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A theorem allowing the derivation of deterministic evolution equations from stochastic evolution equations. tensorial, spinorial, and other extensions

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The proof of a new extension of a theorem that allows to construct deterministic evolution equations from a set of discrete stochastic evolution equation is developed. The present extension allows to handle evolution equations of dynamical variables that are tensors of any rank. Due that the almost paradigmatic field that uses tensors is relativity, an illustrative example is given and the equations that allows to find the geodesics is derived from a set of discrete stochastic evolution equations. Extension to dynamical variables described by spinor indices or "arbitrary labels" are given.

Keywords: Evolution equations; stochastic processes.

La demostración de una nueva extensión de un teorema, que permite la construcción de ecuaciones de evolución deterministas a partir de un conjunto de ecuaciones de evolución discretas estocásticas, es desarrollada. La extensión presente, permite manejar ecuaciones de evolución de variables dinámicas que son tensores de cualquier rango. Como el más paradigmático campo que usa tensores es la relatividad, un ejemplo ilustrativo es dado y las ecuaciones que permiten hallar las geodésicas es derivado de un conjunto de ecuaciones de evolución discretas estocásticas. Extensiones a variables dinámicas descriptas por índices espinoriales o "etiquetas arbitrarias" son dados.

Descriptores: Ecuaciones de evolución; procesos estocásticos.

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1. Introduction

A lot of work concerning the derivation of evolution equation that evolve Markovianly as well as non-Markovianly was done during decades. An illustrative list, including both type of equations, can be found in Refs. 1 to 23. In this paper, perhaps the generalist version of the proof of a theorem that allows to obtain continuum deterministic evolution equations from a set of discrete stochastic evolution equations, is derived. The dynamical variables used are tensors of any rank that may be in general complex numbers and consequently wide variety of problems may be studied with the present approach. This paper, together with another three [24-26], must be viewed as companions papers and in each of them some features and examples were worked in some detail to show the basic steps that allows the construction of the deterministic differential equations. The present approach allows to handle models whose dynamical variables evolves Markovianly as well as non-Markovianly and can be, in general, a mix of different types as in Ref. 24 where the equations of quantum electrodynamics were derived. On the other hand, in the present work the differential equations for the geodesics is also derived from a non-Markovian discrete stochastic evolution equations.

The paper is organized as follows. In Sec. 2 an introduction of the evolution rules corresponding to models with an updating of the dynamical variables that depends on the values of the dynamical variables at an arbitrary many previous time steps and with subsets that are of different type, is considered. In Sec. 3 a theorem allowing to connect two sets of stochastic evolution equations with another two sets that contain deterministic weights is proved. For the cases of non-Markovian discrete stochastic evolution equations, containing sets of different types of dynamical variables, this connection is proved. The Markovian case, can be obtained as a special one, with updating that depends just on the first previous time step. In Sec. 4 the general procedure will be applied to obtain the differential equation for the geodesics. In the appendix, for the sake of completeness, the derivation is made by standard procedures. In Sec. 5 a discussion of the possible extension of the theorem to evolution equations containing spinor indices and in general to dynamical variables of arbitrary sets of labels are sketched to show the possibilities of the use of the theorem. The conclusions are given at the end of this section.

2. Stochastic evolution updating for a set of complex dynamical variables that are components of tensors: Basic definitions

A four-dimensional lattice Λ consisting of a set of points $\mathbf{x}$, with periodic boundary conditions in an interval $[-L_{0i}/2, +L_{0i}/2]$ for $i = 1, ..., 4$ (L_{0i} being finite or infinite), will be considered, and a set of complex dynamical variables $q^{(r_0)r_1 \ldots r_{\alpha_{s_0}}}_{(s_0)s_1 \ldots s_{\beta_{s_0}}}(t, \mathbf{x})$ will be used to describe the value of each dynamical variable belonging to the type s_0 in a realization r_0, that are components of a tensor $r_{\alpha_{s_0}}$ time contra-variant and $s_{\beta_{s_0}}$ times covariant, of order $\alpha_{s_0} + \beta_{s_0}$, at coordinate $\mathbf{x} = x_1, x_2, x_3, x_4$ and at evolution parameter t. The separation between sites or lattice constant is a_1, a_2, a_3, a_4 and the separation between two successive updates is a_0. The length of the lattice corresponding to each coordinate is $L_i = a_i L_{0i}$ and sites of the lattice is $M = (2L_{01}+1)(2L_{02}+1)(2L_{03}+1)(2L_{04}+1)$. The evolution equation for the set of dynamical variable can be expressed, as in Ref. [24], in the following general form

$$q^{(r_0)r_1\ldots r_{\alpha s_0}}_{(s_0)s_1\ldots s_{\beta s_0}}(t+a_0,\mathbf{x}) = q^{(r_0)r_1\ldots r_{\alpha s_0}}_{(s_0)s_1\ldots s_{\beta s_0}}(t,\mathbf{x}) + G^{(r_0)}_{(s_0)}(t,...,t-l_k a_0, X_0,...,X_{l_{0k}}, X_j, X_\xi),$$
$$\forall s_0 \in \{1,...,S\}, t \geq 0, \mathbf{x} \in \Lambda, \tag{1}$$

where G denote the set of rules that define a given model, S is the different many types of variables and X_0,...,$X_{l_{0k}}$ denote the set of complex dynamical variables

$$q^{(r_0)r_1\ldots r_{\alpha s_0}}_{(s_0)s_1\ldots s_{\beta s_0}}(t,\mathbf{x}),..,q^{(r_0)r_1\ldots r_{\alpha s_0}}_{(s_0)s_1\ldots s_{\beta s_0}}(t-l_{0k}a_0,\mathbf{x}),$$

respectively. The set of both discrete and continuous stochastic variables that confer stochasticity to the evolution equations are $X_j = \{j\}$ and $X_\xi = \{\xi\}$, respectively. Note that both, $j = j^{(r_0)}(t)$ and $\xi = \xi^{(r_0)}(t)$, depend on the particular realization r_0 and on the evolution parameter t. Below, usually the dependence on t is neglected and in j also the dependence on r_0, to save printing. The sets of dynamical variables depends on the particular realization r and previous time $t,...,t-l_{0k}a_0$. The previous times is $k+1$ and the set is $l_{0\alpha} = 0,...,k$, for any $0 \geq \alpha \geq k$. The stochastic variables are chosen in such a way that all of them are statistically independent and a factorization of each product that contain stochastic variables is then possible. Let us assume, for the sake of simplicity, that the set of S dynamical variables are separated in subsets of S_1 and S_2 dynamical variables such that $S = S_1 + S_2$. Moreover, let us assume that amount of elements of S_1 and S_2, is one for each subset and the two types of dynamical variables are $s_0 = 1, 2$. Both two subset correspond to dynamical variables that evolves Non-Markovianly. The stochastic evolution equations of the form

$$\begin{aligned}
q^{(r_0)\mathbf{r}_{\alpha 1}}_{(1)\mathbf{s}_{\beta 1}}(t+a_0,\mathbf{x}) &= q^{(r_0)\mathbf{r}_{\alpha 1}}_{(1)\mathbf{s}_{\beta 1}}(t,\mathbf{x}) + \sum_{\{s_{01},\mathbf{l}_1\}} w^{(r_0)}_{s_{01},\mathbf{l}_1}\; q^{(r_0)\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) \\
&+ \sum_{\{s_{01},s_{02},\mathbf{l}_2\}} w^{(r_0)}_{s_{01},s_{02},\mathbf{l}_2} q^{(r_0)\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) \times q^{(r_0)\mathbf{r}_{\alpha s_{02}}}_{(s_{02})\mathbf{s}_{\beta s_{02}}}(t-l_{02}a_0,\mathbf{x}_{12}+\Delta\mathbf{x}_{12}) + ..., \\
&\forall\; s_{01}, s_{02}, ... \in \{1,2\}, t \geq 0, x_1,...,x_4 \in \Lambda, \\
q^{(r_0)\mathbf{r}_{\alpha 2}}_{(2)\mathbf{s}_{\beta 2}}(t+a_0,\mathbf{x}) &= q^{(r_0)\mathbf{r}_{\alpha 2}}_{(2)\mathbf{s}_{\beta 2}}(t,\mathbf{x}) + \sum_{\{s_{02},\mathbf{l}_1\}} w^{(r_0)}_{s_{02},\mathbf{l}_1} q^{(r_0)\mathbf{r}_{\alpha s_{02}}}_{(s_{02})\mathbf{s}_{\beta s_{02}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) \\
&+ \sum_{\{s_{02},s_{01},\mathbf{l}_2\}} w^{(r_0)}_{s_{02},s_{01},\mathbf{l}_2} q^{(r_0)\mathbf{r}_{\alpha s_{02}}}_{(s_{02})\mathbf{s}_{\beta s_{02}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) \times q^{(r_0)\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{02}a_0,\mathbf{x}_{12}+\Delta\mathbf{x}_{12}) + ..., \\
&\forall\; s_{01}, s_{02}, ... \in \{1,2\}, t \geq 0, x_1,...,x_4 \in \Lambda,
\end{aligned} \tag{2}$$

where the short hand notation $\mathbf{r}_{\alpha s_0} = r_1 ... r_{\alpha_{s_0}}$, $\mathbf{s}_{\beta s_0} = s_1 ... s_{\beta_{s_0}}$ for $s_0 = 1, 2$ and

$$\mathbf{x}_{1\gamma} + \Delta\mathbf{x}_{1\gamma} = x_1 + l_{1\gamma}a_1, x_2 + l_{2\gamma}a_2, x_3 + l_{3\gamma}a_3 + l_{4\gamma}a_4,$$

for any γ, was used. To it derive the non-Markovian deterministic equations, Eq. (2) will be used as the starting set of stochastic evolution equations. The short hand notation $\mathbf{l}_1 = l_{01}, l_{11}, l_{21}, l_{31}, l_{41}$ and $\mathbf{l}_2 = l_{01}, l_{11}, l_{21}, l_{31}, l_{41}, l_{02}, l_{12}, l_{22}, l_{32}, l_{41}$ was also used to save printing. The equations amounts to $\mathcal{M} = SM$. The stochastic weights and the dynamical variables, in Eq. (2), are labeled with an index r_0 emphasizing that the value depends on a specific realization. The stochastic weights can, in general, be a complex number with a real $w^{'(r_0)}_{s_{01},...s_{0k},\mathbf{l}_k}$ and an imaginary part $w^{''(r_0)}_{s_{01},...s_{0k},\mathbf{l}_k}$, for any k, with $\mathbf{l}_k = l_{01}, l_{11}, l_{12}, l_{13}, ..., l_{0k}, l_{1k}, l_{2k}, l_{3k}$. To be more formal, an arbitrary weight can be denoted by $w^{(r_0)}_u$ where u is some set of indices $u_1, ..., u_\rho$, non-necessarily of the same type, as in Eq. (2). A general expression of a weight as a product of Kronecker deltas and Heaviside's functions can be written as

$$w^{(r_0)}_u = \prod_{\{k'\}} \delta_{i_{k'},j_{k'}} \left(\prod_{\{v'\}} \theta\left(P_{v'} - \xi^{(r_0)}_{v'}\right)\right) \theta\left(P^{'}_{c'} - \xi^{'(r_0)}_{c'}\right) + i \prod_{\{k''\}} \delta_{i_{k''},j_{k''}} \left(\prod_{\{v''\}} \theta\left(P_{v''} - \xi^{(r_0)}_{v''}\right)\right) \theta\left(P^{''}_{c''} - \xi^{''(r_0)}_{c''}\right). \tag{3}$$

where $\{k'\}$ and $\{v'\}$ are sets of indexes that are used to label discrete and continuous factors, respectively. These indexes correspond to the real part of the complex weight $w_u^{(r_0)}$. c' denote the index that connect the real part of the stochastic weight with the real part of the deterministic weight of some other approach. In the same way, $\{k''\}$, $\{v''\}$ and c'' denote the indexes corresponding to the imaginary part of $w_u^{(r_0)}$. The imaginary unit is i.

There are some key questions that allows the construction of deterministic evolution equations from an average over realizations of a stochastic evolution equation. First, the stochastic weights must be expressed as products of some delta- and theta-functions whose arguments contain discrete as well as continuous stochastic variables, respectively. The definition of these functions are: $\delta_{x,y}$ is equal to 1 if $x = y$ and 0 otherwise, and $\theta\,(x - y)$ is equal to 1 if $x - y \geq 0$ and 0 if $x - y < 0$, for any x and y. Second, all these stochastic variables (discrete and continuous) are statistically independent, allowing the factorization of the averages. Third, two of the theta-functions, corresponding to the real and imaginary part of the stochastic weights, contain in its argument the functions $P^{'}_{c'}$ and $P^{''}_{c''}$ that allows to connect the average over realizations of all the stochastic weights with the deterministic weights of any other deterministic approach (e.g. master equation, etc). For the interpretation of these functions that define the weights see the first example in Sec. 4 of [24].

The above general definition of a generic stochastic weight allows to demonstrate the following theorem.

3. A theorem connecting the average over realizations of the stochastic weights with the deterministic weights

In the general case, corresponding to an updating that depends on more than one previous time steps, the theorem and the proof can be made in an almost verbatim way, with the appropriate changes in the notation, that the one made in Ref. [25]. For the sake of completeness it is reproduced the theorem and the proof below.

Theorem. *A set of deterministic evolution equations is obtained after an average over realizations of a set of stochastic evolution equations as those given in* Eq. (2) *with stochastic coefficients of the general form of those given in* Eq. (3). *The connection with a set of deterministic evolution equation, obtained with other approach, is made after an appropriate election of the functions $P^{'}_{c'}$ and $P^{''}_{c''}$.*

Proof. The proof is obtained in two steps in a very simple way. First, using standard results of statistical mechanics (see the appendix of [24]), the general deterministic equations are obtained after average over realizations on both two sides of Eq. (2), in the following general form

$$\begin{aligned}
q^{\mathbf{r}_{\alpha 1}}_{(1)\mathbf{s}_{\beta 1}}(t+a_0,\mathbf{x}) &= q^{\mathbf{r}_{\alpha 1}}_{(1)\mathbf{s}_{\beta 1}}(t,\mathbf{x}) + \sum_{\{s_{01},\mathbf{l}_1\}} w_{s_{01},\mathbf{l}_1}\, q^{\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) \\
&\quad + \sum_{\{s_{01},s_{02},\mathbf{l}_2\}} w_{s_{01},s_{02},\mathbf{l}_2}\quad q^{\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) \\
&\qquad \times q^{\mathbf{r}_{\alpha s_{02}}}_{(s_{02})\mathbf{s}_{\beta s_{02}}}(t-l_{02}a_0,\mathbf{x}_{12}+\Delta\mathbf{x}_{12}) + ..., \\
&\qquad \forall s_{01},s_{02},...\in\{1,2\}, t\geq 0, x_1,...,x_4\in\Lambda, \\
q^{\mathbf{r}_{\alpha 2}}_{(2)\mathbf{s}_{\beta 2}}(t+a_0,\mathbf{x}) &= q^{\mathbf{r}_{\alpha 2}}_{(2)\mathbf{s}_{\beta 2}}(t,\mathbf{x}) + \sum_{\{s_{02},\mathbf{l}_1\}} w_{s_{02},\mathbf{l}_1}\quad q^{\mathbf{r}_{\alpha s_{02}}}_{(s_{02})\mathbf{s}_{\beta s_{02}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) \\
&\quad + \sum_{\{s_{02},s_{01},\mathbf{l}_2\}} w_{s_{02},s_{01},\mathbf{l}_2}\quad q^{\mathbf{r}_{\alpha s_{02}}}_{(s_{02})\mathbf{s}_{\beta s_{02}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) \\
&\qquad \times q^{\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{02}a_0,\mathbf{x}_{12}+\Delta\mathbf{x}_{12}) + ..., \\
&\qquad \forall s_{01},s_{02},...\in\{1,2\}, t\geq 0, x_1,...,x_4\in\Lambda,
\end{aligned} \tag{4}$$

where

$$w_{s_{01},\mathbf{l}_1} = \overline{w^{(r_0)}_{s_{01},\mathbf{l}_1}},\quad w_{s_{02},\mathbf{l}_1} = \overline{w^{(r_0)}_{s_{02},\mathbf{l}_1}},...$$

are the weights corresponding to the product of one, two,... dynamical variables. Note that it was used a (1,0)-closure for the product of two dynamical variables, that is the simplest closure that can be used in the infinite hierarchy of evolution equations. The deterministic weights can be written in the usual form $w_{s_{01},...,s_{0k},\mathbf{l}_k} = w^{'}_{s_{01},...,s_{0k},\mathbf{l}_k} + i\, w^{''}_{s_{01},...,s_{0k},\mathbf{l}_k}$ and $w_{s_{02},...,s_{0k},s_{01},\mathbf{l}_k} = w^{'}_{s_{02},...,s_{0k},s_{01},\mathbf{l}_k} + i\, w^{''}_{s_{02},...,s_{0k},s_{01},\mathbf{l}_k}$. Note that the factorization of the averages over realization was used because it was assumed that the discrete and continuous stochastic variables in all w's are statistically independent and also are independent of all the dynamical variables. For a demonstration that the product of two functions of complex stochastic variables factorizes, see the appendix of [24].

Second, the last step needed to obtain the connection between two approaches is to make an average over realizations on both two sides of Eq. (3). The result is

$$\begin{aligned}\overline{w_u^{(r_0)}} &= \prod_{\{k'\}} \overline{\delta_{i_{k'},j_{k'}}} \left(\prod_{\{v'\}} \overline{\theta\left(P_{v'} - \xi_{v'}^{(r)}\right)} \right) \overline{\theta\left(P'_{c'} - \xi_{c'}^{\prime(r)}\right)} \\ &\quad + i \prod_{\{k''\}} \overline{\delta_{i_{k''},j_{k''}}} \left(\prod_{\{v''\}} \overline{\theta\left(P_{v''} - \xi_{v''}^{(r)}\right)} \right) \overline{\theta\left(P''_{c''} - \xi_{c''}^{\prime\prime(r)}\right)} \\ &= \prod_{\{k'\}} \frac{1}{M_{k'}} \left(\prod_{\{v'\}} P_{v'} \right) P'_{c'} + i \prod_{\{k''\}} \frac{1}{M_{k''}} \left(\prod_{\{v''\}} P_{v''} \right) P''_{c''}, \end{aligned} \tag{5}$$

where $M_{k'}$ and $M_{k''}$ are the amount of element of the k-th discrete set. Note that it was assumed that all the intervals of variation of all the continuous stochastic variables is $[0, 1]$. If some of the intervals is different, the result of Eq. (56) in the appendix of [24] must be used. The connection with another approach is easily obtained. Equating the coefficients of the expressions of the weights $\left(\overline{w_u^{(r_0)}} = w_c\right)$, $P'_{c'}$ and $P''_{c''}$ can be found as

$$P'_{c'} = \frac{w'_c}{\prod_{\{k'\}} \frac{1}{M_{k'}} \left(\prod_{\{v'\}} P_{v'} \right)}, \tag{6}$$

$$P''_{c''} = \frac{w''_c}{\prod_{\{k''\}} \frac{1}{M_{k''}} \left(\prod_{\{v''\}} P_{v''} \right)}, \tag{7}$$

where w'_c and w''_c are the real and imaginary part of w_c, respectively. If the deterministic evolution equation is expressed as a partial differential equation as those given in the example in Sec. 4, $P'_{c'}$ and $P''_{c''}$, in Eqs. (6,7), must be multiplied by a_0 to recover the correct deterministic weights. These expressions allows to establish the complete equivalence with the deterministic weights corresponding to some other approach.

4. Illustrative example

The usual way to obtain the geodesics is by using of a variational approach (see Appendix). In this section the use of the results of the theorem will be applied to obtain the evolution equations for the geodesics.

4.1. The discrete stochastic evolution rules approach to obtain relativistic evolution equations

In order to obtain the evolution equations for the geodesics that provides the deterministic evolution obtained in the Appendix, let us to identify the two tensors that are needed in this case. The first type of dynamical variable $q_{(1)k}^{(r_0)}(t)$, is the coordinate itself $x_k^{(r_0)}(t)$ and depend solely on the evolution parameter t. The second dynamical variable is the metric $q_{(2)}^{uv}(\mathbf{x})$ or $q_{(2)uv}(\mathbf{x})$ that solely depend on the coordinates usually written as $g^{uv}(\mathbf{x})$ or $g_{uv}(\mathbf{x})$ for any u and v, but do not on t or on the realizations r_0. Note that the coordinate is a co-variant tensor of rank one and the metric is a tensor of rank two. Because of the fact that exist just one tensor that evolves stochastically just one equation is needed. Another feature to note is that in the argument of the metric the variables x_u, for any u, are the coordinate obtained after an average over realizations

$$x_u = x_u(t) = \overline{x_u^{(r_0)}(t)}.$$

With the above consideration the discrete stochastic evolution equation can be written as

$$\begin{aligned} x_k^{(r_0)}(t + a_0) &= x_k^{(r_0)}(t) + w_{1,1,0,0,0}^{(r_0)} x_k^{(r_0)}(t - a_0) - w_{1,2,0,0,0}^{(r_0)} x_k^{(r_0)}(t - 2a_0) - w_{s_{1,2}}^{(r_0)} g^{kh}(\mathbf{x}) \left(g_{hi}(x_j + a_1) - g_{hi}(x_j)\right) \\ &\times \left(x_i^{(r_0)}(t) - x_i^{(r_0)}(t - a_0)\right) \left(x_j^{(r_0)}(t) - x_j^{(r_0)}(t - a_0)\right) - w_{s_{1,2}}^{(r_0)} g^{kh}(\mathbf{x}) \left(g_{hj}(x_i + a_1) - g_{hj}(x_i)\right) \\ &\times \left(x_i^{(r_0)}(t) - x_i^{(r_0)}(t - a_0)\right) \left(x_j^{(r_0)}(t) - x_j^{(r_0)}(t - a_0)\right) - w_{s_{1,2}}^{(r_0)} \quad g^{kh}(\mathbf{x}) \left(-g_{ij}(x_h + a_1) - g_{ij}(x_h)\right) \\ &\times \left(x_i^{(r_0)}(t) - x_i^{(r_0)}(t - a_0)\right) \left(x_j^{(r_0)}(t) - x_j^{(r_0)}(t - a_0)\right), \end{aligned} \tag{8}$$

where it was used the short hand notation

$$g^{uv}(\mathbf{x}) = g^{uv}(x_1, x_2, x_3, x_4),$$

$$g_{uv}(x_k + a_1) = g_{uv}(x_1, ..., x_k + a_1, ..., x_4)$$

for any $1 < k < 4$ or $g_{uv}(x_k+a_1) = g_{uv}(x_1+a_1, ..., x_4)$ and $g_{uv}(x_4 + a_1) = g_{uv}(x_1, ..., x_4 + a_1)$ for $k = 1$ and $k = 4$, respectively. Note that Eq. (8) was not written in the usual form given in the previous sections. It was written in a compact way because all the weights w's, in the three summands below the first line, are equals and all of them are obtained after successive permutation of indices. Note that also a short hand notation $s_{1,2}$, for the set of subindexes, was used. Of course, if the last three summands were expanded the usual form is easily obtained. The weights are

$$w^{(r_0)}_{1,1,0,0,0} = \theta\left(P_{1,1,0,0,0} - \xi^{(r_0)}_{1,1,0,0,0}\right),$$

$$w^{(r_0)}_{1,2,0,0,0} = \theta\left(P_{1,2,0,0,0} - \xi^{(r_0)}_{1,2,0,0,0}\right)$$

and

$$w^{(r_0)}_{s_{1,2}} = \theta\left(P_{s_{1,2}} - \xi^{(r_0)}_{s_{1,2}}\right).$$

The next step to obtain the deterministic equation as Eq. (17) is to do an average over realizations on both two side of Eq. (8) obtaining

$$\begin{aligned}
x_k(t + a_0) &= x_k(t) \\
&+ w_{1,1,0,0,0}x_k(t - a_0) - w_{1,2,0,0,0}x_k(t - 2a_0) \\
&- w_{s_{1,2}}g^{kh}(\mathbf{x})\left(g_{hi}(x_j + a_1) - g_{hi}(x_j)\right) \\
&\times \left(x_i(t) - x_i(t - a_0)\right)\left(x_j(t) - x_j(t - a_0)\right) \\
&- w_{s_{1,2}}g^{kh}(\mathbf{x})\left(g_{hj}(x_i + a_1) - g_{hj}(x_i)\right) \\
&\times \left(x_i(t) - x_i(t - a_0)\right)\left(x_j(t) - x_j(t - a_0)\right) \\
&- w_{s_{1,2}}g^{kh}(\mathbf{x})\left(-g_{ij}(x_h + a_1) - g_{ij}(x_h)\right) \\
&\times \left(x_i(t) - x_i(t - a_0)\right)\left(x_j(t) - x_j(t - a_0)\right), \qquad (9)
\end{aligned}$$

where, as it was done in previous sections, in the dynamical variables and weights the index r_0 was dropped after the average over realizations. Also, the factorization and the (1,0)-closure was used. The averaged weights are

$$w_{1,1,0,0,0} = P_{1,1,0,0,0}, w_{1,2,0,0,0} = P_{1,2,0,0,0}$$

and $w_{s_{1,2}} = P_{s_{1,2}}$. The last steps necessary to obtain the differential equations as Eq. (17) are to do a Taylor series expansion up to $O(a_0^2)$ and $O(a_1)$ and then equating coefficients with Eq. (17). The Taylor series expansion gives

$$\begin{aligned}
&w_{1,1,0,0,0}x_k(t) - w_{1,2,0,0,0}x_k(t) - a_0 w_{1,1,0,0,0}\frac{dx_k(t)}{dt} + 2a_0 w_{1,2,0,0,0}\frac{dx_k(t)}{dt} - a_0\frac{dx_k(t)}{dt} \\
&+ \frac{1}{2}a_0^2 w_{1,1,0,0,0}\frac{d^2x_k(t)}{dt^2} - 2a_0^2 w_{1,2,0,0,0}\frac{d^2x_k(t)}{dt^2} - \frac{1}{2}a_0^2\frac{d^2x_k(t)}{dt^2} \\
&- \frac{1}{2}a_0^2 a_1 w_{s_{1,2}} g^{kh}\left(\frac{\partial g_{hi}}{\partial x_j} + \frac{\partial g_{hj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_h}\right)\frac{dx_i(t)}{dt}\frac{dx_j(t)}{dt} + O(a_0^3) + O(a_1^2) = 0. \qquad (10)
\end{aligned}$$

It is not difficult to see that the values of the weights that give the same equation as Eq. (17), except for an irrelevant factor $-2a_0^2$, are $w_{1,1,0,0,0} = 1$, $w_{1,2,0,0,0} = 1$ and $w_{s_{1,2}} = 4/a_1$. Note that in this case all the weights are real numbers. Finally, The connecting parameters P_u, for any u, are $P_{1,1,0,0,0}$=1, $P_{1,2,0,0,0}$=1 and $P_{s_{1,2}}$=$4/a_1$. Note that the "physical reason" of the election of the terms in Eq. (10), as well as in Eq. (17), is that solely terms linear in $d^2x_k(t)/dt^2$ and products of the form $(dx_i(t)/dt)(dx_j(t)/dt)$ are those that allows the principle of general covariance [27].

5. Conclusions and other possible generalizations

An extension of the general approach to the case where dynamical variables are tensors of arbitrary rank and of different type was analyzed and a theorem, previously proved for Markovian as well as Non-Markovian evolution equations, was extended. The Markovian case can be obtained as a non-Markovian case where the updating depends on the first previous time step. Another possible generalization is the construction of evolution equations of other "mathematical objects" as spinors. It is simple to see that the evolution equations looks like

$$\begin{aligned}
q^{(r_0)\mathbf{R}_{\alpha 1}}_{(1)\mathbf{S}_{\beta 1}}(t + a_0, \mathbf{x}) &= q^{(r_0)\mathbf{R}_{\alpha 1}}_{(1)\mathbf{S}_{\beta 1}}(t, \mathbf{x}) + \sum_{\{s_{01}, \mathbf{l}_1\}} w^{(r_0)}_{s_{01}, \mathbf{l}_1} q^{(r_0)\mathbf{R}_{\alpha s_{01}}}_{(s_{01})\mathbf{S}_{\beta S_{01}}}(t - l_{01}a_0, \mathbf{x}_{11} + \Delta\mathbf{x}_{11}) \\
&+ \sum_{\{s_{01}, s_{02}, \mathbf{l}_2\}} w^{(r_0)}_{s_{01}, s_{02}, \mathbf{l}_2} q^{(r_0)\mathbf{R}_{\alpha s_{01}}}_{(s_{01})\mathbf{S}_{\beta s_{01}}}(t - l_{01}a_0, \mathbf{x}_{11} + \Delta\mathbf{x}_{11}) q^{(r_0)\mathbf{R}_{\alpha s_{02}}}_{(s_{02})\mathbf{S}_{\beta s_{02}}}(t - l_{02}a_0, \mathbf{x}_{12} + \Delta\mathbf{x}_{12}) + ..., \\
&\forall s_{01}, s_{02}, ... \in \{1, 2\}, t \geq 0, x_1, ..., x_4 \in \Lambda,
\end{aligned}$$

$$q^{(r_0)\mathbf{R}_{\alpha 2}}_{(2)\mathbf{S}_{\beta 2}}(t+a_0,\mathbf{x}) = q^{(r_0)\mathbf{R}_{\alpha 2}}_{(2)\mathbf{S}_{\beta 2}}(t,\mathbf{x}) + \sum_{\{s_{02},\mathbf{l}_1\}} w^{(r_0)}_{s_{02},\mathbf{l}_1} q^{(r_0)\mathbf{R}_{\alpha s_{02}}}_{(s_{02})\mathbf{S}_{\beta s_{02}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11})$$

$$+ \sum_{\{s_{02},s_{01},\mathbf{l}_2\}} w^{(r_0)}_{s_{02},s_{01},\mathbf{l}_2} q^{(r_0)\mathbf{R}_{\alpha s_{02}}}_{(s_{02})\mathbf{S}_{\beta s_{02}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) q^{(r_0)\mathbf{R}_{\alpha s_{01}}}_{(s_{01})\mathbf{S}_{\beta s_{01}}}(t-l_{02}a_0,\mathbf{x}_{12}+\Delta\mathbf{x}_{12}) + ...,$$

$$\forall s_{01}, s_{02}, ... \in \{1,2\}, t \geq 0, x_1, ..., x_4 \in \Lambda, \qquad (11)$$

where $\mathbf{R}$ and $\mathbf{S}$ are sets of spinor indices. Note that it was used a slightly different notation of that used, for example, in Ref. [28]. Even for the case of dynamical variables that posses a mix of tensor and spinor indices, the evolution equations looks like

$$q^{(r_0)\mathbf{r}_{\alpha 1}\mathbf{R}_{\alpha 1}}_{(1)\mathbf{s}_{\beta 1}\mathbf{S}_{\beta 1}}(t+a_0,\mathbf{x}) = q^{(r_0)\mathbf{r}_{\alpha 1}\mathbf{R}_{\alpha 1}}_{(1)\mathbf{s}_{\beta 1}\mathbf{S}_{\beta 1}}(t,\mathbf{x}) + \sum_{\{s_{01},\mathbf{l}_1\}} w^{(r_0)}_{s_{01},\mathbf{l}_1} q^{(r_0)\mathbf{r}_{\alpha s_{01}}\mathbf{R}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta S_{01}}\mathbf{S}_{\beta S_{01}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11})$$

$$+ \sum_{\{s_{01},s_{02},\mathbf{l}_2\}} w^{(r_0)}_{s_{01},s_{02},\mathbf{l}_2} q^{(r_0)\mathbf{r}_{\alpha s_{01}}\mathbf{R}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}\mathbf{S}_{\beta s_{01}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) q^{(r_0)\mathbf{r}_{\alpha s_{02}}\mathbf{R}_{\alpha s_{02}}}_{(s_{02})\mathbf{s}_{\beta s_{02}}\mathbf{S}_{\beta s_{02}}}(t-l_{02}a_0,\mathbf{x}_{12}+\Delta\mathbf{x}_{12}) + ...,$$

$$\forall\, s_{01}, s_{02}, ... \in \{1,2\}, t \geq 0, x_1, ..., x_4 \in \Lambda,$$

$$q^{(r_0)\mathbf{r}_{\alpha 2}\mathbf{R}_{\alpha 2}}_{(2)\mathbf{s}_{\beta 2}\mathbf{S}_{\beta 2}}(t+a_0,\mathbf{x}) = q^{(r_0)\mathbf{r}_{\alpha 2}\mathbf{R}_{\alpha 2}}_{(2)\mathbf{s}_{\beta 2}\mathbf{S}_{\beta 2}}(t,\mathbf{x}) + \sum_{\{s_{02},\mathbf{l}_1\}} w^{(r_0)}_{s_{02},\mathbf{l}_1} q^{(r_0)\mathbf{r}_{\alpha s_{02}}\mathbf{R}_{\alpha s_{02}}}_{(s_{02})\mathbf{s}_{\beta s_{02}}\mathbf{S}_{\beta s_{02}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11})$$

$$+ \sum_{\{s_{02},s_{01},\mathbf{l}_2\}} w^{(r_0)}_{s_{02},s_{01},\mathbf{l}_2} q^{(r_0)\mathbf{r}_{\alpha s_{02}}\mathbf{R}_{\alpha s_{02}}}_{(s_{02})\mathbf{s}_{\beta s_{02}}\mathbf{S}_{\beta s_{02}}}(t-l_{01}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11}) q^{(r_0)\mathbf{r}_{\alpha s_{01}}\mathbf{R}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}\mathbf{S}_{\beta s_{01}}}(t-l_{02}a_0,\mathbf{x}_{12}+\Delta\mathbf{x}_{12}) + ...,$$

$$\forall\, s_{01}, s_{02}, ... \in \{1,2\}, t \geq 0, x_1, ..., x_4 \in \Lambda, \qquad (12)$$

where $\mathbf{r}$ and $\mathbf{s}$ designates the sets of tensor indices and $\mathbf{R}$ and $\mathbf{S}$ the spinor ones. More generally, the theorem is also valid for evolution equations containing whatever set of labels. The reason is that the factorization and the use of a (1,0)-closure sill is possible and consequently deterministic evolution equations can also be obtained. Examples of two-components (spinors) or four-components (four-vectors) "mathematical objects" with a slightly different notation can be found in the second example in Refs. [24] and [26], respectively.

This paper must be understood, presently, as part of four companion papers (the other three are given in Refs. [24-26]) each of them showing different features and were planned in such a way that the illustrative examples show an increasing technical difficulties. In the present case the evolution equations of tensors was applied to obtain the deterministic evolution equations for the geodesics.

Finally, it must be emphasized that another extension of the present theorem is possible, to include evolution of functions of the dynamical variables.Another possibility (and perhaps the most important to be used in physical problems) is the evolution equation for the Lagrangian. Of course in this case the evolution equations to be obtained are the Euler-Lagrange equations. This extension will be submitted elsewhere [29].

Appendix

For the sake of completeness, the geodesics are obtained using the usual variational approach. The length L of a curve from A to B can be obtained by

$$L = \int_A^B ds = \int_{t_2}^{t_1} \sqrt{g_{ij}x'_i x'_j}\, dt, \qquad (13)$$

where s is the arc, $g_{ij} = g_{ij}(x_1, x_2, x_3, x_4)$ is the metric tensor, $x_i = x_i(t)$ is the space-time coordinate and $x'_i = dx_i/dt$. After performing the first variation of L the corresponding Euler equations are

$$\frac{\partial F}{dx_i} - \frac{d}{dt}\frac{\partial F}{dx'_i} = 0 \qquad i = 1, ..., 4, \qquad (14)$$

where $F = \sqrt{g_{ij}x'_i x'_j}$. After introducing F in Eq. (14) it can be obtained

$$\frac{1}{2F} g_{ij}x'_i x'_j - \frac{d}{dt}\left(\frac{g_{ih}x'_i + g_{hj}x'_j}{2F}\right) = 0$$

$$i = 1, ..., 4. \qquad (15)$$

Because of the fact that Eq. (15) is valid for any parameter t, it is possible to make $t = s$ then $F = 1$ and

$$\frac{1}{2} g_{ij,h} x'_i x'_j - \frac{1}{2} \left(g_{ih} x''_i + g_{hj} x''_j \right) - \frac{1}{2} \left(g_{ih,j} + g_{hj,i} \right) x'_i x'_j = 0 \qquad i = 1, ..., 4, \tag{16}$$

where $g_{uv,w} = \partial g_{uv} / \partial x_w$ for any u, v and w. Because $g_{hj} = g_{jh}$ then $g_{ih} x''_i = g_{hj} x''_j$ and after multiplying by g^{hk} and summing over h it is obtained the differential equations for the geodesics as

$$x''_k + \Gamma^k_{ij} x'_i x'_j = 0 \qquad k = 1, ..., 4, \tag{17}$$

where the Christoffel symbol Γ^k_{ij} is

$$\Gamma^k_{ij} = \frac{1}{2} g^{hk} \left(g_{jh,i} + g_{hi,j} - g_{ij,h} \right). \tag{18}$$

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