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Bound state solutions of deformed generalized Deng-Fan potential plus deformed Eckart potential in D-dimensions

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In this article,we present the approximate solution of the D-dimensional Schrödinger equation for deformed generalized Deng-Fan plus deformed Eckart potential using parametric Nikiforov-Uvarov method. We obtain the bound state energy eigenvalues and the corresponding wave function for arbitrary l state. Special cases of this potential are also discussed.

Keywords: Deformed generalized Deng-Fan potential; deformed Eckart potential; Nikiforov-Uvarov; D-dimensions.

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1. Introduction

The solutions of the Schrödinger equation (SE) have been extensively studied in recent times. These solutions play a vital role in quantum mechanics since they contain all necessary information governing the quantum mechanical system under consideration. However, apart from the S-wave only few potentials are exactly solvable for $l \neq 0$ such as harmonic potential [1], Coulomb potential [2] and others [3]. Most of the potentials for the description of physical systems are not exactly solvable for $l \neq 0$. Nonetheless, many authors have applied different approximation to the centrifugal term and obtained analytical approximation to the l-wave solutions of the Schrödinger equation with some exponential-like type potential [4-10]. These potentials include the Morse potential [11] Manning-Rosen, Scarf Poschl-Teller and Rosen Morse potentials [12,13]. Various methods have been used to obtain the exact or approximate solutions of the Schrödinger equation for some exponential-type potential. These methods include the Nikiforov-Uvarov method NU [14,15], factorization method [16,17], asymptotic iteration method [18,19] and others [20]. Recently, many researchers have developed interest in D-dimensional solutions [21-25,43,44].

In the present work, we attempt to investigate the deformed generalized Deng-Fan potential plus deformed Eckart potential. Following the work of other authors [21-25,43] , we solved the SE for this potential in D-dimensions using parametric Nikiforov-Uvarov method. The potential under investigation is

$$V(r) = V_0\left(c - \frac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right)^2 - \frac{V_1 e^{-\alpha r}}{1-qe^{-\alpha r}} + \frac{V_0 e^{-\alpha r}}{(1-qe^{-\alpha r})^2} \tag{1}$$

V_0, V_1, V_2 are potential depths, q is the deformation parameter which will take values of 1 and -1, b and c are adjustable constant, α is the range of the potential. If we set $q = c = 1$, $V_1 = V_2 = 0$, the potential reduces to that of the Deng-Fan potential [26,33] with $b = e^{\alpha r_c} - 1$ where r_c is the equilibrium internuclear distance and with $V_0 = 0$ the potential becomes Eckart potential [27-29]. We display the behaviour of this potential with for $q = \pm 1$ in Fig. 1-2.

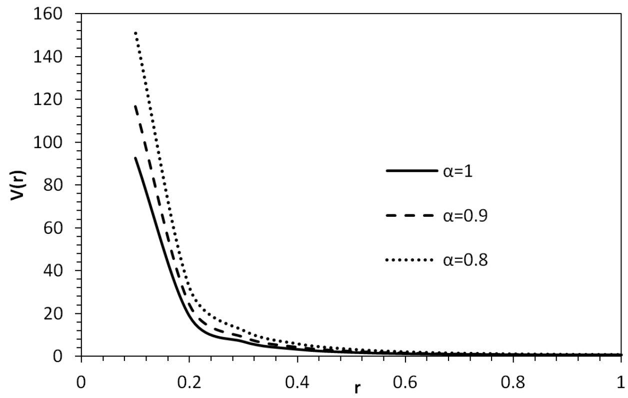

FIGURE 1. Variation of the potential as a function of r for $V_0 = 4$ Mev, $V_1 = 0.1$ Mev, $V_2 = 0.5$ Mev $q = 1$, $c = 0.5$, $b = 0.4$ and various values of $\alpha = 1.0$, 0.9 and 0.8 fm^{-1}.

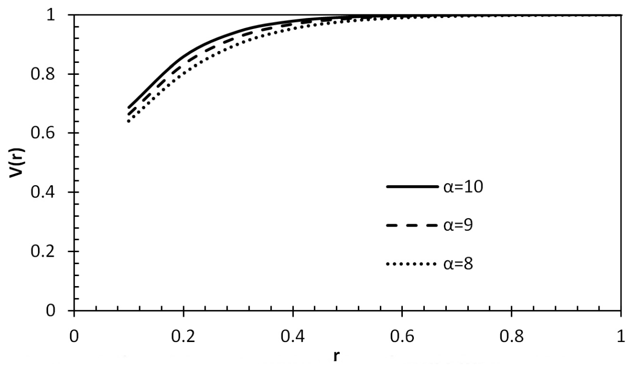

FIGURE 2. Variation of the potential as a function of r for $V_0 = 4$ Mev, $V_1 = 0.1$ Mev, $V_2 = 0.5$ Mev $q = -1$, $c = 0.5$, $b = 0.4$ and various values of $\alpha = 10$, 9 and 8 fm^{-1}.

Three special cases can be obtained from this potential as follows:
(i) Setting $c=0$, $q=-1$, $V_2=0$, $b=1$ Eq. (1) reduces to

$$V_{GWS}=\frac{-V_1e^{-\alpha r}}{1+e^{-\alpha r}}+\frac{V_0e^{-\alpha r}}{(1+e^{-\alpha r})^2} \tag{2}$$

which is known as generalized Woods-Saxon potential [30,31] which is used to describe interaction between nuclei [32]. For $V_0=0$ we obtain the Woods-Saxon potential as

$$V_{WS}(r)=\frac{-V_1e^{-\alpha r}}{1+e^{-\alpha r}} \tag{3}$$

(ii) Setting $V_0=c=V_2=0$ and $q=1$ Eq. (1) reduces to

$$V_H(r)=\frac{-V_1e^{-\alpha r}}{1-e^{-\alpha r}} \tag{4}$$

which is known as Hulthen potential [9,34]
(iii) If and $q=1$ Eq. (1) reduces to

$$V_{MR}(r)=b^2V_0\left(\frac{e^{-\alpha r}}{1-e^{-\alpha r}}\right)^2 \tag{5}$$

which is a known form of Manning-Rosen potential for A=0 [35].

In this work, the centrifugal term will be approximated as [36]

$$\frac{1}{r^2}\approx\frac{\omega e^{-\alpha r}}{1-qe^{-\alpha r}}+\frac{\lambda e^{-\alpha r}}{(1-qe^{-\alpha r})^2} \tag{6}$$

ω and λ are adjustable dimensionless parameter. This approximation scheme is valid for large and small α. To show that Eq. (6) is a good approximation scheme we compare $1/r^2$ and the approximation scheme with different values of αin Figs. 3-4 for $q=1,\ -1$ respectively.

The organization of the paper is as follows: In Sec. 2, we give a brief review of the Nikiforov-Uvarov method in its parametric form. In section 3, we highlight the SE in D-space. In Sec. 4, we present the bound state solution of the SE in D-dimension. In Sec. 5 we give a brief discussion. Finally a brief conclusion is given in Sec. 6.

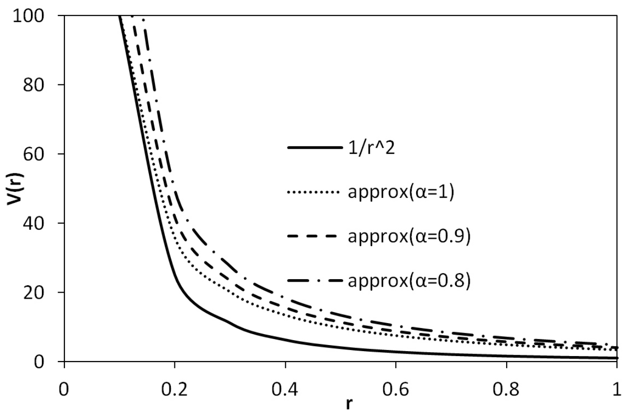

FIGURE 3. Comparison between $1/r^2$ and the approximation scheme as functions of r for $\omega=5.00$, $\lambda=0.53$, $q=1$ and various values of $\alpha=1.0$, 0.9 and 0.8 fm^{-1}.

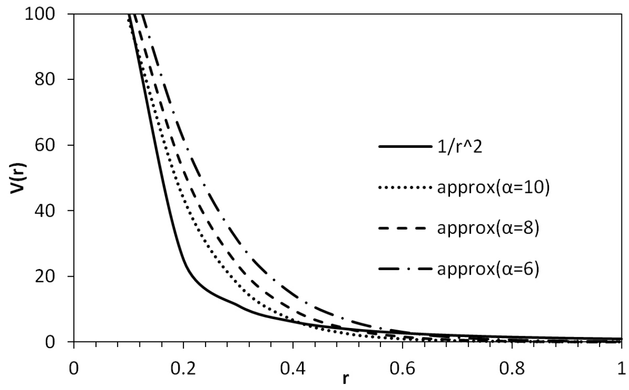

FIGURE 4. Comparison between $1/r^2$ and the approximation scheme as functions of r for $\omega=5.00$, $\lambda=0.53$, $q=-1$ and various values of $\alpha=10$, 9 and 8 fm^{-1}.

2. Review of Nikiforov-Uvarov method and its parametric form

The Nikiforov-Uvarov method (NU) is based on the solution of a generalized second order linear differential equation with special orthogonal function [14,37]. The SE

$$\psi''(r)+[E-V(r)]\,\psi(r)=0 \tag{7}$$

can be solved by the NU method by transforming this equation into hypergeometric-type using the transformation s=$s(x)$

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) \tag{8}$$

In order to find the solution to Eq. (8) we set the wave function as

$$\psi(s)=\varphi(s)\chi_n(s) \tag{9}$$

Substituting Eq. (9) into Eq. (8) reduces Eq. (2) to

$$\sigma(s)\chi_n''(s)+\tau(s)\chi_n'(s)+\lambda\chi_n(s)=0 \tag{10}$$

where the wave function $\varphi(s)$ is a logarithmic function

$$\frac{\varphi'(s)}{\varphi(s)}=\frac{\pi(s)}{\sigma(s)} \tag{11}$$

where $\chi_n(s)$ is the hypergeometric-type function which satisfies the Rodrigue relation

$$\chi_n(s)=\frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right] \tag{12}$$

and B_n is the normalization constant and the weight function $\rho(s)$ satisfies the condition

$$(\sigma(s)\rho(s))'=\tau(s)\rho(s) \tag{13}$$

The required $\pi(s)$ and λ for the NU method are defined as

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right) - \tilde{\sigma}(s) + k\sigma(s)} \quad (14)$$

and

$$\lambda = k + \pi'(s) \quad (15)$$

respectively. It is necessary that the term under the square root sign in Eq. (13) be the square of a polynomial. The eigenvalues in Eq. (15) take the form

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \quad n = 0, 1, 2 \ldots \quad (16)$$

where

$$\tau(s) = \widetilde{\tau}(s) + 2\pi(s) \quad (17)$$

The derivative of Eq. (16) is less than zero for bound state. The energies are obtained by comparing Eqs. (15) and (16).

The parametric generalization of the NU method that is valid for both central and non-central exponential potential [38] can be derived by comparing the generalized hypergeometric-type equation.

$$\psi''(s) + \frac{(c_1 - c_2 s)}{s(1 - c_3 s)}\psi(s) + \frac{1}{s^2(1 - c_3 s)^2}\left[-\xi_1 s^2 + \xi_2 s - \xi_3\right]\psi(s) = 0 \quad (18)$$

Comparing Eq. (8) and Eq. (18) we obtain the following parametric polynomials

$$\widetilde{\tau}(s) = (c_1 - c_2 s) \quad (19)$$

$$\widetilde{\sigma} = -\xi_1 s^2 + \xi_2 s - \xi_3 \quad (20)$$

$$\sigma(s) = s(1 - c_3 s) \quad (21)$$

Substituting Eq. (19), Eq. (20) and Eq. (21) into Eq. (14) we obtain,

$$\pi(s) = c_4 - c_5 s \pm \sqrt{[(c_6 - c_3 k_\pm) s^2 + (c_7 + k_\pm) s + c_8]} \quad (22)$$

where

$$c_4 = \frac{1}{2}(1 - c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3),$$

$$c_6 = c_5^2 + \xi_1, \quad (23)$$

$$c_7 = 2c_4 c_5 - \xi_2, \quad c_8 = c_4^2 + \xi_3 \quad (24)$$

Also

$$k_\pm = -(c_7 + 2c_3 c_8) \pm 2\sqrt{c_8 c_9} \quad (25)$$

where

$$c_9 = c_3 c_7 + c_3^2 c_8 + c_6 \quad (26)$$

Hence, the function $\pi(s)$ becomes

$$\pi(s) = c_4 + c_5 s - \left[(\sqrt{c_9} + c_3\sqrt{c_8})\, s - \sqrt{(c_8)}\right] \quad (27)$$

From the relation in Eq. (17) we have

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)\, s - 2\left[(\sqrt{c_9} + c_3\sqrt{c_8})\, s - \sqrt{c_8}\right] \quad (28)$$

and

$$\tau'(s) = -2c_5 - 2(\sqrt{c_9} - c_3\sqrt{c_8}) \quad (29)$$

Solving Eqs. (15) and (16), we obtain the parametric energy equation as

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)\, c_5 + (2n+1) \times \left[\sqrt{c_9} + c_3\sqrt{c_8}\right] + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \quad (30)$$

The weight function is obtained as

$$\rho(s) = s^{c_{10}}(1 - c_3 s)^{c_{11}} \quad (31)$$

And together with Eq. (12) we obtain

$$\chi_n(s) = P_n^{(c_{10}, c_{11})}(1 - 2c_3 s) \quad (32)$$

where $P_n^{(c_{10},c_{11})}$ are the Jacobi polynomials and the superscripts c_{10} and c_{11} are given by

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} \quad (33)$$

$$c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} \quad (34)$$

The other part of the wave function is obtained from Eq. (11) as

$$\varphi(s) = s^{c_{12}}(1 - c_3 s)^{c_{13}} \quad (35)$$

where

$$c_{12} = c_4 + \sqrt{c_8} \quad (36)$$

$$c_{13} = -c_4 + \frac{1}{c_3}(\sqrt{c_9} - c_5) \quad (37)$$

Thus the total wave function becomes

$$\psi_n(s) = N_n S^{c_{12}}(1 - c_3 s)^{c_{13}} P_n^{(c_{10}, c_{11})}(1 - 2c_3 s) \quad (38)$$

where N_n is the normalization constant.

3. Schrödinger Equation in D-dimension

The D-dimension space SE is [39]

$$\frac{\hbar^2}{2\mu}\left[\nabla_D^2+V(r)\right]\Psi_{nlm}\left(r,\Omega_D\right)=E_{\rm nl}\Psi_{\rm nlm} \quad (39)$$

where

$$\nabla_D^2=\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right)-\frac{\Lambda_D^2}{r^2}\left(\Omega_D\right) \quad (40)$$

is the Laplacian operator. The second term of Eq. (40) is the multidimensional space centrifugal term. Ω_D represent the angular coordinates. The operator Λ_D^2 yields hyperspherical harmonic as its eigenfunction. With this we write the wave function as

$$\Psi_{\rm nlm}\left(r,\Omega_D\right)=R_{\rm nl}(r)Y_l^m\left(\Omega_D\right) \quad (41)$$

$R_{\rm nl}$ is the radial part of the equation and Y_l^m is the angular part called hyper-spherical harmonics. The $Y_l^m\left(\Omega_D\right)$ obey the eigenvalue equation

$$\Lambda_D^2Y_l^m\left(\Omega_D\right)=l\left(l+D-2\right)Y_l^m\left(\Omega_D\right) \quad (42)$$

Substituting Eqs. (40) and (41) into Eq. (39) we obtain the radial equation as

$$\frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial R(r)}{\partial r}\right)+\frac{2\mu}{\hbar^2}\left[E_{nl}-\frac{l+D-2}{r^2}\right]R(r)=0 \quad (43)$$

4. Bound State solution of Schrödinger equation

In order to obtain state solution we have to make the transformation

$$R(r)=r^{-\left(\frac{D-1}{2}\right)}U(r) \quad (44)$$

Substituting Eqs. (1), (6) and (44) into Eq. (43), we derive the bound state SE for $l\neq 0$ for the potential under consideration as

$$U''(r)+\left\{\begin{array}{l}\frac{2\mu E}{\hbar^2}-\frac{2\mu}{\hbar^2}\left[V_0\left(c-\frac{be^{-\alpha r}}{1-qe^{-\alpha r}}\right)^2-\frac{V_1e^{-\alpha r}}{1-qe^{-\alpha r}}+\frac{V_0e^{-\alpha r}}{(1-qe^{-\alpha r})^2}\right]\\ -\left[\left(\frac{(D-1)(D-3)}{4}+l(l+D-2)\right)\left(\frac{\omega e^{-\alpha r}}{1-qe^{-\alpha r}}+\frac{\lambda e^{-\alpha r}}{(1-qe^{-\alpha r})^2}\right)\right]\end{array}\right\}U(r)=0 \quad (45)$$

Using the transformation $s=e^{-\alpha r}$ in Eq. (45) we obtain

$$\frac{d^2U}{ds^2}+\frac{1}{s}\frac{dU}{ds}+\frac{1}{s^2}\left[-\varepsilon^2+\frac{\gamma s}{(1-qs)}-\frac{\beta s}{(1-qs)^2}-\frac{\phi s^2}{(1-qs)^2}\right]U=0 \quad (46)$$

This can further be simplified to hypergeometric-type equation as

$$\frac{d^2U}{ds^2}+\frac{(1-qs)}{s(1-qs)}\frac{1}{s^2(1-qs)^2}[-(q^2\varepsilon^2+q\gamma+\phi)s^2+(2q\varepsilon^2+\gamma-\beta)s-\varepsilon^2]U=0 \quad (47)$$

where

$$-\varepsilon^2=\frac{2\mu}{\hbar^2\alpha^2}\left(E-c^2V_0\right) \quad (48)$$

$$\gamma=\frac{2\mu}{\hbar^2\alpha^2}\left[2bcV_0+V_1-\frac{\omega\hbar^2}{2\mu}\left(\frac{(D-1)(D-3)}{4}+l(l+D-2)\right)\right] \quad (49)$$

$$\beta=\frac{2\mu}{\hbar^2\alpha^2}\left[V_2+\frac{\lambda\hbar^2}{2\mu}\left(\frac{(D-1)\left(D-3\right)}{4}+l\left(l+D-2\right)\right)\right] \quad (50)$$

$$\phi=\frac{2\mu b^2V_0}{\hbar^2\alpha^2} \quad (51)$$

Comparing Eq. (47) and Eq. (18), we obtain the following

$$c_1=1,\quad c_2=c_3=q,\quad \xi_1=q^2\varepsilon^2+q\gamma+\phi,\quad \xi_2=2q\varepsilon^2+\gamma-\beta,\xi_3=\varepsilon^2,\quad c_4=0,\quad c_5=-\frac{q}{2},$$

$$c_6=q^2\varepsilon^2+q\gamma+\phi+\frac{q^2}{4},\quad c_7=-2q\varepsilon^2+\gamma-\beta,\quad c_8=\varepsilon^2,\quad c_9=\phi+q\beta+\frac{q^2}{4},$$

$$c_{10}=1+2\varepsilon,\quad c_{11}=\sqrt{1+\frac{4}{q^2}\left(\phi+q\beta\right)},\quad c_{12}=\varepsilon\quad \text{and}\quad c_{13}=\frac{1}{2}\left[1+\sqrt{1+\frac{4}{q^2}\left(\phi+q\beta\right)}\right]$$

From Eq. (30), we obtain

$$\varepsilon_{nlD}^{q}=\frac{\gamma+\frac{\phi}{q}}{2q\left(n+\sigma_q\right)}-\frac{n+\sigma_q}{2} \tag{52}$$

Using Eq. (48) we obtain the energy eignevlaues as

$$E_{nlD}^{q}=-\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{\gamma+\frac{1}{q}\phi}{2q\left(n+\sigma_q\right)}-\frac{n+\sigma_q}{2}\right]^2+c^2V_0 \tag{53}$$

where

$$\sigma_q=\frac{1}{2}\left[1+\sqrt{1+\frac{4}{q^2}\left(\phi+q\beta\right)}\right] \tag{54}$$

From Eq. (35) the first part of the wave function is obtained as

$$\varphi(s)=s^{\varepsilon}\left(1-qs\right)^{\frac{1}{2}\left[1+\sqrt{1+\frac{4}{q^2}\left(\phi+q\beta\right)}\right]} \tag{55}$$

and the weight function from Eq. (31) as

$$\rho(s)=s^{1+2\varepsilon}\left(1-qs\right)^{\sqrt{1+\frac{4}{q^2}\left(\phi+q\beta\right)}} \tag{56}$$

We also obtain the second part of the wave function as

$$\chi_n(s)=P_n^{\left(1+2\varepsilon,\sqrt{1+\frac{4}{q^2}\left(\phi+q\beta\right)}\right)}\left(1-2qs\right) \tag{57}$$

Thus the total wave function from Eq. (38) is

$$\begin{aligned}U_{nlD}^{q}(r)&=N_{nlD}e^{-\varepsilon\alpha r}\\&\times\left(1-qe^{-\alpha r}\right)^{\frac{1}{2}\left[1+\sqrt{1+\frac{4}{q^2}\left(\phi+q\beta\right)}\right]}\\&\times P_n^{\left(1+2\varepsilon,\sqrt{1+\frac{4}{q^2}\left(\phi+q\beta\right)}\right)}\left(1-2qe^{-\alpha r}\right)\end{aligned} \tag{58}$$

The Jacobi polynomials are reported in several literatures [40-42], where is the normalization constant and is calculated as

$$N_{nlD}^{q}=\left[\frac{q^{2\varepsilon}(1+2\varepsilon)\left(2\varepsilon+\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}+2n+2\right)n!\Gamma\left(2\varepsilon+\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}+n+2\right)}{2\left(\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}+2n+1\right)\Gamma(2\varepsilon+n+2)\Gamma\left(\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}+n+1\right)}\right]^{\frac{1}{2}} \tag{59}$$

We have used the normalization condition alongside Eq. ET II 285(5,9) and Eq. ET II 285(5,9) of Ref. 42 to obtain Eq. (59). The derivation of Eq. (59) is given in the appendix

5. Special cases

We now take a look at the special cases discussed in Sec. 1. This is done by making adjustment to the constants in the potential.

(i) Generalized Woods-Saxon potential: If we set V_2=c=0, $\omega=\alpha^2$, $q=-1$, $D=3$. Our potential in Eq. (1) reduces to the generalized Woods-Saxon potential in Eq. (2) and the energy in Eq. (53) becomes

$$\begin{aligned}E_{nl}^{GWS}=-\frac{\hbar^2\alpha^2}{2\mu}\Bigg\{&\frac{n+Q}{2}\\&-\left[\frac{\frac{2\mu}{\hbar^2\alpha^2}\left(b^2V_0-V_1\right)+l\left(l+1\right)}{2\left(n+Q\right)}\right]\Bigg\}^2\end{aligned} \tag{60}$$

where

$$Q=\frac{1}{2}\left[1+\sqrt{1+4\left(\frac{2\mu b^2V_0}{\hbar^2\alpha^2}-\frac{\lambda}{\alpha^2}l(l+1)\right)}\right] \tag{61}$$

and the wave function in Eq. (58) becomes

$$\begin{aligned}U_{nl}^{GWS}&=B_{nl}e^{-\zeta\alpha r}\left(1+e^{-\alpha r}\right)^{\frac{1}{2}\left[1+\sqrt{1+4(\phi-\beta)}\right]}\\&\times P_n^{\left(1+2\zeta,\sqrt{1+4(\phi-\beta)}\right)}\left(1+2e^{-\alpha r}\right)\end{aligned} \tag{62}$$

where

$$\zeta=\frac{n+Q}{2}-\left[\frac{\frac{2\mu}{\hbar^2\alpha^2}\left(b^2V_0-V_1\right)+l\left(l+1\right)}{2\left(n+Q\right)}\right] \tag{63}$$

B_{nl} is the normalization constant in the GWS model.

If we proceed by setting $\omega=V_0=0$ and $\lambda=-\alpha^2$, the potential in Eq. (5) reduces to Woods-Saxon potential in Eq. (3) and energy becomes

$$E_{nl}^{WS}=-\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{n+l+1}{2}-\frac{\mu V_1}{\hbar^2\alpha^2\left(n+l+1\right)}\right]^2 \tag{64}$$

and the wave function is

$$\begin{aligned}U_{nl}^{WS}&=C_{nl}e^{-\delta\alpha r}\left(1+e^{-\alpha r}\right)^{-(l+1)}\\&\times P_n^{(1+2\delta,2l+1)}\left(1+2e^{-\alpha r}\right)\end{aligned} \tag{65}$$

where

$$\delta=\frac{n+l+1}{2}-\frac{\mu V_1}{\hbar^2\alpha^2\left(n+l+1\right)} \tag{66}$$

(ii) Hulthen potential: If we make the choice $\omega=V_0=V_2=c=0$, $\lambda=\alpha^2$, $q=1$ and $D=3$ the potential reduces to Hulthen potential given in Eq. (4) and the energy eigenvalues reduces to

$$E_{nl}^{H}=-\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{2\mu V_1}{\hbar^2\alpha^2\left(n+l+1\right)}-\frac{n+l+1}{2}\right]^2 \tag{67}$$

This consistent with result obtained in [9,36]. The wave function for the Hulthen potential is obtained from (58) as

$$U_{nl}^{H} = D_{nl} e^{-\nu\alpha r}\left(1-e^{-\alpha r}\right)^{l+1} \times P_n^{(1+2\nu,2l+1)}\left(1-2e^{-\alpha r}\right) \tag{68}$$

where

$$\nu = \frac{\mu V_1}{\hbar^2\alpha^2 (n+l+1)} - \frac{n+l+1}{2} \tag{69}$$

(iii) Manning-Rosen potential: Setting $\omega = V_0 = V_2 = c = 0$, $\lambda = \alpha^2$, $D = 3$ and $q = 1$ the potential reduces to the form of Manning-Rosen potential given in Eq. (5) whereas the energy becomes

$$E_{nl}^{MR} = -\frac{\hbar^2\alpha^2}{2\mu}\left[\frac{\mu b^2 V_1}{\hbar^2\alpha^2 (n+\rho)} - \frac{(n+\rho)}{2}\right]^2 \tag{70}$$

where

$$\rho = \frac{1}{2}\left[1+\sqrt{(2l+1)^2 + \frac{8\mu b^2 V_0}{\hbar^2\alpha^2}}\right] \tag{71}$$

and the wave function is

$$U_{nl}^{MR} = F_{nl} e^{-\eta\alpha r}\left(1-e^{-\alpha r}\right)^{\frac{1}{2}\left[1+\sqrt{(2l+1)^2+\frac{8\mu b^2 V_0}{\hbar^2\alpha^2}}\right]} \times P_n^{\left(1+2\eta,\sqrt{(2l+1)^2+\frac{8\mu b^2 V_0}{\hbar^2\alpha^2}}\right)}\left(1-2e^{-\alpha r}\right) \tag{72}$$

where

$$\eta = \frac{\mu b^2 V_1}{\hbar^2\alpha^2 (n+\rho)} - \frac{(n+\rho)}{2} \tag{73}$$

The normalization constants B_{nl}, C_{nl}, D_{nl} and F_{nl} can be obtained from Eq. (59) by making the necessary substitutions.

6. Conclusion

In this work, we study the Deformed generalized Deng-Fan plus Eckart potential for non-vanishing angular momentum in D-dimensions with the aid of an approximation scheme using parametric Nikiforov-Uvarov method. The eigenvlaues of this potential and that of its special cases – Woods-Saxon, Hulthen and Manning-Rosen potentials, are obtained and the wave functions of each potential are expressed in terms of the Jacobi polynomials.

APPENDIX

Derivation of the Normalization Constant (Eq. (59))

Let $y = 1-2qe^{-\alpha r}$, therefore Eq. (58) becomes

$$U_{nlD}^{q} = N_{nlD}\left(\frac{1-y}{2q}\right)^{\varepsilon}\left(\frac{1+y}{2}\right)^{\frac{1}{2}\left[1+\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}\right]} \times p_n^{\left(1+2\varepsilon,\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}\right)}(y) \tag{A.1}$$

Applying the normalization condition to Eq. (A.1) we obtain

$$N_{nlD}^2\left(\frac{1}{2q}\right)^{2\varepsilon}\left(\frac{1}{2}\right)^{\left[1+\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}\right]} \times \int_{-1}^{1}(1-y)^{2\varepsilon}(1+y)^{\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}}(1+y) \times \left[p_n^{\left(1+2\varepsilon,\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}\right)}(y)\right]^2 dy = 1 \tag{A.2}$$

We have used the interval of the Jacobi polynomials, [1,-1], in the integral. Writing $(1-y)$ as $[2-(1-y)]$ and expanding Eq. (A.2) we obtain,

$$N_{nlD}^2\left(\frac{1}{q}\right)^{2\varepsilon}\left(\frac{1}{2}\right)^{\left[1+2\varepsilon+\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}\right]}\left\{2\int_{-1}^{1}(1-y)^{2\varepsilon}(1+y)^{\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}}\left[P_n^{\left(1+2\varepsilon,\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}\right)}(y)\right]^2 dy - \int_{-1}^{1}(1-y)^{1+2\varepsilon}(1+y)^{1+\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}}\left[P_n^{\left(1+2\varepsilon,\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}\right)}(y)\right]^2 dy\right\} = 1 \tag{A.3}$$

We now use the standard orthogonal integrals of Jacobi polynomials as follows

$$\int_{-1}^{1}(1-y)^{\alpha-1}(1+y)^{\beta}\left[P_n^{(\alpha,\beta)}(y)\right]^2 dy = \frac{2^{\alpha+\beta}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\alpha\Gamma(\alpha+\beta+n+1)} \tag{A.4}$$

$$\int_{-1}^{1}(1-y)^{\alpha}(1+y)^{\beta}\left[P_n^{(\alpha,\beta)}(y)\right]^2 dy = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\alpha\Gamma(\alpha+\beta+n+1)} \tag{A.5}$$

Equations (A.4) and (A.5) are respectively invoked from Eqs. ET II 285(6) and ET II 285 (5,9) of Ref. 42 (pages 806 and 807 respectively). α and β in Eqs. (A.4) and (A.5) have nothing to do with the α and β in this work. Comparing the superscripts of Eqs. (A.4) and (A.5) with that of Eq. (A.3), we obtain the normalization constant as given Eq. (59) as

$$N_{nlD}^{q}=\left[\frac{q^{2\varepsilon}(1+2\varepsilon)\left(2\varepsilon+\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}+2n+2\right)n!\Gamma\left(2\varepsilon+\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}+n+2\right)}{2\left(1+\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}+2n+1\right)\Gamma(2\varepsilon+n+2)\Gamma\left(\sqrt{1+\frac{4}{q^2}(\phi+q\beta)}+n+1\right)}\right]^{\frac{1}{2}}$$

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