde{\mu}}{2}}\left(1+e^{-2\alpha(r-r_c)}\right)^{\frac{1+\tilde{\vartheta}}{2}}P_n^{(\mu',\tilde{\vartheta})}\left(1+2e^{-2\alpha(r-r_c)}\right)Y_{lm}(\theta,\varphi) \tag{70a}$$

where

$$\tilde{\mu}=2\sqrt{\tilde{\varepsilon}-\tilde{\Lambda}},\qquad \tilde{\vartheta}=\sqrt{\frac{1}{4}-\tilde{\delta}-\tilde{\gamma}-\tilde{\Lambda}}\qquad \tilde{\Lambda}=\frac{\mu V_2}{2\alpha^2\hbar^2}-\frac{l(l+1)}{4\alpha^2}(a_1+a_2+a_3)$$
$$\tilde{\delta}=\frac{\mu V_2}{2\alpha^2\hbar^2}-\frac{l(l+1)a_1}{4\alpha^2}\qquad \tilde{\gamma}=\frac{\mu V_1}{2\alpha^2\hbar^2}-\frac{l(l+1)}{2\alpha^2}\left(a_1+\frac{a_2}{2}\right) \tag{70b}$$

5.3. Generalized Woods-Saxon Potential

If we set $V_2=a=0$, $V_1=-\frac{V_0}{4}$, $b=1$ in Eq. (1), we obtain generalized Woods-Saxon potential [30] as

$$V(r)=\frac{-V_0e^{-2\alpha(r-r_c)}}{(1+e^{-2\alpha(r-r_c)})^2}+\frac{V_0e^{-2\alpha(r-r_c)}}{(1+e^{-2\alpha(r-r_c)})^2}. \tag{71}$$

Substituting these parameters into Eqs. (55) and (60), we obtain the energy eigenvalues and the corresponding wave function for this potential as

$$E_{nl}=V_2+\frac{l(l+1)\hbar^2}{2\mu}(a_1+a_2+a_3)-\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{\mu V_0}{2\alpha^2\hbar^2}+\frac{l(l+1)}{4\alpha^2}(a_2+a_3)+\left(n+\frac{1}{2}+\sqrt{\frac{\mu V_0}{2\alpha^2\hbar^2}+\frac{l(l+1)a_3}{4\alpha^2}+\frac{1}{4}}\right)^2}{n+\frac{1}{2}+\sqrt{\frac{\mu V_0}{2\alpha^2\hbar^2}+\frac{l(l+1)a_3}{4\alpha^2}+\frac{1}{4}}}\right]^2 \tag{72}$$

and

$$\psi(r,\theta,\varphi)=N_{nl}\frac{1}{r}\left(e^{-2\alpha(r-r_c)}\right)^{\frac{\mu'}{2}}\left(1+e^{-2\alpha(r-r_c)}\right)^{\frac{(1+\Omega)}{2}}\times P_n^{(\mu',\Omega)}\left(1+2e^{-2\alpha(r-r_c)}\right)Y_{lm}(\theta,\varphi), \tag{73a}$$

where

$$\mu'=2\sqrt{\varepsilon-\Lambda},\qquad \Omega=\sqrt{\frac{1}{4}-\delta-\gamma'-\Lambda}\qquad \gamma'=\frac{l(l+1)}{2\alpha^2}\left(a_1+\frac{a_2}{2}\right), \tag{73b}$$

Λ and had been defined in subsec. (5.1).

5.4. Poschl-Teller Potential

If we chose $V_0=V_2=0$ in Eq. (1), we obtain the Poschl potential [31]

$$V(r)=\frac{-4V_1e^{-2\alpha(r-r_c)}}{(1+e^{-2\alpha(r-r_c)})^2}. \tag{74}$$

Substituting these parameters into Eqs. (55) and (60), we obtain the energy spectrum and the corresponding wave function as

$$E_{nl}=\frac{l(l+1)\hbar^2}{2\mu}(a_1+a_2+a_3)-\frac{\alpha^2\hbar^2}{2\mu}\times\left[\frac{\frac{l(l+1)}{4\alpha^2}(a_2+a_3)+\left(n+\frac{1}{2}+\sqrt{\frac{\mu V_1}{2\alpha^2\hbar^2}+\frac{l(l+1)a_3}{4\alpha^2}+\frac{1}{4}}\right)^2}{n+\frac{1}{2}+\sqrt{\frac{\mu V_1}{2\alpha^2\hbar^2}+\frac{l(l+1)a_3}{4\alpha^2}+\frac{1}{4}}}\right]^2 \tag{75}$$

and

$$\psi(r,\theta,\varphi) = N_{nl}\frac{1}{r}\left(e^{-2\alpha(r-r_c)}\right)^{\frac{\mu'}{2}}\left(1+e^{-2\alpha(r-r_c)}\right)^{\frac{1+\Omega'}{2}} \times P_n^{(\mu',\Omega')}\left(1+2e^{-2\alpha(r-r_c)}\right)Y_{lm}(\theta,\varphi) \quad (76a)$$

where

$$\mu' = 2\sqrt{\varepsilon^2-\Lambda}, \quad \Omega^1 = 2\sqrt{\frac{1}{4}-\delta'-\tilde{\gamma}-\Lambda}$$
$$\delta' = \frac{l(l+1)a_1}{4\alpha^2}, \quad (76b)$$

Λ had been defined already in Subsec. 5.1 and $\tilde{\gamma}$ is defined in Subsec. 5.2.

5.5. Standard Eckart Potential

If we set $V_0 = 0$, $V_1 = \frac{-V_1}{4}$ and $V_2 = -V_2$, $b = 1$, $a = 0$ in Eq. (1) we obtain standard Eckart potential [32] as

$$V(r) = \frac{V_1 e^{-2\alpha(r-r_c)}}{\left(1+e^{-2\alpha(r-r_c)}\right)^2} - V_2\left(\frac{1-e^{-2\alpha(r-r_c)}}{1+e^{-2\alpha(r-r_c)}}\right). \quad (77)$$

Substituting these parameters into Eqs. (55) and (60), we obtain the energy spectrum and the corresponding wave function as

$$E_{nl} = -V_2 + \frac{l(l+1)\hbar^2}{2\mu}(a_1+a_2+a_3) - \frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{l(l+1)}{4\alpha^2}(a_2+a_3) + \left(n+\frac{1}{2}+\sqrt{\frac{\mu V_1}{2\alpha^2\hbar^2}+\frac{l(l+1)a_3}{4\alpha^2}+\frac{1}{4}}\right)^2}{n+\frac{1}{2}+\sqrt{\frac{\mu V_1}{2\alpha^2\hbar^2}+\frac{l(l+1)a_3}{4\alpha^2}+\frac{1}{4}}}\right]^2 \quad (78)$$

and

$$\psi(r,\theta,\varphi) = N_{nl}\frac{1}{r}\left(e^{-2\alpha(r-r_c)}\right)^{\frac{\epsilon'}{2}}\left(1+e^{-2\alpha(r-r_c)}\right)^{\frac{1+\tilde{\Omega}}{2}} \times P_n^{(\epsilon',\tilde{\Omega})}(1+2e^{-2\alpha(r-r_c)})Y_{lm}(\theta,\varphi) \quad (79)$$

where

$$\epsilon' = 2\sqrt{\varepsilon^2-\tilde{C}}, \quad \tilde{\Omega} = \sqrt{\frac{1}{4}-\tilde{A}-\tilde{B}-\tilde{C}}, \quad (80)$$

and

$$\tilde{A} = \frac{-\mu V_2}{2\alpha^2\hbar^2} - \frac{l(l+1)a_1}{4\alpha^2}$$
$$\tilde{B} = \frac{\mu V_1}{2\alpha^2\hbar^2} - \frac{l(l+1)}{4\alpha^2}\left(a_1+\frac{a_2}{2}\right)$$
$$\tilde{C} = \frac{\mu V_2}{2\alpha^2\hbar^2} - \frac{l(l+1)}{2\alpha^2}(a_1+a_2+a_3). \quad (81)$$

6. Conclusion

In this paper, we solved explicitly the Schrödinger equation for the modified Hylleraas plus exponential Rosen Morse potential for arbitrary states by using the parametric form of the Nikiforov-Uvarov method. By using the elegant approximation for the centrifugal term, we obtain approximately the energy eigenvalues and the unnormalized wave function expressed in terms of the Jacobi polynomials for arbitrary wave states. We obtained some well known potential such as Woods-Saxon, Rosen Morse, generalized Woods-Saxon, Poschl-Teller and standard Eckart potentials by choosing appropriate parameters in the modified Hylleraas plus exponential Rosen Morse .

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