

## The diagonal Bernoulli differential estimation equation

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The Bernoulli Differential Equation traditionally applies a linearization procedure instead of solving the direct form, and viewed in state space has unknown parameters, focusing all attention on it. This equation viewed in state space with unknown matrix parameters had a natural transformation and introduced a diagonal description. In this case, the problem is to know the matrix parameters. This procedure is a new technique for solving the state space Bernoulli Differential Equation without using linearization into diagonal filtering application. Diagonal filtering is a kind of quadratic estimation. This is a procedure which uses observed signals with noises and produces the best estimation for unknown matrix parameters. More formally, diagonal filtering operates recursively on streams of noisy input signals to produce an optimal estimation of the underlying state system. The recursive nature allows running in Real-time bounded temporally using the present input signal and the previously calculated state and no additional past information. From a theoretical standpoint, the diagonal filtering assumption considered that the black-box system model includes all error terms and signals having a Gaussian distribution, described as a recursive system in a Lebesgue sense. Diagonal filtering has numerous applications in science and pure solutions, but generally, the applications are in tracking and performing the stochastic system.

*Keywords:* Filtering; matrix theory; control theory; stochastic processes.

Al resolver la ecuación diferencial de Bernoulli tradicionalmente se aplica un proceso de linealización en lugar de un método directo considerando que tiene parámetros desconocidos. Este artículo considera una transformación natural al espacio de estados e introduce la descripción diagonal; en este caso, el problema es conocer la matriz de parámetros. El procedimiento es una nueva técnica para resolver la ecuación diferencial de Bernoulli sin usar la linealización aplicando el filtrado en forma diagonal. Con el cual se realiza la estimación con base en el segundo momento de probabilidad. Éste es un procedimiento que utiliza a las señales observables con ruido, produce la mejor estimación para los parámetros desconocidos. Formalmente, éste opera recursivamente sobre la señal de entrada con ruido, produciendo una estimación óptima de los parámetros internos del sistema. Debido a la naturaleza recursiva del procedimiento, éste puede implementarse en tiempo-real ya que su respuesta está acotada temporalmente, usando para ello tan solo a la señal de entrada presente y el estado calculado anteriormente, sin información previa adicional. Desde un punto de vista teórico, la hipótesis principal del filtrado en forma diagonal es que el sistema subyacente es un sistema dinámico y que todos los términos, tanto de error como de la señal de entrada, tienen una distribución de Gauss. El filtro diagonal es un sistema recursivo en el sentido de Lebesgue que estima parámetros. Tiene numerosas aplicaciones en ciencias aplicadas y desarrollos teóricos. Una aplicación común es el seguimiento de las trayectorias en los sistemas dinámicos.

*Descriptores:* Teoría matricial; teoría de control; procesos estocásticos.

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### 1. Introduction

Applications mainly from dynamics, population biology and electrical theory are used to show how ordinary differential equations appear in science and applied science formulation problems. Many physics problems can be modeled in the first order of a nonlinear ordinary differential system such as the Bernoulli Differential Equation. An example is gases and liquids flow. The Bernoulli Equation with soft modifications incorporates viscous losses, compressibility and unsteady behaviour found in other more complex calculations. When viscous effects are incorporated, the result is called the Energy Equation.

The Bernoulli Differential Equation is distinguished by the degree. For instance, the equation having is applied to logistic model growth in biology [1] and chaos behavior [2], with forming Gizbun or quadratic equations commonly used

to analyze corrosion processes [3]. The differential equation is also a nonlinear part of the Klein-Gordon form which is widely used. Among these are: the dynamics of elementary particles and stochastic resonances studies [4], energy transportation [5], squeezed laser excitation [6].

As usually explained in mathematical handbooks [7], solving the Bernoulli Differential Equation is always through a linearization procedure recommended by Jacob Bernoulli [8]. The transformation from the nonlinear form to a linear differential equation is performed using a basic function, and later using the common method solving it [9]. However, due to its simplicity, the Bernoulli equation may not provide an accurate enough answer for many situations, but it is a good starting point. It can certainly provide a first parameters estimation. Instead of traditional linearization, this paper develops diagonal filtering as a new technique to

solve the Bernoulli Differential Equation without linearization procedures.

The filtering problem with respect to a black-box system is to find an optimal steady state description of unknown parameters affecting the proposed model, observing the convergence [10,11]. The filter estimation can be described optimally considering the gradient properties where each requirement contributes to the functional error [10].

Diagonal filtering as a fine system is built without losing its properties considering two steps: estimation and identification. The first process refers to evaluation of uncertain gain variables, like an unknown characteristic matrix parameters on the basis of the observable signal as an explicit description. As the estimation problem commonly assumes a suitable mathematical affine model with respect to the observable answer system described by discrete-time dynamic models and characterized by relationships among observable variables represented by a matrix difference equations [12]. In the second, the matrix parameters estimation is used in identification internal states and applied in the reconstruction reference model observable vector [13].

The whole process compounded by estimation and identification is known as a digital filter. The convergence rate is obtained through the functional error established between the observable signal and reference model [14].

A system viewed as a black-box with bounded inputs and outputs vectors with perturbations has an affine model corresponding in output [10]. Therefore, the internal dynamics and gains based on an affine model could be possibly known using the estimation and identification techniques [13].

The states space representation is a set of first-order differential equations [15] that relate to the mathematical differences model to the state space dimension [16]. Its description is a homogeneous differences equation, with the solution depending on the estimation results [17]. The state space representation with respect to the reference differences model allows observing the unknown matrix parameters [18].

The internal unknown matrix parameters can be estimated based on the second probability moment considering the instrumental variable method [17]. In this sense, the state variables and instrumental variable must be uncorrelated. Traditional estimation techniques are based on pseudo-inverse methodology [18].

The estimation cases include some unknown initial values; the second step uses the old values to get the following approximation [19].

The iteration continues until the diagonal matrix converges to the system dimensions. It is expected that the estimator values are convenient to define best estimations. For this, it is necessary to compute the eigenvalues and eigenvectors system, requiring an evaluation of some pseudo-inverse method that is expensive in computational complexity [20,21]. Thus, this procedure is not suitable at all, and in many cases with the same experiment gives different results. To avoid this, an alternative method is an optimal diagonal filter considering the observable signals given in diagonal

structure [22,23]. This reduces the computational complexity, because there is no necessity to implement any Penrose procedure [24,25].

The purpose of this paper is to show an application of diagonal filtering for solving the Bernoulli Differential Equation and is structured in the following manner: Sec. 1 is the present description; Sec. 2 describes the basic formalism of the diagonal filtering for a Laplacian form, Sec. 3 shows the simulation results and discussion, Sec. 4 determines the conclusions, and finally, Sec. 5 includes the theorem proofs.

### 2. Main results

In mathematics an ordinary differential equation is called a Bernoulli Eq. (1) when each term has an exponential indicating its grade in an algebraic sense. Bernoulli equations are special because they are nonlinear differential with known exact solutions. Nevertheless, they can be transformed into a linear equation using the diagonal forms.

$$y' + P(x)y = Q(x)y^\alpha \tag{1}$$

Few differential forms have a simply analytical solution, and their behaviour must be studied under certain conditions. The enormous importance of differential description is mainly due to the fact that the investigation of many problems in physics, chemistry and other applied sciences, can be reduced to the equations solution, which requires significant technical development such as modeling and simulation.

Let (1) be the Bernoulli equation with  $\alpha$ ,  $P(x)$ ,  $Q(x) \in R$ , its state space representation associated in diagonal form is given by (2).

**Theorem 1.** *The system (1) with respect to a reference signal is viewed as a black-box scheme and has a state space estimator described in (2).*

$$\begin{bmatrix} \dot{x}_{11} & 0 & 0 & 0 \\ 0 & \dot{x}_{22} & 0 & 0 \\ 0 & 0 & \dot{x}_{12} & 0 \\ 0 & 0 & 0 & \dot{x}_{21} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & P_1(s) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q(s) \end{bmatrix} \times \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_1^\alpha & 0 \\ 0 & 0 & 0 & x_2^\alpha \end{bmatrix} \tag{2}$$

Where  $\dot{x}_1 = \dot{x}_{11} + \dot{x}_{12}$ ,  $\dot{x}_2 = \dot{x}_{22} + \dot{x}_{21}$ ,  $P_1(s) = -P(s)$ .

**Proof 1.** See Appendix.

After the state space model is formed, the goal is to estimate the simplified unknown parameters

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & P_1(s) & 0 \\ 0 & 0 & Q(s) \end{bmatrix} \text{ denoted by } \tilde{A}_\varphi$$

In a short representation, the Bernoulli state space has the form (3).

$$\dot{\Phi}_\varphi = \tilde{A}_\varphi \Phi_\varphi \tag{3}$$

Where

$$\dot{\Phi}_\varphi = \begin{bmatrix} \dot{x}_{11} & 0 & 0 \\ 0 & \dot{x}_{22} & 0 \\ 0 & 0 & \dot{x}_{21} \end{bmatrix}$$

and

$$\Phi_\varphi = \begin{bmatrix} x_2 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_2^\alpha \end{bmatrix}$$

**Theorem 2.** The estimator  $\hat{A}_\varphi$  has a stochastic description based on  $\dot{\Phi}_\varphi$  and observable state,  $\Phi_\varphi$  as (4).

$$\hat{A}_\varphi = E \left\{ \dot{\Phi}_\varphi \check{M}_\varphi \right\} \tag{4}$$

The symbol  $\check{M}_\varphi$  represents the correlation matrix.

**Proof 2.** See Appendix.

**Theorem 3.** Let be  $\hat{A}_\varphi$  be the stochastic estimator in diagonal form as (4), then its recursive form in a Lebesgue sense given by (5), with  $k \in Z_+$ .

$$\hat{A}_{\varphi k} = s_k \left[ \check{M}_{\varphi k}^T + \frac{1}{2} \left( \check{M}_{\varphi k}^T - \check{M}_{\varphi k-1}^T \right) \right] + \hat{A}_{\varphi k-1} \tag{5}$$

**Proof 3.** See Appendix.

In Ref. 21, Medel *et al.* presented a method for constructing an optimal m-dimensional stochastic estimator for a black-box model in a diagonal form. Taking into account the previous results, the case 2-dimensional in a Laplacian structure for a diagonal filtering is used.

Let (6) be a differential equation for a 2-differentiable real-valued function  $\varphi \in R^2$ ,  $a_i \in R, i \in Z_+$

$$\nabla^2 \varphi - a_1 \nabla \varphi - a_0 \varphi = 0, \tag{6}$$

The problem is to know the unobservable matrix parameters

$$\begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix}$$

Corollary 1 establishes the state space representation for (6) and the solution for the associated Bernoulli equation as Theorem 1, considering (1) as a primitive function for the Laplacian equation.

**Corollary 1.** Let (7) be a Laplacian equation, where  $a_0, a_1, \alpha \in R$  are constants.

$$\nabla^2 \varphi - a_1 \nabla \varphi - a_0 \varphi = a_0 \alpha \varphi^{\alpha-1} \nabla \varphi, \tag{7}$$

Transforming (7) to the Bernoulli equation, the minimum number of state variables required to represent it is equal to differential equation system order. Then the simplified state space representation in diagonal form is giving by (8).

$$\begin{bmatrix} \dot{x}_{11} & 0 & 0 \\ 0 & \dot{x}_{22} & 0 \\ 0 & 0 & \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \times \begin{bmatrix} x_2 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_1^\alpha \end{bmatrix} \tag{8}$$

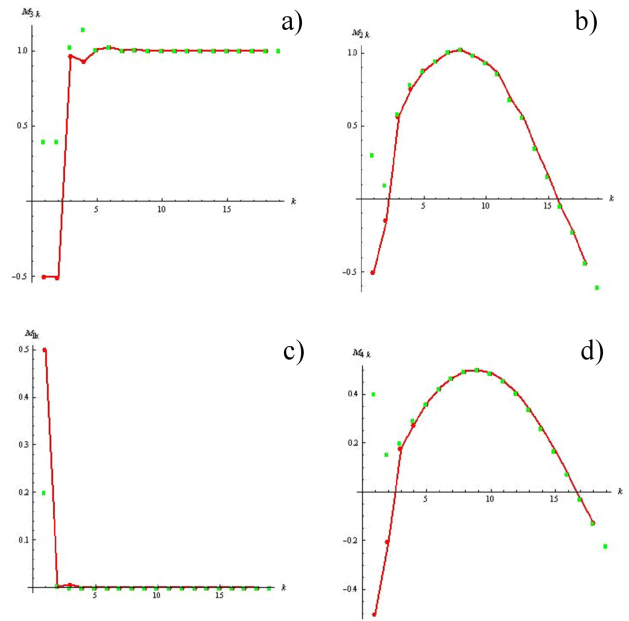


FIGURE 1. Observable signal (red) and its identification components (green): a) First  $\check{m}_{1k}$  and  $\hat{m}_{1k}$ , b) Second  $\check{m}_{2k}$  and  $\hat{m}_{2k}$ , c) Third  $\check{m}_{3k}$  and  $\hat{m}_{3k}$ , d) Fourth  $\check{m}_{4k}$  and  $\hat{m}_{4k}$ .

Where  $a_3 = a_0 \alpha \varphi^{-1}$

**Proof C1.** See Appendix.

### 3. Simulation results and Discussion

Considering the Laplacian Eq. (7) and its state space representation in differences (3), Theorem 2 describes the algorithm for constructing the optimal stochastic estimator for a diagonal filter. Figure 1 separately shows each of the system components. Figures 1 a), b), c) and d) show in red, the observable signal,  $\check{m}_{ik}, i = \overline{1,4}$  and in green their identification,  $\hat{m}_{ik}, i = \overline{1,4}$ . Figure 2 a) shows the estimated values for be  $A_k$ . Figure 2 b) shows functional error  $J_k$ .

Diagonal filtering works in a two-step process. In the first step, the diagonal filtering produces an estimator using the innovation technique and the current state variables, along with their uncertainties. In the second step, once the outcome of the next measurement, necessarily corrupted with some amount of error, including random noise is observed, these estimates are updated using a weighted average, with more weight being given to estimate with higher certainty.

All measurements and calculations based on models are estimated to some degree. The diagonal filtering averages an identification of a system state with a new measurement using matrix parameters estimation. The parameters are calculated from the covariance matrix described by the instrumental variable that guarantees a simple inversion matrix. The result is a new state estimation that lies in between the predicted and measured state, and has a better estimated uncertainty than either. This process is repeated for every step,

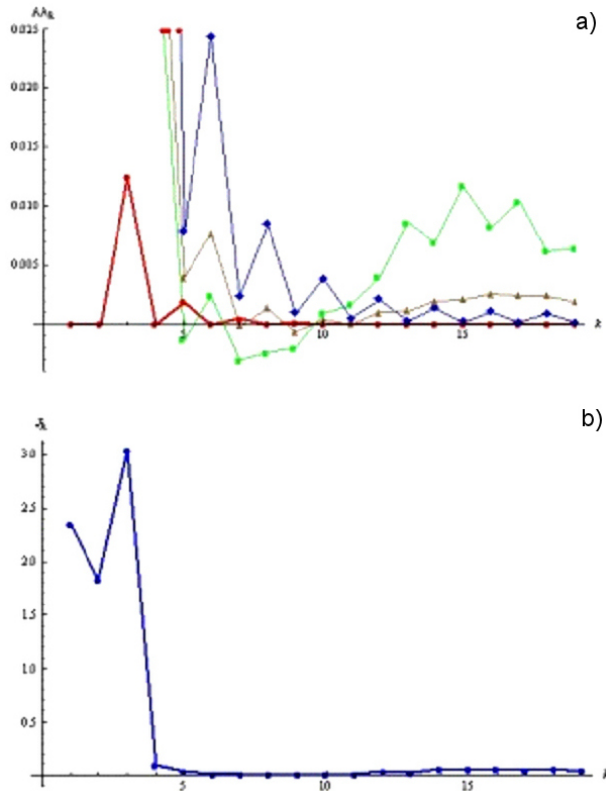


FIGURE 2. a) Estimated values for be  $\hat{A}_k$ , b) Functional error  $J_k$ .

with the new estimation and its covariance informing the prediction used in the following iteration. This means that the diagonal filtering works recursively and requires only the last best guess, not the entire data system state to calculate a new state.

When performing the actual calculations for the filter, as discussed below, the state estimation and covariance are coded into matrices to handle the multiple dimensions involved in a single set of calculations. This allows representation of linear relationships between different state variables in any covariance transition model

### 4. Conclusion

Diagonal structure is an efficient recursive filter that estimates the internal state of a linear dynamic system from noisy signals. It can be used in a wide range of science applications and will be an important topic in control theory and control systems. Diagonal filtering is a solution to what is probably the most fundamental problem in description systems.

In most applications, the internal state is much larger than the few observable parameters which are measured. However, using the method for a m-dimensional estimator [21], diagonal filtering can estimate the entire internal state. For discrete filters the computational complexity is more or less proportional to the number of filter coefficients.

Practical implementation of diagonal filtering is often simple due to the ability obtaining a good estimate of the matrix parameters and is optimal in all cases.

### Appendix

This section presents the proofs for the Theorems described in the paper.

**Proof 1.** For all  $\alpha \in R$ , defining  $x_1 = x_2$  and  $x_2 = y'$  where  $x_2 = P_1(s) + Q(s)x_2^\alpha$  with  $P_1(s) = -P_1(s)$ , then (1) can be written in a matrix form as (9).

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & P_1(s) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Q(s) \end{bmatrix} \begin{bmatrix} x_1^\alpha \\ x_2^\alpha \end{bmatrix} \tag{9}$$

(9) Transformed in a diagonal form as in [21] obtaining (10).

$$\begin{bmatrix} \dot{x}_{11} & 0 & 0 & 0 \\ 0 & \dot{x}_{22} & 0 & 0 \\ 0 & 0 & \dot{x}_{12} & 0 \\ 0 & 0 & 0 & \dot{x}_{21} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & P_1(s) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q(s) \end{bmatrix} \times \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_1^\alpha & 0 \\ 0 & 0 & 0 & x_2^\alpha \end{bmatrix} \tag{10}$$

The simplified diagonal form for (10) is given by (11).

$$\begin{bmatrix} \dot{x}_{11} & 0 & 0 \\ 0 & \dot{x}_{22} & 0 \\ 0 & 0 & \dot{x}_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \times \begin{bmatrix} x_2 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_2^\alpha \end{bmatrix} \tag{11}$$

**Proof 2.** The simplified diagonal form for (3) is given by (11). Let

$$\vartheta_\varphi = \begin{bmatrix} x_2 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_2^\alpha \end{bmatrix}^T$$

be an instrumental variable. Multiplying (11) by  $\vartheta_\varphi$ , with second probability moment given by (12).

$$E \{ \dot{\Phi}_\varphi \vartheta_\varphi \} = \tilde{A}_\varphi E \{ M_\varphi \} \tag{12}$$

Where  $M_\varphi = \Phi_\varphi \vartheta_\varphi$ ,  $\det(M_\varphi) \neq 0$  and  $\check{M}_\varphi = \vartheta_\varphi M_\varphi^{-1}$ . Finally, the optimal stochastic estimator is given by (4).

**Proof 3.** From (4), computing the recursive form for the stochastic estimator be  $\hat{A}_{\varphi_k}$ . Considering  $\check{M}_\varphi$  and  $\dot{\Phi}_\varphi$  are diagonal matrices, then  $\check{M}_\varphi = \check{M}_\varphi^T \dot{\Phi}_\varphi = \Phi_\varphi^T$  describes in (13)

$$\hat{A}_{\varphi_k} = E \{ \check{M}_\varphi^T \dot{\Phi}_\varphi \} \tag{13}$$

The integral form in (14),

$$\hat{A}_{\varphi_k} = \int_{\Phi_\phi} \check{M}_{\varphi}^T d\Phi_{\varphi} \tag{14}$$

And using the Lebesgue form in (15),

$$\hat{A}_{\varphi_k} = \lim_{k \rightarrow S} \sum_{i=1}^k \left[ \check{M}_{\varphi_{i-1}}^T (\Phi_{\varphi_i} - \Phi_{\varphi_{i-1}}) + \frac{1}{2} (\check{M}_{\varphi_i}^T - \check{M}_{\varphi_{i-1}}^T) (\Phi_{\varphi_i} - \Phi_{\varphi_{i-1}}) \right] \tag{15}$$

By linearity property in (16).

$$\hat{A}_{\varphi_k} = \sum_{i=1}^k \lim_{|\Phi_{\varphi_i} - \Phi_{\varphi_{i-1}}| \rightarrow S_i} \left[ \check{M}_{\varphi_{i-1}}^T S_i + \frac{1}{2} (\check{M}_{\varphi_i}^T - \check{M}_{\varphi_{i-1}}^T) S_i \right] \tag{16}$$

Evaluating the last term,  $i = k$ , in (17)

$$\begin{aligned} \hat{A}_{\varphi_k} &= \check{M}_{\varphi_{k-1}}^T S_k + \frac{1}{2} (\check{M}_{\varphi_k}^T - \check{M}_{\varphi_{k-1}}^T) S_k \\ &+ \sum_{i=1}^{k-1} \lim_{|\Phi_{\varphi_i} - \Phi_{\varphi_{i-1}}| \rightarrow S_i} \left[ \check{M}_{\varphi_{i-1}}^T S_i + \frac{1}{2} (\check{M}_{\varphi_i}^T - \check{M}_{\varphi_{i-1}}^T) S_i \right] \end{aligned} \tag{17}$$

Simplifying in (18)

$$\hat{A}_{\varphi_k} = \check{M}_{\varphi_{k-1}}^T S_k + \frac{1}{2} (\check{M}_{\varphi_k}^T - \check{M}_{\varphi_{k-1}}^T) S_k + \hat{A}_{\varphi_{k-1}}. \tag{18}$$

**Proof C1.** Let  $x_1, x_2$  be the state variables with states as  $x_1 = \varphi, x_2 = \nabla\varphi$ . Therefore,  $x_1 = \nabla\varphi$ , corresponding to  $x_2, \dot{x}_2 = \nabla^2\varphi$ , is equal to (7) and writing in diagonal form by Theorem 1 gives (8) with  $a_3 = a_0\alpha x_2$ . Where  $a_0 = a_3/\alpha x_2, \forall \alpha \neq 0, \alpha \in R$ .

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